Automorphisms of rational projective $\mathbb{K}^*$-surfaces

Dissertation

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Timo Hummel
aus Göppingen

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Joseph Haydn, Die Schöpfung, Hob. XXI:2.

Introduction

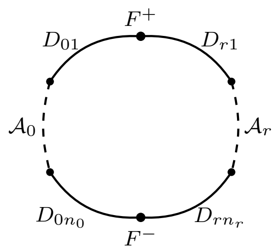
We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. By a $\mathbb{K}^*$*-surface* we mean a normal irreducible surface X endowed with an effective morphical action $\mathbb{K}^* \times X \to X$ of the multiplicative group $\mathbb{K}^*$. The geometry of $\mathbb{K}^*$-surfaces has been intensely studied by many authors; see for instance [**18**–**20**, **41**–**44**]. We consider the automorphism group $\mathrm{Aut}(X)$ of a rational projective $\mathbb{K}^*$-surface X. This is an affine algebraic group and our first aim is to give a detailed explicit description of the unit component of $\mathrm{Aut}(X)$ in terms of basic data of the action.

We will apply the results to the study of almost homogeneous $\mathbb{K}^*$-surfaces in general and, more concretely, develop classifications in the almost homogeneous log del Pezzo case.

In order to formulate our main result of Chapter 1, let us recall the basic geometric features of $\mathbb{K}^*$-surfaces. One calls a fixed point *elliptic (hyperbolic, parabolic)* if it lies in the closure of infinitely many (precisely two, precisely one) non-trivial $\mathbb{K}^*$-orbit(s). Elliptic and hyperbolic fixed points are isolated, whereas the parabolic fixed points form a closed smooth curve with at most two connected components. Every projective normal $\mathbb{K}^*$-surface X has a *source* and a *sink*, that means irreducible components $F^+, F^- \subseteq X$ of the fixed point set admitting open $\mathbb{K}^*$-invariant neighborhoods $U^+, U^- \subseteq X$ such that

$$\lim_{t \to 0} t \cdot x \ \in \ F^+ \text{ for all } x \in U^+, \qquad \lim_{t \to \infty} t \cdot x \ \in \ F^- \text{ for all } x \in U^-,$$

where these limits are the respective values at the points 0 and ∞ of the unique morphism $\mathbb{P}_1 \to X$ extending the orbit map $t \mapsto t \cdot x$. The source, and as well the sink, consists either of a single elliptic fixed point or it is a smooth irreducible curve of parabolic fixed points; we write x^+ and x^- in the elliptic case and D^+ and D^- in the parabolic case. Apart from the source and the sink, we find at most hyperbolic fixed points. The raw geometric picture of a rational projective $\mathbb{K}^*$-surface X is as follows:



The general orbit $\mathbb{K}^* \cdot x \subseteq X$ has trivial isotropy group $\mathbb{K}^*_x$ and its closure connects the source and the sink in the sense that it contains one fixed point from F^+ and one from F^-. Besides the general orbits, we have the special non-trivial orbits. Their closures are rational curves $D_{ij} \subseteq X$ forming the *arms* $\mathcal{A}_i = D_{i1} \cup \ldots \cup D_{in_i}$ of X, where $i = 0, \ldots, r$, the intersections $F^+ \cap D_{i1}$ and $D_{in_i} \cap F^-$ consist each of a fixed point and any two subsequent D_{ij}, D_{ij+1} intersect in a hyperbolic fixed point. To every such rational curve D_{ij} we associate an integer, namely the order l_{ij} of the $\mathbb{K}^*$-isotropy group of the general point of D_{ij}.

Every $\mathbb{K}^*$-surface X admits a minimal equivariant resolution $\pi\colon \tilde{X} \to X$ of singularities. If there is a parabolic fixed point curve $D^+ \subseteq X$, then we consider the points $x_i \in X$ lying in $D^+ \cap D_{i1}$. If x_i is singular, then the fibre $\pi^{-1}(x_i)$ of the minimal resolution is a connected part $E_{i1} \cup \ldots \cup E_{iq_i}$ of an arm of $\tilde{X}$, where the curve E_{i1} intersects the proper transform of D^+. We define

$$c_i(D^+) \ := \ \left(-E_{i1}^2 - \cfrac{1}{-E_{i2}^2 - \cfrac{1}{\ldots - E_{iq_i}^2}} \right)^{-1}$$

if x_i is singular and $c_i(D^+) = 0$ else. We call an elliptic fixed point $x \in X$ *simple*, if $\pi^{-1}(x)$ is contained in an arm of $\tilde{X}$. If X admits a simple elliptic fixed point, then we may assume that this is x^-. The fibre $\pi^{-1}(x^-) = E_1 \cup \ldots \cup E_q$ is a connected part of an arm of $\tilde{X}$ and E_q contains a smooth elliptic fixed point of $\tilde{X}$. In this situation, we define

$$c(x^-) \ := \ \left(-E_q^2 - \cfrac{1}{-E_{q-1}^2 - \cfrac{1}{\ldots - E_1^2}} \right)^{-1}$$

if x^- is singular and $c(x^-) := 0$ else. Finally, a point $x \in X$ is called *quasismooth* if it is a toric surface singularity; see Definition 6.7 and Corollary 6.12 for more background. In case of a quasismooth simple elliptic fixed point $x^- \in X$, we can always assume the numeration of the arms $\mathcal{A}_0, \ldots, \mathcal{A}_r$ to be *normalized* in the sense that $l_{0n_0} \geq \ldots \geq l_{rn_r}$ and $l_{in_i} = l_{jn_j}$ implies $D_{in_i}^2 \leq D_{jn_j}^2$ whenever $i < j$ and $n_i, n_j \geq 2$. In this situation, the exceptional curves $E_1, \ldots, E_q \subseteq \tilde{X}$ belong to the arm $\tilde{\mathcal{A}}_0 \subseteq \tilde{X}$ mapping onto the arm $\mathcal{A}_0 \subseteq X$; see Proposition 9.12. We denote by $\tilde{l}_{0\tilde{n}_0}$ the order of the isotropy group of the general point of $E_q = \tilde{D}_{0\tilde{n}_0} \subseteq \tilde{\mathcal{A}}_0$. We are ready to state the first result of this thesis.

THEOREM 0.1. *Let X be a non-toric rational projective $\mathbb{K}^*$-surface. Then the unit component of the automorphism group* $\mathrm{Aut}(X)$ *of X is given as a semidirect product*

$$\mathrm{Aut}(X)^0 \ = \ (\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta) \rtimes_\psi \mathbb{K}^*, \qquad \rho \in \mathbb{Z}_{\geq 0}, \ \zeta \in \{0, 1\}.$$

If X has neither a non-negative fixed point curve nor a quasismooth simple elliptic fixed point, then $\rho = \zeta = 0$ holds. Otherwise, precisely one of the following holds.

(i) *There is a non-negative fixed point curve. Then we can assume that this curve is $D^+ \subseteq X$. In this situation, we have $\zeta = 0$ and*

$$\rho \ = \ \max(0, \ (D^+)^2 + 1 - \sum_{i=0}^{r} c_i(D^+)).$$

The group homomorphism $\psi\colon \mathbb{K}^* \to \mathrm{Aut}(\mathbb{K}^\rho)$ *fixing the semidirect product structure is given by* $t \mapsto t^{-1}E_\varrho$.

(ii) *There is exactly one quasismooth simple elliptic fixed point. We can assume that this is* x^- *and the numeration of the arms is normalized. Then*

$$\rho \;=\; \max\left(0,\; \left\lfloor l_{1n_1}^{-1} \min_{i\neq 0}(l_{in_i}D_{in_i}^2 + (l_{in_i} - l_{1n_1})D_{in_i}D_{1n_1}) - c(x^-)\right\rfloor + 1\right)$$

holds. Moreover, we have $\zeta = 1$ *if and only if for all* $i \neq 1$ *the following inequalities are satisfied*

$$l_{in_i}D_{in_i}^2 \;\geq\; (l_{0n_0} - l_{in_i})D_{in_i}D_{0n_0}.$$

The semidirect product structure on $\mathrm{Aut}(X)$ *is determined by the following. For* $\zeta = 1$, *the homomorphism* $\varphi\colon \mathbb{K} \to \mathrm{Aut}(\mathbb{K}^\rho)$ *is given by*

$$s \mapsto A = (a_{\mu\alpha}), \text{ where } a_{\mu\alpha} \;=\; \begin{cases} \binom{\alpha-1}{\mu-1}s^{\alpha-\mu}, & \alpha \geq \mu, \\ 0 & \alpha < \mu. \end{cases}$$

Moreover, the group homomorphism $\psi\colon \mathbb{K}^* \to \mathrm{Aut}(\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta)$ *is given by*

$$t \;\mapsto\; \begin{cases} \mathrm{diag}\left(t^{\tilde{l}_{0\tilde{n}_0}}, \ldots, t^{\tilde{l}_{0\tilde{n}_0}-(\rho-1)l_{1n_1}}\right), & \zeta = 0, \\ \mathrm{diag}(t^{l_{0n_0}}, \ldots, t^{l_{0n_0}-(\rho-1)l_{1n_1}}, t^{l_{1n_1}}), & \zeta = 1. \end{cases}$$

Automorphism groups of rational surfaces have also been considered by several other authors. For instance, Sakamaki [**46**] studied the case of cubic surfaces without parameters. More generally, Cheltsov and Prokhorov [**10**] and, independently, also Martin and Stadlmayr [**37**] determined the Gorenstein log del Pezzo surfaces with infinite automorphism groups. It turns out that 50 out of the 53 listed surfaces of [**10**, **37**] are in fact $\mathbb{K}^*$-surfaces and the descriptions of the automorphism groups obtained there are in accordance with Theorem 0.1. Note that Theorem 0.1 does not make any assumptions on the singularities or the canonical divisor and thus goes far beyond the Gorenstein log del Pezzo case. Of course, one may ask why Theorem 0.1 excludes the toric surfaces. For the sake of completeness, we treat them in Proposition 4.9.

Let us take a closer look at the action of the automorphism group. We discuss the question when a non-toric rational $\mathbb{K}^*$-surface X admits an almost transitive action, that means a morphical action of an algebraic group G having an open orbit. In this case, X is an equivariant compactification of a G-homogeneous space and it is natural to ask when it is even an equivariant compactification of an algebraic group. Theorem 0.1 allows us to give answers in terms of isotropy group orders and intersection numbers.

THEOREM 0.2. *Consider a non-toric rational projective* $\mathbb{K}^*$*-surface* X*. Then the following statements are equivalent.*

(i) *The surface* X *admits an almost transitive action of a two-dimensional algebraic group* G.

(ii) *The surface* X *is almost homogeneous in the sense that the action of the automorphism group* $\mathrm{Aut}(X)$ *on* X *is almost transitive.*

(iii) *There is a quasismooth simple elliptic fixed point, say $x^- \in X$, and, assuming the numeration of the arms to be normalized, one of the following two series of inequalities is valid:*

$$l_{in_i} D_{in_i}^2 + (l_{in_i} - l_{1n_1}) D_{in_i} D_{1n_1} \ \geq \ l_{1n_1} c(x^-), \quad i = 1, \ldots, r,$$

$$l_{in_i} D_{in_i}^2 \ \geq \ (l_{0n_0} - l_{in_i}) D_{in_i} D_{0n_0}, \quad i = 0, 2, \ldots, r.$$

Assume that one of the above statements holds and let G be a two-dimensional algebraic group acting effectively and almost transitively on X.

(iv) *If only one of the series of inequalities of (iii) is valid, then G is a non-abelian semidirect product $\mathbb{K} \rtimes_{\varphi} \mathbb{K}^*$. Moreover:*
   (a) *If the first series of inequalities holds, then $\varphi(t)(s) = t^{l_{1n_1}} s$ and for general $x \in X$, the isotropy group G_x is cyclic of order l_{0n_0}. Thus, X is an equivariant G-compactification if and only if $l_{0n_0} = 1$.*
   (b) *If the second series of inequalities holds, then $\varphi(t)(s) = t^{l_{0n_0}} s$ and for general $x \in X$, the isotropy group G_x is cyclic of order l_{1n_1}. Thus, X is an equivariant G-compactification if and only if $l_{1n_1} = 1$.*

(v) *If the series of inequalities of (iii) both are valid, then the groups G from (iv) (a) and (iv) (b) both act, and, moreover, X is an equivariant compactification of the vector group $G = \mathbb{K}^2$.*

The case of almost homogeneous Gorenstein del Pezzo surfaces has been investigated in [**15**, **16**]. Theorem 0.2 is in accordance with the results obtained there and it delivers in addition the general isotropy groups. Note that there exist normal surfaces X which equivariantly compactify the abelian group $\mathbb{K} \times \mathbb{K}^*$. But any such X is a toric surface according to [**4**, Theorem 2]. More explicit statements on the almost homogenoeus case are given in Section 13. For instance Proposition 13.12 specifies up to conjugation all the semidirect products $\mathbb{K} \times_{\psi} \mathbb{K}^* \subseteq \mathrm{Aut}(X)$ acting almost transitively. Moreover, the almost transitive $\mathbb{K}^2$-actions on X from Theorem 0.2 (v) are so-called *additive actions* on X in the sense of [**5**, **17**]. In Proposition 13.17, we determine up to conjugation by elements from $\mathbb{K}^*$ all the additive actions on X.

Using our result on the combinatorics of rational projective $\mathbb{K}^*$-surfaces with non-trivial automorphism group we proceed with a classification result for almost homogeneous log del Pezzo $\mathbb{K}^*$-surfaces, i.e. log del Pezzo surfaces whose automorphism group acts with an open oprbit.

A *del Pezzo surface* is a normal projective surface X with ample anticanonical divisor $-\mathcal{K}_X$. For a resolution of singularities $\varphi\colon Y \to X$ consider the *ramification formula*

$$\mathcal{K}_Y \ = \ \varphi^* \mathcal{K}_X + \sum a_i E_i,$$

where the E_i are the prime components of the exceptional divisor and a_i are called the *discrepancies* of the resolution. The surface X is called

(i) *log terminal*, if $a_i > -1$ for all i.
(ii) *ε-log terminal*, if $a_i > -1 + \varepsilon$ for all i for a given $0 < \varepsilon < 1$.
(iii) *canonical*, if $a_i \geq 0$ for all i.

(iv) *terminal*, if $a_i > 0$ for all i.

A log terminal del Pezzo surface is called, in short, *log del Pezzo*. Those surfaces have been studied intensely beginning in the late 19th century with the smooth case.

In [**6**] Alexeev contributed to the study of these surfaces showing that for each ε there are only finitely many families of ε-log terminal del Pezzo surfaces. Subsequently, concrete classification work was mostly based on using an important invariant of X, its *Gorenstein index*. This is the smallest positive integer ι_X such that $\iota_X K_X$ is Cartier. Restricting to $\iota(X) \leq 2$, all log del Pezzo surfaces have been classified in [**7**] by giving the corresponding intersection graphs of a certain resolution of singularities. The theory of $K3$ surfaces played a substantial role in their work. In a completely different manner, log del Pezzo surfaces with $\iota(X) = 2$ were classified in [**39**]. This approach was adopted in [**22**] extending his ideas to cover the case that $\iota(X) = 3$. Additionally, the applied method can be used to perform the classification for arbitrary Gorenstein index.

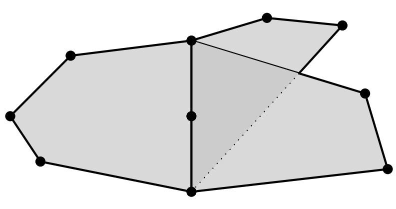
The classification process boils down to a completely combinatorial problem when restricting to surfaces that allow an effective action of a non-trvial torus. We achieve concrete algorithms that can be implemented to a computer algebra system, `Maple` for instance, and the results can be analyzed more thoroughly. If the acting torus has dimension 2, the surface is toric and can be described by its associated *Fano polygon*, i.e. a lattice polygon such that the origin is contained in its relative interior. The following one-to-one correspondence provides the combinatoric picture:

$$\{\text{Fano polygons}\} \quad \longleftrightarrow \quad \{\text{Fano toric surfaces}\}$$

For a survey of classification results for Fano toric surface see [**34**]. Moreover the *Graded Ring Database* (see [**9**]) lists the polygons of toric log del Pezzo surfaces with Gorensteinindex up to 17.

For a one-dimensional torus action we arrive at the study of $\mathbb{K}^*$-surfaces. Different approaches have been used to classify parts of this broader class of surfaces, for example the use of polyhedral divisors. This way, results for the case of Picard number 1 and $\iota(X) \leq 3$ were obtained in [**47**].

Our approach relies on the anticanonical complex, a combinatorial object that generalizes Fano polytopes to a wider class of varieties. It has been introduced in [**8**, **31**] and has shown to be a very useful tool for classifications of Fano varieties with torus actions: Results for threefolds with a two torus action using the anticanonical complex have been obtained in [**8**, **40**], Furthermore there have been works on generalizations of this tool in [**1**] for non-complete and non-$\mathbb{Q}$-Gorenstein, [**31**] for higher complexity and [**38**] for non-degenerate toric complete intersections.



Anticanonical complex.

In the surface case we introduce the notion of *LDP complex*, a combinatorial object that uniquely describes a log del Pezzo $\mathbb{K}^*$-surface. It is a polyhedral complex of two-dimensional polygons in a higher dimensional ambient rational vector space and for every $\mathbb{K}^*$-surface it coincides with its anticanonical complex. Adding two arithmetic condition, this object yields the following correspondence, analagous to the well known correspondece for toric Fano varieties:

THEOREM 0.3. *There is a one-to-one correspondence between LDP complexes and log del Pezzo $\mathbb{K}^*$-surfaces.*

$$\{\textit{LDP complexes}\} \quad \longleftrightarrow \quad \{\textit{log del Pezzo } \mathbb{K}^*\textit{-surfaces}\}$$

As in the case of Fano polytopes, the LDP complex fixes the described $\mathbb{K}^*$-surface up to isomorphism and there is a combinatorial description for ε-log canonicity and almost homogeneity. Moreover, we achieve that, under certain arithmetic conditions, removing vertices of an LDP complex is the same as contracting divisors of the corresponding log del Pezzo surface.

In Chapter 2, we focus on the classification of almost homogeneous LDP complexes and present work obtained in cooperation with Daniel Hättig. In [**3**] a first classification of almost homogeneous log del Pezzo $\mathbb{K}^*$-surfaces has been given for Picard number one and varieties with at worst one singularity. Furthermore as mentioned before, classification results for del Pezzo surfaces with infinite automorphism groups have been recently achieved in [**10**] and in [**37**]. Moreover equivariant compactifications of two-dimensional algebraic groups have been studied in [**15, 16**]. The authors restricted to du Val singularities.

We develope a concrete algorithm to classify almost homogeneos $1/k$-log canonical $\mathbb{K}^*$-surfaces after finitely many steps, therefore settling the question of classifying all log del Pezzo $\mathbb{K}^*$-surfaces with at worst $1/k$-log canonical singularities. This algorithm has been implemented in `Maple`, providing a classification of all nontoric $1/k$-log canonical almost homogeneous del Pezzo $\mathbb{K}^*$-surfaces for any given $k \in \mathbb{Z}_{\geq 1}$. In particular, for increasing k all log terminal almost homogeneous del Pezzo $\mathbb{K}^*$-surfaces can be obtained using this algorithm.

We shortly describe the main ingredients of the actual classification process. The algorithmic classification follows three steps:

(i) Find all almost k-hollow polygons.
(ii) Find all combinatorially minimal almost homogeneous, almost k-hollow LDP complexes.
(iii) Build all almost homogeneous, almost k-hollow LDP complexes.

The key observation is that removing certain vertices shrinks the LDP complex, hence the LDP complex obtained remains almost k-hollow. Therefore, it suffices to find all almost k-hollow LDP complexes that do not admit any further removal of vertices and build complexes starting with these combinatorially minimal ones by reversing the removal process.

Observe that this also shows that for $\mathbb{K}^*$-surfaces contraction preserves the property to be log del Pezzo and there are at worst $1/k$-log canonical singularities, see Theorem 7.7. Furthermore the first step requires the classification of almost k-hollow polygons. For the case $k = 3$ we achieved the following result:

THEOREM 0.4. *The following statements hold:*

(i) *There are exactly* 47902 *toric* $1/3$-*log canonical del Pezzo surfaces.*
(ii) *There are exactly* 91 *non-toric* $1/3$-*log canonical combinatorially minimal almost homogeneous del Pezzo* $\mathbb{K}^*$-*surfaces.*
(iii) *There are exactly* 21968 *non-toric* $1/3$-*log canonical almost homogeneous del Pezzo* $\mathbb{K}^*$-*surfaces.*

In particular, we showed that the unit component of the automorphism group of a non-toric $1/3$-log canonical almost homogeneous del Pezzo $\mathbb{K}^*$-surfaces is of dimension at most 7.

Contents

CHAPTER 1

The unit component of the automorphism group

1. Outline of the chapter

Let us give an outline of this chapter, showing its basic ingredients and some main ideas. Our working environment is the *Cox ring based approach* of [**25**, **29**] to rational T-varieties X of complexity one, that means that X is normal, rational and comes with an effective torus action $T \times X \to X$ such that the general T-orbit is of codimension one in X. One of the basic features of this approach is that it provides a natural T-equivariant closed embedding $X \subseteq Z$ into a toric variety Z. In Section 2, we present a brief general reminder.

We will also make use of the description of the automorphism group of a toric variety via the *Demazure roots* of its defining fan; see [**11**, **14**] and Section 3 for a quick summary. First applications are the tools provided in Section 4 and the explicit description of the automorphism goups of toric surfaces given there. The understanding of $\mathrm{Aut}(X)$ for a complete rational T-variety X of complexity one is not yet as developed as in the toric case. However, the main results of [**3**] show that $\mathrm{Aut}(X)^0$ is generated by T and the additive one-parameter groups, also called root groups, arising from *Demazure P-roots*; see also Section 3. In Theorem 5.4 we provide a presentation of the automorphisms arising from Demazure P-roots as restrictions of automorphisms of the ambient toric variety Z which are explicitly given in Cox coordinates. The explicit nature of the result is crucial for our purposes. The general question to which extent a variety inherits its automorphisms from a suitable ambient variety is interesting as well. For Mori dream spaces X, a positive result concerning $\mathrm{Aut}(X)^0$ is given in [**28**, Thm. 4.4]; see also [**45**] for further results in the case of quasismooth Fano weighted complete intersections.

From Section 6 on, we focus on rational projective $\mathbb{K}^*$-surfaces. We first recall basics on their geometry and relate defining data to self intersection numbers, see Sections 6 and 7. In Theorem 8.4 we figure out geometric implications of the existence of a quasismooth simple elliptic fixed point: a non-toric rational projective $\mathbb{K}^*$-surface X can have at most one such fixed point and if there is one, then any parabolic fixed point curve is contractible or its intersection with any arm of X is a singularity of X. In Section 9, we introduce horizontal and vertical *P-roots*, which basically means adapting the more involved notion of a Demazure P-root to the surface case. Together with $\mathbb{K}^*$, the root groups arising from the P-roots generate $\mathrm{Aut}(X)^0$. We link existence of P-roots to the geometry of X. From [**3**] we infer that $\mathrm{Aut}(X)$ acts with an open orbit if and only if there is a horizontal P-root.

Proposition 9.6 shows that the presence of a horizontal P-root forces a quasismooth simple elliptic fixed point. By Proposition 9.17, existence of vertical P-roots exclude quasismooth simple elliptic fixed points. Consequently, $\mathrm{Aut}(X)$ does not act almost transitively if we have vertical P-roots. Each vertical root is uniquely associated with a parabolic fixed point curve in the sense that the corresponding root group moves that curve. Proposition 9.18 shows that if there are vertical roots, then they are all associated with the same fixed point curve.

Starting with Section 10, we study the structure of the unit component of $\mathrm{Aut}(X)$. The first task is to figure out relations among the root groups arising from the P-roots. A sufficiently detailed study allows us to figure out minimal generating systems of P-roots. Proposition 10.2 does this for the case that $\mathrm{Aut}(X)$ acts with an open orbit and Proposition 10.3 settles the remaining case. In Section 11, we show in terms of the combinatorics of defining data that for the minimal resolution of singularities $\tilde{X} \to X$ of a rational projective $\mathbb{K}^*$-surface, the groups $\mathrm{Aut}(\tilde{X})^0$ and $\mathrm{Aut}(X)^0$ coincide. Whereas the latter can as well be deduced from the general existence of a functorial resolution in characteristic zero, our investigation is more specific and allows us to relate the root groups of X with those of $\tilde{X}$ in an explicit manner. In Section 12, we prove Theorem 0.1. The basic idea is to gain the desired information on $\mathrm{Aut}(X)^0$ and its action on X via morphisms $X \leftarrow \tilde{X} \to X'$, where $\tilde{X} \to X$ is the minimal resolution and $\tilde{X} \to X'$ a suitable birational contraction to a certain toric surface that allows to keep track on the relevant root groups. Finally, in Section 13 we study almost transitive actions on X, specify the acting two-dimensional groups and prove Theorem 0.2.

The last two Sections show useful examples and that our results are in accordance with recent classification results of [**10**] and [**15, 16**]. The results of this chapter are published in [**27**].

2. T-varieties of low complexity

Here we provide the necessary background on toric varieties and rational varieties with torus action of complexity one. Throughout the whole thesis, the ground field $\mathbb{K}$ is algebraically closed and of characteristic zero. We simply write $\mathbb{K}$ for the additive group of the ground field, $\mathbb{K}^*$ for the multiplicative one and $\mathbb{T}^n$ for the n-fold direct product $(\mathbb{K}^*)^n$.

By a torus we mean an algebraic group T isomorphic to some $\mathbb{T}^n$. A quasitorus is a direct product of a torus and a finite abelian group. By a T-variety X we mean a normal, irreducible variety X with an effective action of a torus T given by a morphism $T \times X \to X$. The complexity of a T-variety X is the difference $\dim(X) - \dim(T)$.

We turn to toric varieties, which by definition are the T-varieties of complexity zero. The basic feature of toric varieties is that they are completely described via lattice fans. We assume the reader to be familiar with the foundations of this theory as explained for instance in [**12, 13, 23**].

We will intensely use the Cox ring and Cox's quotient construction for toric varieties [**11**]. Recall that for any normal variety X with only constant global invertible functions and finitely generated divisor class group $\mathrm{Cl}(X)$, one associates a Cox sheaf

$$\mathcal{R} \;=\; \bigoplus_{D\in \mathrm{Cl}(X)} \mathcal{O}_X(D),$$

see [**2**, Chap. 1] for details. The Cox ring $\mathcal{R}(X)$ is the $\mathrm{Cl}(X)$-graded algebra of global sections of the Cox sheaf. If the Cox ring is finitely generated, we can establish the following picture

$$\begin{tikzcd}
\mathrm{Spec}_X\,\mathcal{R} \arrow[r, equal] & \hat{X} \arrow[d, "/\!\!/H"'] & \subseteq & \bar{X} \arrow[r, equal] & \mathrm{Spec}_X\,\mathcal{R}(X) \\
& X & & &
\end{tikzcd}$$

where $\bar{X}$ is the total coordinate space coming with an action of the characteristic quasitorus $H = \mathrm{Spec}\,\mathbb{K}[\mathrm{Cl}(X)]$ and the characteristic space $\hat{X}$ which is an open H-invariant subset of $\bar{X}$ and has X as a good quotient for the induced H-action. In the case of toric varieties, this picture can be established in terms of defining lattice fans as follows.

Construction 2.1. Let Z be the toric variety defined by a fan Σ in a lattice N such that the primitive generators $v_1, \ldots, v_r$ (of the rays) of Σ span the rational vector space $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. We have mutually dual exact sequences

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^r \xrightarrow{\;P\;} N$$

$$0 \longleftarrow K \xleftarrow[\;Q\;]{} \mathbb{Z}^r \xleftarrow[\;P^*\;]{} M \longleftarrow 0$$

where $P\colon \mathbb{Z}^r \to N$ sends the i-th canonical basis vector $e_i \in \mathbb{Z}^r$ to the i-th primitive generator $v_i \in N$; we also speak of the generator matrix $P = [v_1, \ldots, v_r]$ of Σ. The lower sequence gives rise to an exact sequence

$$1 \longrightarrow H \longrightarrow \mathbb{T}^r \xrightarrow{\;P\;} T_Z \longrightarrow 1$$

involving the quasitorus $H = \mathrm{Spec}\,\mathbb{K}[K]$ and the acting torus $T_Z = \mathrm{Spec}\,\mathbb{K}[M]$ of Z. Moreover, the divisor class group and the Cox ring of Z are given as

$$\mathrm{Cl}(Z) \;=\; K, \qquad\qquad \mathcal{R}(Z) \;=\; \mathbb{K}[T_1, \ldots, T_r],$$

where the $\mathrm{Cl}(Z)$-grading of $\mathcal{R}(Z)$ is given by $\deg(T_i) = Q(e_i)$. Finally, we obtain a fan $\hat{\Sigma}$ in $\mathbb{Z}^r$ consisting of certain faces of the positive orthant, namely

$$\hat{\Sigma} \;:=\; \{\delta_0 \preceq \mathbb{Q}^r_{\geq 0};\; P(\delta_0) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

The toric variety $\hat{Z}$ associated with $\hat{\Sigma}$ is the characteristic space of Z, sitting as an open toric subset in the total coordinate space $\bar{Z} := \mathbb{K}^r$. As P is a map of the fans $\hat{\Sigma}$ and Σ, it defines a toric morphism $p\colon \hat{Z} \to Z$, the good quotient for the action of the quasitorus $H = \ker(p) \subseteq \mathbb{T}^r$ on $\hat{Z}$.

Remark 2.2. Construction 2.1 allows to put hands on the points of a toric variety: every $x \in Z$ can be written as $x = p(z)$, where $z \in \hat{Z}$ is a point with closed H-orbit in $\hat{Z}$. Such a presentation is unique up to multiplication by elements of H and we call $z = (z_1, \ldots, z_r)$ *Cox coordinates* for the point $x \in Z$.

We will use the Cox ring based approach to torus actions as developed for the case of rational T-varieties of complexity one in [**25**, **29**], and, more recently, in widest possible generality in [**26**]. Let us first have a look at an example, showing some of the main ideas.

Example 2.3. Consider the surface X in the weighted projective space $\mathbb{P}_{2,7,1,13}$ given as the zero set of a weighted homogeneous trinomial equation:

$$X \;=\; V(T_{01}^7 + T_{12}^2 + T_{21}T_{22}) \;\subseteq\; \mathbb{P}_{2,7,1,13},$$

where each of the variables appears in exactly one monomial as indicated by the double-indexing T_{ij}. Then X comes with a $\mathbb{K}^*$-action, given by

$$t \cdot [z] \;=\; [z_{01}, z_{11}, t^{-1}z_{21}, tz_{22}].$$

The ambient space $\mathbb{P}_{2,7,1,13}$ is a toric variety. Its defining fan Σ lives in $\mathbb{Z}^3$ and its rays are generated by the columns v_{01}, v_{11}, v_{21} and v_{22} of the matrix

$$P \;=\; \begin{bmatrix} -7 & 2 & 0 & 0 \\ -7 & 0 & 1 & 1 \\ -4 & 1 & 1 & 0 \end{bmatrix}.$$

This setting reflects the key features of the $\mathbb{K}^*$-action on our surface X. For instance, setting $D_{ij} := X \cap V(T_{ij})$, we obtain the arms of X as

$$\mathcal{A}_0 \;=\; D_{01}, \qquad \mathcal{A}_1 \;=\; D_{11}, \qquad \mathcal{A}_2 \;=\; D_{21} \cup D_{22}.$$

Moreover, the order l_{ij} of the isotropy group of the general point in D_{ij} shows up in the upper two rows of the matrix P, as we have

$$l_{01} \;=\; 7, \qquad l_{11} \;=\; 2, \qquad l_{21} \;=\; 1, \qquad l_{22} \;=\; 1.$$

Finally, X inherits many geometric properties from its ambient space $Z := \mathbb{P}_{2,7,1,13}$. Most significantly, the Cox ring of X is the factor algebra

$$\mathcal{R}(X) \;=\; \mathcal{R}(Z)/\langle g \rangle \;=\; \mathbb{K}[T_{01}, T_{11}, T_{21}, T_{22}]/\langle T_{01}^7 + T_{12}^2 + T_{21}T_{22} \rangle,$$

where the grading of the Cox rings $\mathcal{R}(X)$ and $\mathcal{R}(Z)$ by the divisor class group $\mathrm{Cl}(X) = \mathrm{Cl}(Z) = \mathbb{Z}$ are given by

$$\deg(T_{01}) \;=\; 2, \quad \deg(T_{02}) \;=\; 7, \quad \deg(T_{11}) \;=\; 1, \quad \deg(T_{21}) \;=\; 13.$$

This picture extends as follows. The arbitrary rational projective $\mathbb{K}^*$-surface X comes embedded into a certain toric variety, is given by specific trinomial equations as above and the key features of the $\mathbb{K}^*$-action as well as the geometry of X can be extracted from the defining data. Here comes the construction provided in [**25**, **29**]; see also [**2**, Sec. 3.4].

Construction 2.4. Fix $r \in \mathbb{Z}_{\geq 1}$, a sequence $n_0, \ldots, n_r \in \mathbb{Z}_{\geq 1}$, set $n := n_0 + \ldots + n_r$, and fix integers $m \in \mathbb{Z}_{\geq 0}$ and $0 < s < n + m - r$. The input data are matrices

$$A = [a_0, \ldots, a_r] \in \mathrm{Mat}(2, r{+}1; \mathbb{K}), \qquad P = \begin{bmatrix} L & 0 \\ d & d' \end{bmatrix} \in \mathrm{Mat}(r{+}s, n{+}m; \mathbb{Z}),$$

where A has pairwise linearly independent columns and P is built from an $(s \times n)$-block d, an $(s \times m)$-block d' and an $(r \times n)$-block L of the form

$$L \;=\; \begin{bmatrix} -l_0 & l_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \dots & l_r \end{bmatrix}, \qquad l_i \;=\; (l_{i1}, \dots, l_{in_i}) \;\in\; \mathbb{Z}^{n_i}_{\geq 1}$$

such that the columns v_{ij}, v_k of P are pairwise different primitive vectors generating $\mathbb{Q}^{r+s}$ as a cone. Consider the polynomial algebra

$$\mathbb{K}[T_{ij}, S_k] \;:=\; \mathbb{K}[T_{ij}, S_k;\ 0 \leq i \leq r,\ 1 \leq j \leq n_i, 1 \leq k \leq m].$$

Denote by $\mathfrak{I}$ the set of all triples $I = (i_1, i_2, i_3)$ with $0 \leq i_1 < i_2 < i_3 \leq r$ and define for any $I \in \mathfrak{I}$ a trinomial

$$g_I \;:=\; g_{i_1,i_2,i_3} \;:=\; \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}, \qquad T_i^{l_i} \;:=\; T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}.$$

Consider the factor group $K := \mathbb{Z}^{n+m}/\mathrm{im}(\mathrm{P}^*)$ and the projection $Q\colon \mathbb{Z}^{n+m} \to K$. We define a K-grading on $\mathbb{K}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) \;:=\; w_{ij} \;:=\; Q(e_{ij}), \qquad \deg(S_k) \;:=\; w_k \;:=\; Q(e_k).$$

Then the trinomials g_I just introduced are K-homogeneous, all of the same degree. In particular, we obtain a K-graded factor algebra

$$R(A, P) \;:=\; \mathbb{K}[T_{ij}, S_k] \;/\; \langle g_I;\ I \in \mathfrak{I} \rangle.$$

The ring $R(A, P)$ just constructed is a normal complete intersection ring and its ideal of relations is, for example, generated by $g_{i,i+1,i+2}$, where $i = 0, \dots, r-2$. The varieties X with torus action of complexity one are constructed as quotients of $\operatorname{Spec} R(A, P)$ by the quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$. Each of them comes embedded into a toric variety.

Construction 2.5. Situation in Construction 2.4. Consider the common zero set of the defining relations of $R(A, P)$:

$$\bar{X} \;:=\; V(g_I;\ I \in \mathfrak{I}) \;\subseteq\; \bar{Z} \;:=\; \mathbb{K}^{n+m}$$

Let Σ be any fan in the lattice $N = \mathbb{Z}^{r+s}$ having the columns of P as the primitive generators of its rays. Construction 2.1 leads to a commutative diagram

$$\begin{tikzcd} \bar{X} \arrow[r, phantom, "\subseteq"] & \bar{Z} \\ \hat{X} \arrow[u, phantom, "\cup" sloped] \arrow[r] \arrow[d, "p", "/\!\!/ H"'] & \hat{Z} \arrow[u, phantom, "\cup" sloped] \arrow[d, "p"', "/\!\!/ H"] \\ X \arrow[r] & Z \end{tikzcd}$$

with a variety $X = X(A, P, \Sigma)$ embedded into the toric variety Z associated with Σ. Dimension, divisor class group and Cox ring of X are given by

$$\dim(X) = s + 1, \qquad \mathrm{Cl}(X) \;\cong\; K, \qquad \mathcal{R}(X) \;\cong\; R(A, P).$$

The subtorus $T \subseteq \mathbb{T}^{r+s}$ of the acting torus of Z associated with the sublattice $\mathbb{Z}^s \subseteq \mathbb{Z}^{r+s}$ leaves X invariant and the induced T-action on X is of complexity one.

Remark 2.6. In Construction 2.5, the group $H \cong \operatorname{Spec} \mathbb{K}[\operatorname{Cl}(X)]$ is the characteristic quasitorus and $\bar{X} \cong \operatorname{Spec} \mathcal{R}(X)$ is the total coordinate space of X. Moreover, $p\colon \widehat{X} \to X$ is the characteristic space over X.

Remark 2.7. As in the toric case, Construction 2.5 yields *Cox coordinates* for the points of $X = X(A, P, \Sigma)$. Every $x \in X \subseteq Z$ can be written as $x = p(z)$, where $z \in \hat{X} \subseteq \hat{Z}$ is a point with closed H-orbit in $\hat{X}$ and this presentation is unique up to multiplication by elements of H.

Remark 2.8. We say that the matrix P from Construction 2.4 is *irredundant* if we have $l_{i1}n_i \geq 2$ for $i = 0, \ldots, r$. In Construction 2.5, we may assume without loss of generality that P is irredundant. An $X(A, P, \Sigma)$ with irredundant P is a toric variety if and only if $r = 1$ holds.

The results of [**2**, **25**, **29**] tell us in particular the following; see also [**26**] for a generalization to higher complexity.

Theorem 2.9. *Every normal rational projective variety with a torus action of complexity one is equivariantly isomorphic to some $X(A, P, \Sigma)$.*

3. Demazure roots and automorphisms

Here we present the necessary general background and facts on automorphisms of toric varieties and rational varieties with a torus action of complexity one.

The approach to automorphisms via Demazure roots involves locally nilpotent derivations. Let us briefly recall some basics from [**21**]. A *derivation* on an integral affine $\mathbb{K}$-algebra R is a $\mathbb{K}$-linear map $\delta\colon R \to R$ satisfying the Leibniz rule

$$\delta(fg) \ = \ \delta(f)g + f\delta(g).$$

A derivation $\delta\colon R \to R$ is *locally nilpotent* if every $f \in R$ admits an $n \in \mathbb{N}$ with $\delta^n(f) = 0$. Any locally nilpotent derivation $\delta\colon R \to R$ defines a representation

$$\bar{\lambda}_\delta^\star\colon \mathbb{K} \to \operatorname{Aut}(R), \qquad \bar{\lambda}_\delta^\star(s)(f) := \exp(s\delta)(f) := \sum_{k=0}^{\infty} \frac{s^k}{k!}\delta^k(f).$$

In fact this yields a bijection between the locally nilpotent derivations of R and the rational representations of $\mathbb{K}$ by automorphisms of R. Consequently,

$$\bar{\lambda}_\delta\colon \mathbb{K} \ \to \ \operatorname{Aut}(\operatorname{Spec} R), \qquad s \ \mapsto \ \operatorname{Spec}(\bar{\lambda}_\delta^\star(s))$$

is a group homomorphism and, by construction, each of the automorphisms $\bar{\lambda}_\delta(s)$ of $\operatorname{Spec}(R)$ has $\bar{\lambda}_\delta(s)^* = \bar{\lambda}_\delta^\star(s)$ as its comorphism.

As for any complete rational variety, the automorphism group of a toric variety is an affine algebraic group. Its structure has been studied by Demazure [**14**] and Cox [**11**]. The following is a key concept.

Definition 3.1. Notation as in 2.1. A *Demazure root* at the primitive generator $v_i \in N$ of Σ is an integral linear form $u \in M$ satisfying the conditions

$$\langle u, v_i \rangle \ = \ -1, \qquad\qquad \langle u, v_j \rangle \ \geq \ 0 \text{ for all } j \neq i.$$

Construction 3.2. Notation as in 2.1. Let $u \in M$ be a Demazure root at the primitive generator $v_i \in N$ of Σ. The associated locally nilpotent derivation δ_u on $\mathbb{K}[T_1, \ldots, T_r]$ is defined by its values on the variables:

$$\delta_u(T_j) \; := \; \begin{cases} T_i T^{P^*(u)}, & j = i, \\ 0, & j \neq i, \end{cases} \qquad \text{where} \quad T^{P^*(u)} = T_1^{\langle u, v_1 \rangle} \cdots T_r^{\langle u, v_r \rangle}.$$

Observe that $\delta_u^2 = 0$ holds. Moreover, we have $Q(P^*(u)) = 0$ and thus δ_u preserves the K-grading of $\mathbb{K}[T_1, \ldots, T_r]$. The corresponding rational representation of $\mathbb{K}$ on $\mathbb{K}[T_1, \ldots, T_n]$ is given by

$$\bar{\lambda}_u^\star(s)(T_j) \; = \; \begin{cases} T_i + sT_i T^{P^*(u)}, & j = i, \\ T_j, & j \neq i, \end{cases}$$

Now, if $\bar{\lambda}_u \colon \mathbb{K} \to \operatorname{Aut}(\bar{Z})$ leaves $\hat{Z}$ invariant, for instance, if Z is complete, then $\bar{\lambda}_u$ descends to a homomorphism $\lambda_u \colon \mathbb{K} \to \operatorname{Aut}(Z)$, the *root group* associated with the Demazure root u. In Cox coordinates, we have

$$\lambda_u(s)(z) \; = \; z + s z_i z^{P^*(u)} e_i.$$

Theorem 3.3. *See* [**11**, Cor. 4.7]. *Let Z be a complete toric variety arising from a fan Σ in a lattice N. Then* $\operatorname{Aut}(Z)^0$ *is generated as group by the acting torus T_Z of Z and the images $\lambda_u(\mathbb{K})$, where u runs through the Demazure roots of Σ.*

The concept of Demazure roots was extended in [**3**] to the case of normal rational varieties X with an effective torus action $T \times X \to X$ of complexity one. Let us recall the basic notions and facts.

Definition 3.4. See [**3**, Def. 5.2]. Let P be a matrix as in Construction 2.4. Consider the columns $v_{ij}, v_k \in N = \mathbb{Z}^{r+s}$ of P and the dual lattice M of N.

(i) A *vertical Demazure P-root* is a tuple (u, k_0) with a linear form $u \in M$ and an index $1 \leq k_0 \leq m$ satisfying

$$\begin{aligned} \langle u, v_{ij} \rangle \; &\geq \; 0 \qquad \text{for all } i, j, \\ \langle u, v_k \rangle \; &\geq \; 0 \qquad \text{for all } k \neq k_0, \\ \langle u, v_{k_0} \rangle \; &= \; -1. \end{aligned}$$

(ii) A *horizontal Demazure P-root* is a tuple (u, i_0, i_1, C), where $u \in M$ is a linear form, $i_0 \neq i_1$ are indices with $0 \leq i_0, i_1 \leq r$, and $C = (c_0, \ldots, c_r)$ is a sequence with $1 \leq c_i \leq n_i$ such that

$$\begin{aligned} l_{ic_i} \; &= \; 1 \qquad \text{for all } i \neq i_0, i_1, \\ \langle u, v_{ic_i} \rangle \; &= \; \begin{cases} 0, & i \neq i_0, i_1, \\ -1, & i = i_1, \end{cases} \\ \langle u, v_{ij} \rangle \; &\geq \; \begin{cases} l_{ij}, & i \neq i_0, i_1, \quad j \neq c_i, \\ 0, & i = i_0, i_1, \quad j \neq c_i, \\ 0, & i = i_0, \qquad j = c_i, \end{cases} \\ \langle u, v_k \rangle \; &\geq \; 0 \qquad \text{for all } k. \end{aligned}$$

EXAMPLE 3.5. Consider the defining matrix P of the $\mathbb{K}^*$-surface discussed before in Example 2.3, that means

$$P \;=\; \begin{bmatrix} -7 & 2 & 0 & 0 \\ -7 & 0 & 1 & 1 \\ -4 & 1 & 1 & 0 \end{bmatrix}.$$

As $m = 0$, there are no vertical Demazure P-roots, but we have a horizontal Demazure P-root (u, i_0, i_1, C) given by

$$u \;=\; (-1, 0, 1), \qquad i_0 \;=\; 0, \qquad i_1 \;=\; 1, \qquad C \;=\; (1, 1, 2).$$

CONSTRUCTION 3.6. See [**3**, Constr. 3.4 and 5.7]. Let A and P be as in Construction 2.4. Given $i_0 \neq i_1$ with $0 \leq i_0, i_1 \leq r$ and $C = (c_0, \ldots, c_r)$ with $1 \leq c_i \leq n_i$ we define $\zeta = \zeta(i_0, i_1, C) = (\zeta_{ij}, \zeta_k) \in \mathbb{Z}^{n+m}$ by

$$\zeta_{ij} \;:=\; \begin{cases} l_{ij}, & i \neq i_0, i_1, \ j \neq c_i, \\ -1, & i = i_1, \qquad j = c_{i_1}, \\ 0 & \text{else}, \end{cases} \qquad\qquad \zeta_k \;:=\; 0, k = 1, \ldots, m.$$

Moreover, to $u \in M$ and the lattice vectors $\zeta \in \mathbb{Z}^{n+m}$ just introduced we assign the following monomials

$$h^u \;=\; \prod_{i,j} T_{ij}^{\langle u, v_{ij} \rangle} \prod_k S_k^{\langle u, v_k \rangle}, \qquad\qquad h^\zeta \;:=\; \prod_{i,j} T_{ij}^{\zeta_{ij}} \prod_k S_k^{\zeta_k}.$$

Every Demazure P-root κ defines a locally nilpotent derivation δ_κ on $\mathbb{K}[T_{ij}, S_k]$. If $\kappa = (u, k_0)$ is vertical, then one sets

$$\delta_\kappa(T_{ij}) \;:=\; 0 \text{ for all } i, j, \qquad \delta_\kappa(S_k) \;:=\; \begin{cases} h^u S_{k_0}, & k = k_0, \\ 0, & k \neq k_0. \end{cases}$$

If $\kappa = (u, i_0, i_1, C)$ is horizontal, then there is a unique vector $\beta = \beta(A, i_0, i_1)$ in the row space of A with $\beta_{i_0} = 0$, $\beta_{i_1} = 1$ and one sets

$$\delta_\kappa(T_{ij}) \;:=\; \begin{cases} \beta_i \frac{h^u}{h^\zeta} \prod_{k \neq i, i_0} \frac{\partial T_k^{l_k}}{\partial T_{k c_k}}, & j = c_i, \\ 0, & j \neq c_i, \end{cases} \qquad \delta_\kappa(S_k) \;:=\; 0, \ k = 1, \ldots, m.$$

In both cases, the derivation δ_κ repects the K-grading. Moreover, δ_κ leaves the ideal of defining relations of $R(A, P)$ invariant and thus induces a locally nilpotent derivation on $R(A, P)$. This gives us

$$\bar{\lambda}_\kappa := \bar{\lambda}_{\delta_\kappa} \colon \mathbb{K} \;\to\; \mathrm{Aut}(\bar{X}),$$

where $\bar{\lambda}_\kappa$ is the additive one-parameter group associated with the locally nilpotent derivation δ_κ of $R(A, P)$. If $\hat{X}$ is invariant under $\bar{\lambda}_\kappa$, for example, if X is complete, then the *root group* associated with κ is

$$\lambda_\kappa := \lambda_{\delta_\kappa} \colon \mathbb{K} \;\to\; \mathrm{Aut}(X).$$

EXAMPLE 3.7. We continue the discussion started in 2.3 and 3.5. Recall that the $\mathbb{K}^*$-surface X comes embedded into a weighted projective space Z via

$$X \;=\; V(T_{01}^7 + T_{11}^2 + T_{21}T_{22}) \;\subseteq\; \mathbb{P}_{2,7,1,13} \;=\; Z.$$

We determine the root group $\lambda_\kappa\colon \mathbb{K} \to \mathrm{Aut}(X)$ arising from the horizontal Demazure-P root $\kappa = (u, i_0, i_1, C)$ given by

$$u \ = \ (-1,0,1), \qquad i_0 \ = \ 0, \qquad i_1 \ = \ 1, \qquad C \ = \ (1,1,2).$$

First we have to write down the monomials h^u and h^ζ and the vector β from Construction 3.6. These are

$$h^u \ = \ T_{01}^3 T_{11}^{-1} T_{21}, \qquad h^\zeta \ = \ T_{11}^{-1} T_{21}, \qquad \frac{h^u}{h^\zeta} \ = \ T_{01}^3, \qquad \beta \ = \ (0,1,-1).$$

Next we describe the derivation $\delta_\kappa\colon R(A,P) \to R(A,P)$. It is determined by its values on the variables T_{ij}, which in turn are given as

$$\delta_\kappa(T_{01}) \ = \ 0, \qquad \delta_\kappa(T_{11}) \ = \ T_{01}^3 \frac{\partial T_2^{l_2}}{\partial T_{22}} \ = \ T_{01}^3 T_{21},$$

$$\delta_\kappa(T_{21}) \ = \ 0, \qquad \delta_\kappa(T_{22}) \ = \ -T_{01}^3 \frac{\partial T_1^{l_1}}{\partial T_{11}} \ = \ -2T_{01}^3 T_{11}.$$

For computing the exponential map, we have to evaluate the powers of δ_κ on the variables. For T_{11} and T_{22} this needs further computation:

$$\begin{array}{lclclcl}
\delta_\kappa^2(T_{11}) & = & \delta_\kappa(T_{01}^3 T_{21}) & = & T_{01}^3 \delta_\kappa(T_{21}) & = & 0, \\
\delta_\kappa^2(T_{22}) & = & \delta_\kappa(-T_{01}^3 T_{11}) & = & -2T_{01}^3 \delta_\kappa(T_{11}) & = & -2T_{01}^6 T_{21}, \\
\delta_\kappa^3(T_{22}) & = & \delta_\kappa(-2T_{01}^6 T_{21}) & = & -2T_{01}^6 \delta_\kappa(T_{21}) & = & 0.
\end{array}$$

Using this, we directly obtain the comorphism $\bar{\lambda}_\kappa(s)^* = \exp(s\delta_\kappa)$. On the variables T_{ij} it is given by

$$\begin{array}{lcl}
T_{01} & \mapsto & T_{01}, \\
T_{11} & \mapsto & T_{11} + sT_{01}^3 T_{21}, \\
T_{21} & \mapsto & T_{21}, \\
T_{22} & \mapsto & T_{22} - 2sT_{01}^3 T_{11} - s^2 T_{01}^6 T_{21}.
\end{array}$$

Consequently, we can represent each automorphism $\lambda_\kappa(s)\colon X \to X$, where $s \in \mathbb{K}$, explicitly in Cox coordinates as

$$[z_{01},\, z_{11},\, z_{21},\, z_{22}] \ \mapsto \ [z_{01},\, z_{11} + sz_{01}^3 z_{21},\, z_{21},\, z_{22} - 2sz_{01}^3 z_{11} - s^2 z_{01}^6 z_{21}].$$

One of the central ingredients of the first chapter is the following result on automorphisms of varieties with a torus action of complecity one.

THEOREM 3.8. *See* [**3**, Thm. 5.5 and Cor. 5.11]. *Let* $X = X(A,P,\Sigma)$ *be a complete variety arising from Construction 2.5. Then* $\mathrm{Aut}(X)^0$ *is generated as a group by the acting torus* T_X *of* X *and the root groups associated with the Demazure-*P *roots.*

Every Demazure P-root in the sense of Definition 3.4 also hosts a Demazure root in the sense of Definition 3.1. We spend a few words on the relations among the associated automorphisms.

REMARK 3.9. Consider a complete variety $X = X(A,P,\Sigma)$ and its ambient toric variety Z as in Construction 2.5.

(i) Let $\kappa = (u, k_0)$ be a vertical Demazure P-root. Then u is a Demazure root at v_{k_0}, each $\lambda_u(s) \in \mathrm{Aut}(Z)$ leaves $X \subseteq Z$ invariant and the restriction of $\lambda_u(s)$ to X equals $\lambda_\kappa(s) \in \mathrm{Aut}(X)$.

(ii) Let $\kappa = (u, i_0, i_1, C)$ be a horizontal Demazure P-root. Then u is a Demazure root at $v_{i_1 c_{i_1}}$. In general, the automorphism $\lambda_u(s) \in \mathrm{Aut}(Z)$ does not leave $X \subseteq Z$ invariant.

EXAMPLE 3.10. Consider once more the $\mathbb{K}^*$-surface $X \subseteq Z = \mathbb{P}_{2,7,1,13}$ from 2.3. As seen in 3.5, we have the horizontal Demazure-P root $\kappa = (u, i_0, i_1, C)$, where

$$u \;=\; (-1, 0, 1), \qquad i_0 \;=\; 0, \qquad i_1 \;=\; 1, \qquad C \;=\; (1, 1, 2).$$

In Example 3.7, we computed the associated root group. In Cox coordinates the automorphisms $\lambda_\kappa(s)$ are given by

$$[z_{01},\, z_{11},\, z_{21},\, z_{22}] \;\mapsto\; [z_{01},\, z_{11} + s z_{01}^3 z_{21},\, z_{21},\, z_{22} - 2s z_{01}^3 z_{11} - s^2 z_{01}^6 z_{21}].$$

The linear form u also defines a Demazure root at v_{11} for Z in the sense of Definition 3.1. By Construction 3.2 the corresponding automorphisms are

$$\lambda_u(s)\colon Z \;\to\; Z, \qquad [z_{01},\, z_{11},\, z_{21},\, z_{22}] \;\mapsto\; [z_{01},\, z_{11} + s T_{01}^3 T_{21},\, z_{21},\, z_{22}]$$

Observe that these ambient automorphisms do not leave the surface $X \subseteq Z$ invariant. For instance, we have

$$x \;=\; [1, 0, -1, 1] \;\in\; X, \qquad \lambda_u(1)(x) \;=\; [1, 1, -1, 1] \;\notin\; X.$$

The toric variety $Z = \mathbb{P}_{2,7,1,13}$ admits two further Demazure roots, each at the primitive ray generator v_{22}, namely

$$u' \;:=\; (0, -1, 1), \qquad\qquad u'' \;:=\; (-1, -1, 2).$$

The corresponding derivations vanish at all variables T_{ij} except for T_{22}, and on T_{22} the evaluations are given as

$$\delta_{u'}(T_{22}) \;=\; T_{01}^3 T_{11}, \qquad\qquad \delta_{u''}(T_{22}) \;=\; T_{01}^6 T_{21}.$$

With the associated automorphisms of Z, we can represent $\lambda_\kappa(s)$ on X as a composition of ambient automorphisms

$$\lambda_\kappa(s) \;=\; \lambda_u(s) \circ \lambda_{u'}(-2s) \circ \lambda_{u''}(-s^2)|_X$$

4. Automorphisms of complete toric surfaces

We consider the toric variety arising from a complete fan and investigate groups of automorphisms generated by the acting torus and root groups arising from Demazure roots at one or two generators. The results are given in Propositions 4.2 and 4.4. As an independent application we present in Proposition 4.9 the automorphism groups of the projective toric surfaces.

REMINDER 4.1. Let G, H be groups and $\varphi\colon H \to \mathrm{Aut}(G)$ a homomorphism. The *semidirect product* is the set $G \times H$ together with the group law

$$(g, h) \cdot (g', h') \;:=\; (g\varphi(h)(g'), hh').$$

The notation for the semidirect product is $G \rtimes_\varphi H$ and we call φ the *twisting homomorphism*. Observe that $G = G \times \{e_H\}$ is a normal subgroup in $G \rtimes_\varphi H$.

PROPOSITION 4.2. *Let Σ be a complete fan in $\mathbb{Z}^n$, denote by $P = [v_1, \ldots, v_r]$ the generator matrix and let Z be the associated toric variety. Moreover, fix $0 \leq i_0 \leq r$ and let $u_1, \ldots, u_\rho$ be pairwise distinct Demazure roots at v_{i_0}.*

(i) *Let* $1 \le j \le k \le \rho$. *Then* $\delta_{u_j}\delta_{u_k} = \delta_{u_k}\delta_{u_j} = 0$ *holds. In particular, for any two* $s_j, s_k \in \mathbb{K}$, *we have*

$$\bar{\lambda}_{u_j}(s_j)^*\bar{\lambda}_{u_k}(s_k)^* \;=\; \bar{\lambda}_{u_k}(s_k)^*\bar{\lambda}_{u_j}(s_j)^*$$

(ii) *Let* $U \subseteq \mathrm{Aut}(Z)$ *be the subgroup generated by the root groups* $\lambda_{u_1}, \ldots, \lambda_{u_\rho}$. *Then we have an isomorphism of algebraic groups*

$$\Theta\colon \mathbb{K}^\rho \;\to\; U, \qquad (s_1, \ldots, s_\rho) \;\mapsto\; \lambda_{u_1}(s_1)\cdots\lambda_{u_\rho}(s_\rho).$$

(iii) *The acting torus* $\mathbb{T}^n \subseteq \mathrm{Aut}(Z)$ *normalizes* $U \subseteq \mathrm{Aut}(Z)$ *and we have an isomorphism of algebraic groups*

$$\Psi\colon \mathbb{K}^\rho \rtimes_\psi \mathbb{T}^n \;\to\; U\mathbb{T}^n, \qquad (s,t) \;\mapsto\; \Theta(s)t,$$

where the twisting homomorphism $\psi\colon \mathbb{T}^n \to \mathrm{Aut}(\mathbb{K}^\rho)$ *sends* $t \in \mathbb{T}^n$ *to the diagonal matrix* $\mathrm{diag}(\chi^{u_1}(t), \ldots, \chi^{u_\varrho}(t))$.

In the proof and also later, we will make use of the following fact; see for instance [**17**, Lemma 1].

Lemma 4.3. *Consider a complete fan* Σ *in* $\mathbb{Z}^n$, *the associated toric variety* Z *and a Demazure root* u *of* Σ. *Then, for all* $t \in \mathbb{T}^n$ *and* $s \in \mathbb{K}$, *the root group* λ_u *associated with* u *satisfies* $t^{-1}\lambda_u(s)t = \lambda_u(\chi^u(t)s)$.

Proof of Proposition 4.2. We prove (i). Due to the definition of δ_{u_j} and δ_{u_k} given in Construction 3.2, we have $\delta_{u_k}(T_i) = \delta_{u_j}(T_i) = 0$ for all $i \neq i_0$. We conclude

$$\delta_{u_j}\delta_{u_k}(T_i) \;=\; 0 \;=\; \delta_{u_k}\delta_{u_j}(T_i),$$

whenever $i \neq i_0$. Moreover, using the Leibniz rule, we compare the evaluations at T_{i_0} and obtain

$$\delta_{u_j}\delta_{u_k}(T_{i_0}) \;=\; \delta_{u_j}\prod_{i\neq i_0} T_i^{\langle u_k, v_i\rangle} \;=\; 0 \;=\; \delta_{u_k}\prod_{i\neq i_0} T_i^{\langle u_j, v_i\rangle} \;=\; \delta_{u_k}\delta_{u_j}(T_{i_0}).$$

As δ_j and δ_k commute, we can apply the homomorphism property of the exponential map which yields

$$\bar{\lambda}_{u_j}(s_j)^*\bar{\lambda}_{u_k}(s_k)^* = \exp(s_j\delta_{u_j} + s_k\delta_{u_k}) = \exp(s_k\delta_{u_k} + s_j\delta_{u_j}) = \bar{\lambda}_{u_k}(s_k)^*\bar{\lambda}_{u_j}(s_j)^*.$$

This proves (i). As a consequence, λ_{u_j} and λ_{u_k} commute. Thus, the map Θ from (ii) is a homomorphism. Let us see why Θ is injective. Assume

$$\Theta(s_1, \ldots, s_\rho) \;=\; \lambda_{u_1}(s_1)\cdots\lambda_{u_\rho}(s_\rho) \;=\; \mathrm{id}_Z.$$

The task is to show $s_1 = \ldots = s_\rho = 0$. In Cox coordinates, the automorphism $\vartheta := \Theta(s_1, \ldots, s_\rho)$ of Z is given by

$$\bar{\vartheta}\colon z \;\mapsto\; \tilde{z}, \qquad \tilde{z}_i \;=\; \begin{cases} z_{i_0} + s_1 z^{P^*u_1} z_{i_0} + \ldots + s_\rho z^{P^*u_\rho} z_{i_0}, & i = i_0, \\ z_i & i \neq i_0. \end{cases}$$

Consider the set $\hat{Z}_0 \subseteq \hat{Z}$ obtained by removing all $V(T_i, T_j)$ from $\bar{Z} = \mathbb{K}^r$, where $i \neq j$. Then $\vartheta = \mathrm{id}_Z$ implies that there is a morphism $h\colon \hat{Z}_0 \to H = \ker(p)$ with

$$\bar{\vartheta}(z) \;=\; h(z)\cdot z \qquad \text{for all } z \in \hat{Z}_0.$$

As H is a quasitorus, h must be constant. Since H acts freely on $V(T_{i_0}) \cap \hat{Z}_0$, we obtain $h(z) = e_H$ for all $z \in \hat{Z}_0$. Consequently,

$$s_1 z^{P^* u_1} + \ldots + s_\rho z^{P^* u_\rho} \;=\; 0 \qquad \text{for all } z \in \mathbb{T}^r.$$

Since $v_1, \ldots, v_r$ generate $\mathbb{Q}^n$ as a vector space, the dual map P^* is injective. Thus, as $u_1, \ldots, u_\rho$ are pairwise distinct, we can conclude

$$s_1 \;=\; \ldots \;=\; s_\rho \;=\; 0.$$

We verify (iii). Using (ii) and Lemma 4.3, we see that $\mathbb{T}^n$ normalizes U. In particular, $U\mathbb{T}^n \subseteq \mathrm{Aut}(Z)$ is a closed subgroup. By the definition of Θ and applying again Lemma 4.3, we obtain

$$\Theta(s)\Theta(\varphi(t)(s')) \;=\; \Theta(s)\Theta(\chi^{u_1}(t)s'_1, \ldots, \chi^{u_\varrho}(t)s'_\varrho) \;=\; \Theta(s)t^{-1}\Theta(s')t.$$

We conclude that Ψ is a group homomorphism. Since U is unipotent and every $t \in \mathbb{T}$ is semisimple, we have $U \cap \mathbb{T}^n = \{\mathrm{id}_Z\}$. Thus, Ψ injective. Using (ii) again, we see that Ψ is surjective. $\square$

Proposition 4.4. *Let Σ be a complete fan with primitive generators $v_1, \ldots, v_r$ and Z the associated toric variety. For $i_0 \neq i_1$ and $\varepsilon \geq 0$, let u, u_ε be Demazure roots at v_{i_0}, v_{i_1} respectively, such that $\langle u, v_{i_1} \rangle = 0$ and $\langle u_\varepsilon, v_{i_0} \rangle = \varepsilon$.*

(i) *For $\mu = 0, \ldots, \varepsilon$, set $u_\mu := u_\varepsilon + (\varepsilon - \mu)u$. Then each u_μ is a Demazure root at v_{i_1} with $\langle u_\mu, v_{i_0} \rangle = \mu$ and for every $0 \leq \alpha \leq \varepsilon$ we have*

$$\lambda_{u_\alpha}(s)\lambda_u(q) \;=\; \lambda_u(q) \prod_{\mu=0}^{\alpha} \lambda_{u_\mu}\left(s\binom{\alpha}{\mu}q^{\alpha-\mu}\right).$$

(ii) *Let $U, V \subseteq \mathrm{Aut}(Z)$ be the subgroups generated by $\lambda_{u_0}, \ldots, \lambda_{u_\varepsilon}$ and by λ_u, respectively. Then V normalizes U and*

$$\Phi\colon \mathbb{K}^{\varepsilon+1} \rtimes_\varphi \mathbb{K} \;\to\; UV, \quad (s_0, \ldots, s_\varepsilon, q) \;\mapsto\; \lambda_{u_0}(s_0) \cdots \lambda_{u_\varepsilon}(s_\varepsilon)\lambda_u(q)$$

is an isomorphism of algebraic groups, where the twisting homomorphism $\varphi\colon \mathbb{K} \to \mathrm{Aut}(\mathbb{K}^{\varepsilon+1})$ is given by the matrix valued map

$$q \mapsto A(q) = (a_{\mu\alpha}(q)), \text{ where } a_{\mu\alpha}(q) \;=\; \begin{cases} \binom{\alpha-1}{\mu-1} q^{\alpha-\mu}, & \alpha \geq \mu, \\ 0, & \alpha < \mu. \end{cases}$$

(iii) *The acting torus $\mathbb{T}^n \subseteq \mathrm{Aut}(Z)$ normalizes $UV \subseteq \mathrm{Aut}(Z)$ and we have an isomorphism of algebraic groups*

$$\Psi\colon (\mathbb{K}^{\varepsilon+1} \rtimes_\varphi \mathbb{K}) \rtimes_\psi \mathbb{T}^n \;\to\; UV\mathbb{T}^n, \qquad (s, q, t) \;\mapsto\; \Phi(s, q)t,$$

where the twisting homomorphism $\psi\colon \mathbb{T}^n \to \mathrm{Aut}(\mathbb{K}^{\varepsilon+1} \rtimes_\varphi \mathbb{K})$ sends $t \in \mathbb{T}^n$ to the diagonal matrix $\mathrm{diag}(\chi^{u_0}(t), \ldots, \chi^{u_\varepsilon}(t), \chi^u(t))$.

Proof. We show (i). The fact that each u_μ is a Demazure root as claimed is directly verified. Now, write $P = [v_1, \ldots, v_r]$. Then we compute in Cox coordinates:

$$\begin{aligned}
\lambda_u(q)(z)^{P^*(u_\alpha)} &= \left(z + qz_{i_0}z^{P^*(u)}e_{i_0}\right)^{P^*(u_\alpha)} \\
&= \left(z_{i_0} + qz_{i_0}\prod_{i=0}^{r} z_i^{\langle u,v_i\rangle}\right)^{\alpha} \prod_{i\neq i_0} z_i^{\langle u_\alpha,v_i\rangle} \\
&= \left(\sum_{\mu=0}^{\alpha}\binom{\alpha}{\mu} q^{\alpha-\mu} z_{i_0}^{\alpha}\prod_{i=0}^{r} z_i^{(\alpha-\mu)\langle u,v_i\rangle}\right)\prod_{i\neq i_0} z_i^{\langle u_\alpha,v_i\rangle} \\
&= \sum_{\mu=0}^{\alpha}\binom{\alpha}{\mu} q^{\alpha-\mu}\prod_{i=0}^{r} z_i^{\langle u_\alpha+(\alpha-\mu)u,v_i\rangle} \\
&= \sum_{\mu=0}^{\alpha}\binom{\alpha}{\mu} q^{\alpha-\mu} z^{P^*(u_\mu)}.
\end{aligned}$$

Next, using $\langle u, v_{i_1}\rangle = 0$, we see that the monomial $z^{P^*(u)}$ does not depend on z_{i_1} and thus, for any $a \in \mathbb{K}$, obtain

$$\begin{aligned}
\lambda_u(q)(z + ae_{i_1}) &= z + ae_{i_1} + q(z + ae_{i_1})^{P^*(u)} \\
&= z + qz^{P^*(u)} + ae_{i_1} \\
&= \lambda_u(q)(z) + ae_{i_1}.
\end{aligned}$$

Set for short $t_\mu := s\binom{\alpha}{\mu}q^{\alpha-\mu}$. Then, applying the two computations just performed, we can verify the displayed formula as follows:

$$\begin{aligned}
\lambda_{u_\alpha}(s)\lambda_u(q)(z) &= \lambda_u(q)(z) + sz_{i_1}\lambda_u(q)(z)^{P^*(u_\alpha)}e_{i_1} \\
&= \lambda_u(q)(z) + \sum_{\mu=0}^{\alpha} t_\mu z_{i_1} z^{P^*(u_\mu)} e_{i_1} \\
&= \lambda_u(q)(\lambda_{u_0}(t_0)(z)) + \sum_{\mu=1}^{\alpha} t_\mu z_{i_1} z^{P^*(u_\mu)} e_{i_1} \\
&\;\;\vdots \\
&= \lambda_u(q)\lambda_{u_\alpha}(t_\alpha)\cdots\lambda_{u_0}(t_0)(z),
\end{aligned}$$

where we used that all the u_μ are Demazure roots at v_{i_1} and hence for every $\mu = 0, \ldots, \alpha - 1$ we have

$$z_{i_1} z^{P^*(u_{\mu+1})} = \lambda(t_\mu)(z)_{i_1}\lambda(t_\mu)(z)^{P^*(u_\mu)}.$$

We turn to (ii). First note that V normalizes U, since by the identity of (i) it normalizes each $\lambda_{u_\alpha}(\mathbb{K})$, where $\alpha = 0, \ldots, \varepsilon$. In particular, UV is a closed subgroup of $\mathrm{Aut}(Z)$. Now, for $s \in \mathbb{K}^{\varepsilon+1}$ and $q \in \mathbb{K}$, set

$$\Psi(s) := \Psi(s, 0) \qquad\qquad \Psi(r) := \Psi(0, q).$$

By the nature of Ψ, we then have $\Psi(s, q) = \Psi(s)\Psi(q)$. Moreover, showing that Ψ respects the multiplication of (s, q) and (s', q') means to verify

$$\Psi(q)\Psi(s') = \Psi(\varphi(q)(s'))\Psi(q).$$

For this, write $s' = s'_0 e_0 + \ldots + s'_\varepsilon e_\varepsilon$ with the canonical basis vectors $e_i \in \mathbb{K}^{\varepsilon+1}$. Then for each $s'_i e_i$, the above equality is a direct consequence of the definition of φ and the identity provided by (i). Thus, Ψ is a homomorphism.

Proposition 4.2 tells us that Ψ maps $\mathbb{K}^{\varepsilon+1}$ isomorphically onto U and $\mathbb{K}$ isomorphically onto V. In particular, Ψ is surjective. Now consider an element $(s, q) \in \ker(\Psi)$ with $s \neq 0$ and $q \neq 0$. As in the proof of Proposition 4.2, we work in Cox coordinates. The element (s, q) restricted to the identity on $V(T_{i_0}, T_{i_1}) \subseteq \bar{Z}$. By our assumptions, v_{i_0} and v_{i_1} form part of a lattice basis of $\mathbb{Z}^n$ and thus H acts freely on $V(T_{i_0}, T_{i_1})$; see [**2**, Prop. 2.1.4.2]. We conclude $s = 0$ and $q = 0$.

Finally we note that the verification of Assertion (iii) runs exactly as for the corresponding statement of Proposition 4.2. $\square$

We enter the surface case. The first step towards our description of the automorphism groups is combinatorial: we specify the fans admitting Demazure roots at two or more primitive generators.

Proposition 4.5. *Let N and M be mutually dual two-dimensional lattices. Consider distinct primitive vectors $v, v', w, w' \in N$ and $u, u' \in M$ such that*

$$\langle u, v\rangle = -1, \qquad \xi := \langle u, v'\rangle \geq 0, \qquad \langle u, w\rangle \geq 0, \qquad \langle u, w'\rangle \geq 0,$$

$$\langle u', v'\rangle = -1, \qquad \xi' := \langle u', v\rangle \geq 0, \qquad \langle u', w\rangle \geq 0. \qquad \langle u', w'\rangle \geq 0.$$

Assume $\xi = 0$ or $\xi' = 0$. Then we have $w \in \operatorname{cone}(-v, -v')$ and, choosing suitable $\mathbb{Z}$-linear coordinates on N, we achieve

$$v = (1, 0) \qquad v' = (0, 1), \qquad u = (-1, \xi), \qquad u' = (\xi', -1).$$

Assume $\xi, \xi' > 0$, $w \notin \operatorname{cone}(v, v')$ and $w' \neq w$. Then each of u, u' annihilates w and w'. Choosing suitable $\mathbb{Z}$-linear coordinates on N and $b \in \mathbb{Z}_{\geq 1}$, we have

$$v = (1, 0), \qquad u = (-1, 1), \qquad w = (-1, -1),$$

$$v' = (b - 1, b), \qquad u' = (1, -1), \qquad w' = (1, 1).$$

Proof. By assumption v and v' generate $N_{\mathbb{Q}}$ as a vector space and w is not a multiple of one of v, v'. Thus, we can write $w = -\eta v - \eta' v'$ with $\eta, \eta' \in \mathbb{Q}$. Evaluating the linear forms u and u' yields

$$\eta - \xi\eta' = \langle u, w\rangle \geq 0, \qquad \eta' - \xi'\eta = \langle u', w\rangle \geq 0.$$

Assume $\xi\xi' = 0$. Then $\eta, \eta' \geq 0$, hence $w \in \operatorname{cone}(-v, -v')$. Moreover, (v, v') is a basis of N as it is sent via (u, u') to a basis of $\mathbb{Z}^2$. Clearly, (v, v') provides the desired coordinates. Now assume $\xi, \xi' > 0$ and $w \notin \operatorname{cone}(v_1, v_2)$. Then

$$\eta \geq \xi\eta' \geq \xi\xi'\eta > 0, \qquad \eta' \geq \xi'\eta \geq \xi'\xi\eta' > 0.$$

We conclude $\xi = \xi' = 1$ and $\eta = \eta'$. This implies $u' = -u$. Consequently, each of the linear forms u and u' annihilates w and w'. With respect to suitable $\mathbb{Z}$-linear coordinates on N, we have

$$v = (1, 0), \qquad v' = (a, b), \text{ where } 0 \leq a < b.$$

Then $\langle u, v\rangle = -1$ implies $u = (-1, x)$ with $x \in \mathbb{Z}$. Moreover, $\langle u, v'\rangle = 1$ gives us $bx = a + 1$. Because of $b > a$, we must have $x = 1$. Consequently, $b = a + 1 \geq 1$ holds. We conclude

$$u = (-1, 1), \quad u' = (1, -1), \quad w = (-1, -1), \quad w' = (1, 1).$$

□

Corollary 4.6. *Let Σ be a complete fan in $\mathbb{Z}^2$ and $P = [v_1, \ldots, v_r]$ the generator matrix of Σ. Let Demazure roots m_1 at v_1 and m_2 at v_2 be given.*

(i) *Assume that $\langle m_1, v_2 \rangle = 0$ or $\langle m_2, v_1 \rangle = 0$ holds. Then, with respect to suitable $\mathbb{Z}$-linear coordinates, we have*

$$v_1 = (1,0), \qquad v_2 = (0,1), \qquad v_i \in \mathrm{cone}(-v_1, -v_2),\ \ i = 3, \ldots, r$$

and, denoting by $\xi(\Sigma)$ the minimum of the slopes of the lines $\mathbb{Q}v_3, \ldots, \mathbb{Q}v_r$, the Demazure roots of Σ at v_1 and v_2 are given as

$$u = (-1,0), \qquad u_\xi = (\xi, -1), \quad 0 \le \xi \le \xi(\Sigma).$$

Assume in addition that some v_i with $i \ge 3$ admits a Demazure root u'. Then, according to the value of $\xi(\Sigma)$ and suitably renumbering, we have

$$\xi(\Sigma) = 0: \quad v_3 = (-1,0), \quad u' = (1,0), \quad v_4 = (0,-1), \quad u'' = (0,1),$$

$$\xi(\Sigma) > 0: \quad v_3 = (-1, -\xi(\Sigma)), \quad u' = (1,0), \quad v_4 = (0,-1)$$

as the lists of the remaining primitive generators and the remaining Demazure roots of the fan Σ, where v_4 is optional in the second case.

(ii) *Assume that $\langle m_1, v_2 \rangle > 0$ and $\langle m_2, v_1 \rangle > 0$ hold. Then $r = 3$ or $r = 4$ and with respect to suitable $\mathbb{Z}$-linear coordinates, we have*

$$v_1 = (1,0), \quad v_2 = (b-1, b), \quad v_3 = (-1,-1), \quad v_4 = (1,1),$$

where $b \ge 1$. The possible Demazure roots of Σ are $u = (-1,1)$ at v_1, next $u' = (1,-1)$ at v_2 and, $u_\xi = (\xi, 1-\xi)$ at v_3, where $0 \le \xi \le b$.

Proof. We show (i). We may assume $\langle m_1, v_2 \rangle = 0$. In this setting, the first part of Proposition 4.5 shows $v_i \in \mathrm{cone}(-v_1, -v_2)$ for $i = 3, \ldots, r$ and provides us with the desired coordinates. Moreover, $0 \le \xi \le \xi(\Sigma)$ holds because $\langle u_\xi \rangle$ evaluates non-negatively on v_i for $i \ge 3$. Now, let v_i with $i \ge 3$ admit a Demazure root u'. We may assume $i = 3$ and then have

$$v_3 \; = \; (-a, -b), \quad a, b \in \mathbb{Z}_{\ge 0}, \qquad\qquad u' \; = \; (c,d), \quad c, d \in \mathbb{Z}_{\ge 0}.$$

So, $\langle u', v_3 \rangle = -1$ means $-ac - bd = -1$. Consequently, $ac = 0$ or $bd = 0$. Consider the case $bd = 0$. Then $a = c = 1$. According to $b = 0$ and $d = 0$, we arrive at

$$u' \; = \; (1,0), \quad v_3 \; = \; (-1,0), \qquad\qquad u' \; = \; (1,0), \quad v_3 \; = \; (-1,-b).$$

Observe $b = \xi(\Sigma)$ and that we must add $v_4 = (0,-1)$ if $b = 0$ and may add it if $b > 0$. The case $ac = 0$ delivers nothing new. We turn to (ii). Here, we may assume $v_3 \notin \mathrm{cone}(v_1, v_2)$. Then the second part of Proposition 4.5 brings us to the setting of (ii) and, similarly as above, the statement on the Demazure roots is obtained via straightforward calculation. □

Remark 4.7. The toric surfaces behind the fans of Corollary 4.6 provided we have at least three primitive allowing a Demazure root are the following. In (i), we have $\mathbb{P}_1 \times \mathbb{P}_1$ if $\xi(\Sigma) = 0$ and for $\xi(\Sigma) > 0$ we obtain the weighted projective plane $\mathbb{P}_{1,1,b}$ for $r = 3$ and the Hirzebruch surfaces

Z_b, where $b \in \mathbb{Z}_{\geq 1}$, for $r = 4$. The latter two are also the ones showing up in (ii).

Now we begin to invesigate of the structure of the automorphism group in the surface case.

PROPOSITION 4.8. *Situation as in Corollary 4.6 (ii). Then for $r = 3$ and $r = 4$ the following holds. For $0 \leq \xi \leq b$, the root subgroups given by u, u' and u_ξ satisfy*

$$\lambda_{u_\xi}(r)\lambda_u(s)\lambda_{u'}(s') \;=\; \lambda_u(s)\lambda_{u'}(s')\prod_{\nu=0}^{b}\lambda_{u_\nu}\left(r\sum_{\mu=0}^{\min(\nu,\xi)}\binom{\xi}{\mu}\binom{b-\mu}{b-\nu}s^{\xi-\mu}(s')^{\nu-\mu}\right).$$

Let $U \subseteq \mathrm{Aut}(X)$ be the subgroup generated by $\lambda_{u_0}, \ldots, \lambda_{u_b}, \lambda_u$ and $\lambda_{u'}$. Then we have an isomorphism of algebraic groups

$$\begin{array}{rcl} \Psi_2\colon \mathbb{K}^{b+1} \rtimes_{\varphi_2} \mathbb{K}^2, & \to & U, \\ (r_0, \ldots, r_b, s, s') & \mapsto & \lambda_u(s) \circ \lambda_{u'}(s')\lambda_{u_0}(r_0)\cdots\lambda_{u_b}(r_b), \end{array}$$

Here, the twisting homomorphism $\varphi_2\colon \mathbb{K}^2 \to \mathrm{Aut}(\mathbb{K}^{b+1})$ is given by the matrix valued map $(s, s') \mapsto B(s, s') = (b_{ji}(s, s'))$, where $b_{ji}(s, s') = 0$ for $i < j$ and

$$b_{ji}(s, s') \;=\; \sum_{\mu=0}^{\min(j-1,i-1)}\binom{i-1}{\mu}\binom{b-\mu}{b-j+1}s^{i-1-\mu}(s')^{j-1-\mu} \qquad \textit{for } i \geq j.$$

PROOF. First recall that both u and u' evaluate to zero on v_3. Furthermore, one directly computes

$$\langle u_\xi, v_1\rangle = \xi, \quad u_\xi+(\xi-\mu)u = u_\mu, \quad \langle u_\mu, v_2\rangle = b-\mu, \quad u_\mu+(b-\mu-\nu)u' = u_{b-\nu}.$$

Thus, applying Proposition 4.4 to the pairs roots u, u_ξ and u', u_ξ, we can verify the first assertion:

$$\begin{aligned} \lambda_{u_\xi}(r)\lambda_u(s)\lambda_{u'}(s') &= \lambda_u(s)\prod_{\mu=0}^{\xi}\lambda_{u_\mu}\left(r\binom{\xi}{\mu}s^{\xi-\mu}\right)\lambda_{u'}(s') \\ &= \lambda_u(s)\lambda_{u'}(s')\prod_{\mu=0}^{\xi}\prod_{\nu=0}^{b-\mu}\lambda_{b-\nu}\left(r\binom{\xi}{\mu}s^{\xi-\mu}\binom{b-\mu}{\nu}(s')^{b-\mu-\nu}\right) \\ &= \lambda_u(s)\lambda_{u'}(s')\prod_{\nu=0}^{b}\prod_{\mu=0}^{\min(\nu,\xi)}\lambda_{u_\nu}\left(r\binom{\xi}{\mu}s^{\xi-\mu}\binom{b-\mu}{b-\nu}(s')^{\nu-\mu}\right) \\ &= \lambda_u(s)\lambda_{u'}(s')\prod_{\nu=0}^{b}\lambda_{u_\nu}\left(r\sum_{\mu=0}^{\min(\nu,\xi)}\binom{\xi}{\mu}\binom{b-\mu}{b-\nu}s^{\xi-\mu}(s')^{\nu-\mu}\right) \end{aligned}$$

For second assertion, we proceed similarly as in the proof of Proposition 4.4. The property that Ψ_2 is a group homomorphism reduces to the following, whih is a consequnece of the first assertion.

$$\Psi_2(r)\Psi_2(s, s') \;=\; \Psi_2(\varphi_2(s, s'))\Psi_2(r).$$

By definition, Ψ_2 is surjective. To see injectivity, take $(s, s', r) \in \ker(\Psi_2)$. Then, working Cox coordinates, we find an $h \in H$ with $\Psi_2(s, s', r)(z) = h\cdot z$.

we conclude $s = 0$, $s' = 0$ and $r = 0$, using the fact that the H-action is explicitly known in our situation. $\square$

Proposition 4.9. *Let Σ be a complete fan in $\mathbb{Z}^2$, denote by $\ell(\Sigma)$ the number of primitive ray generators admitting a Demazure root and by $\rho(\Sigma)$ the total number of Demazure roots of Σ. Then the unit component of the automorphism group of the toric surface Z defined by Σ is given as follows:*

Z	$\ell(\Sigma)$	$\mathrm{Aut}^0(Z)$	$\rho(\Sigma)$
$-$	0	$\mathbb{T}^2$	*0*
$-$	1	$\mathbb{K}^\rho \rtimes_{\psi_1} \mathbb{T}^2$	ρ
$-$	2	$(\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}) \rtimes_{\psi_2} \mathbb{T}^2$	$\rho + 1$
$\mathbb{P}_{1,1,b}$, $b \geq 2$	3	$(\mathbb{K}^{b+1} \rtimes_{\varphi'} \mathbb{K}^2) \rtimes_{\psi_3} \mathbb{T}^2$	$b + 3$
Z_b, $b \geq 1$	3	$(\mathbb{K}^{b+1} \rtimes_{\varphi'} \mathbb{K}^2) \rtimes_{\psi_3} \mathbb{T}^2$	$b + 3$
$\mathbb{P}_2$	3	$\mathrm{PGL}_3(\mathbb{K})$	6
$\mathbb{P}_1 \times \mathbb{P}_1$	4	$\mathrm{PGL}_2(\mathbb{K}) \times \mathrm{PGL}_2(\mathbb{K})$	4

The twisting homomorphisms φ and φ' are given by upper triangular matrices $\varphi(s) = A = (a_{ji})$ and $\varphi'(s, s') = B = (b_{ji})$, where for $i \geq j$ we have

$$a_{ji} \;=\; \binom{i-1}{j-1} s^{i-j}, \qquad b_{ji} \;=\; \sum_{\mu=0}^{\min(j-1,i-1)} \binom{i-1}{\mu} \binom{b-\mu}{b-j+1} s^{i-1-\mu} (s')^{j-1-\mu}.$$

Furthermore the twisting homomorphism for the toric factors are given by the following diagonal matrices:

$$\begin{aligned}
\psi_1(t_1, t_2) &= \mathrm{diag}(t_1^{-1}, \ldots, t_1^{-1}) \\
\psi_2(t_1, t_2) &= \mathrm{diag}(t_2^{-1}, t_1 t_2^{-1}, \ldots, t_1^{\rho-1} t_2^{-1}, t_1^{-1}), \\
\psi_3(t_1, t_2) &= \mathrm{diag}(t_2, t_1, t_1^2 t_2^{-1}, \ldots, t_1^{\rho} t_2^{1-\rho}, t_1 t_2^{-1}, t_1^{-1} t_2).
\end{aligned}$$

Proof. The cases of $\mathbb{P}_2$ and $\mathbb{P}_1 \times \mathbb{P}_1$ are well known. So, let Z arise from a complete fan admitting Demazure roots and denote by $U \subseteq Z$ the subgroup generated generated by all the root groups. If only one primitive generator of Σ allows Demazure roots, then Proposition 4.2 yields $U \cong \mathbb{K}^\rho$. If there are roots at exactly two primitive ray generators, Proposition 4.4 shows that $U \cong \mathbb{K}^\rho \rtimes_\varphi \mathbb{K}$ is as claimed. In the cases $Z = \mathbb{P}_{1,1,b}$ or $Z = Z_b$, Proposition 4.8 tells us that $U \cong K^\rho \rtimes_{\varphi'} \mathbb{K}^2$ is as in the assertion. Finally, using Lemma 4.3 we see that also the twisting homomorphisms ψ_1, ψ_2 and ψ_3 are the right ones. $\square$

5. Representation via toric ambient automorphisms

The main result of this section, Theorem 5.4, is an important ingredient for the explicit handling of automorphisms in the subsequent sections; it represents automorphisms of a variety with torus action of complexity one as restrictions of explicitly accessible automorphisms of its ambient toric variety.

Construction 5.1. Situation as in Construction 2.4. Let $\kappa = (u, i_0, i_1, C)$ be a horizontal Demazure P-root. Define linear forms

$$u_{\nu,\iota} \;:=\; \nu u + e'_{i_1} - e'_\iota \;\in\; M, \qquad \nu = 1, \ldots, l_{i_1 c_{i_1}}, \quad \iota \neq i_0, i_1,$$

where $e_0' := 0$ and $e_1', \ldots, e_r' \in M$ are the first r canonical basis vectors. So, the values of $u_{\nu,\iota}$ on the columns v_{ij} and v_k of P are given by

$$\langle u_{\nu,\iota}, v_{ij}\rangle \;=\; \begin{cases} \nu\langle u, v_{ij}\rangle + l_{ij}, & i = i_1, \\ \nu\langle u, v_{ij}\rangle - l_{ij} & i = \iota, \\ \nu\langle u, v_{ij}\rangle & i \neq i_1, \iota, \end{cases} \qquad \langle u_{\nu,\iota}, v_k\rangle = \nu\langle u, v_k\rangle,\ k = 1, \ldots, m.$$

LEMMA 5.2. *Let δ and δ' be derivations on the polynomial ring $\mathbb{K}[T_1, \ldots, T_n]$.*

(i) *We have $\delta(a) = 0$ for every $a \in \mathbb{K}$.*
(ii) *If $\delta(T_i) = \delta(0)$ holds for $i = 1, \ldots, n$, then $\delta = 0$ holds.*
(iii) *If $\delta'(T_i) = \delta(T_i)$ holds for $i = 1, \ldots, n$, then $\delta' = \delta$ holds.*
(iv) *If $\delta'\delta(T_i) = 0$ holds for $i = 1, \ldots, n$, then $\delta'\delta = 0$ holds.*
(v) *If $\delta'\delta(T_i) = \delta\delta'(T_i)$ holds for $i = 1, \ldots, n$, then $\delta'\delta = \delta\delta'$ holds.*

PROOF. The statements are directly verified by using the defining properties of derivations and fact that $\mathbb{K}[T_1, \ldots, T_n]$ is generated as a $\mathbb{K}$-algebra by the variables $T_1, \ldots, T_n$. $\square$

LEMMA 5.3. *Consider linear forms $u_{\nu,\iota}$ and $u_{\nu',\iota'}$ as in Construction 5.1 and the associated derivations $\delta_{u_{\nu,\iota}}$ and $\delta_{u_{\nu',\iota'}}$ on $\mathbb{K}[T_{ij}, S_k]$ as provided by Construction 3.2.*

(i) *The linear form $u_{\nu,\iota} \in M$ is a Demazure root at $v_{\iota c_\iota} \in N$ in the sense of Definition 3.1.*
(ii) *The derivation $\delta_{u_{\nu,\iota}}$ annihilates all variables T_{ij} and S_k except $T_{\iota c_\iota}$, where we have*
$$\delta_{u_{\nu,\iota}}(T_{\iota c_\iota}) \;=\; f_{u,\nu,\iota} T_{i_1 c_{i_1}}^{l_{i_1 c_{i_1}} - \nu}$$
with a monomial $f_{u,\nu,\iota}$ in the variables T_{ij}, S_k but not depending on any T_{ic_i} with $i \neq i_0$.
(iii) *We have $\delta_{u_{\nu',\iota'}}\delta_{u_{\nu,\iota}} = \delta_{u_{\nu,\iota}}\delta_{u_{\nu',\iota'}} = 0$. In particular, the derivations $\delta_{u_{\nu,\iota}}$ and $\delta_{u_{\nu',\iota'}}$ commute.*
(iv) *For any two $s, s' \in \mathbb{K}$, the automorphisms $\bar\lambda_{u_{\nu,\iota}}(s)^*$ and $\bar\lambda_{u_{\nu',\iota'}}(s')^*$ of $\mathbb{K}[T_1, \ldots, T_n]$ satisfy*
$$\bar\lambda_{u_{\nu',\iota'}}(s')^* \circ \bar\lambda_{u_{\nu,\iota}}(s)^* \;=\; \exp(s'\delta_{u_{\nu',\iota'}} + s\delta_{u_{\nu,\iota}}).$$
In particular, $\bar\lambda_{u_{\nu,\iota}}(s)^*$ and $\bar\lambda_{u_{\nu',\iota'}}(s')^*$ as well as the associated automorphisms $\bar\lambda_{u_{\nu,\iota}}(s)$ and $\bar\lambda_{u_{\nu',\iota'}}(s')$ of $\bar Z$ commute.

PROOF. For (i), we need that $u_{\nu,\iota}$ evaluates to -1 on $v_{\iota c_\iota}$ and is nonnegative on all other columns of P. By Definition 3.4, the latter is clear for all v_k and v_{ij} with $(i,j) \neq (i_1, c_{i_1})$ or $i \neq \iota$. For the open cases, we compute

$$\langle u_{\nu,\iota}, v_{i_1 c_{i_1}}\rangle \;=\; \nu\langle u, v_{i_1 c_{i_1}}\rangle + l_{i_1 c_{i_1}} \;=\; l_{i_1 c_{i_1}} - \nu,$$
$$\langle u_{\nu,\iota}, v_{\iota j}\rangle \;=\; \nu\langle u, v_{\iota j}\rangle - l_{\iota j} \quad \begin{cases} \geq\ (\nu - 1) l_{\iota j}, & j \neq c_\iota, \\ =\ -1, & j = c_\iota. \end{cases}$$

We turn to (ii). We infer directly from Construction 3.2 that $\delta_{u_{\nu,\iota}}$ annihilates all variables T_{ij}, S_k except $T_{\iota c_\iota}$ and satisfies

$$\delta_{u_{\nu,\iota}}(T_{\iota c_\iota}) \;=\; T_{\iota c_\iota} \prod T_{ij}^{\langle u_{\nu,\iota}, v_{ij}\rangle} \prod S_k^{\langle u_{\nu,\iota}, v_k\rangle} \;=\; f_{u,\nu,\iota} T_{i_1 c_{i_1}}^{l_{i_1 c_{i_1}} - \nu},$$

where by the compution proving (i) the monomial $f_{u,\nu,\iota}$ depends neither on $T_{i_1,c_{i_1}}$ nor on T_{ι,c_ι} and the T_{ic_i} with $i \neq i_0, i_1, \iota$ have by Definition 3.4 the exponent

$$\langle u_{\nu,\iota}, v_{ic_i} \rangle \ = \ \nu \langle u, v_{ic_i} \rangle \ = \ 0.$$

We show (iii). From (ii) we infer that $\delta_{u_{\nu',\iota'}} \delta_{u_{\nu,\iota}}$ as well as $\delta_{u_{\nu,\iota}} \delta_{u_{\nu',\iota'}}$ annihilate all variables T_{ij} and S_k; use Lemma 5.2 (ii) and (iii). Thus, Lemma 5.2 (v) gives

$$\delta_{u_{\nu',\iota'}} \delta_{u_{\nu,\iota}} \ = \ \delta_{u_{\nu,\iota}} \delta_{u_{\nu',\iota'}} \ = \ 0.$$

To obtain (iv), first note that due to (iii), the linear endomorphisms $s\delta_{u_{\nu,\iota}}$ and $s'\delta_{u_{\nu',\iota'}}$ of $\mathbb{K}[T_1, \ldots, T_n]$ commute. Thus, the assertion follows from the definition $\bar{\lambda}_\delta(s)^* := \exp(s\delta)$ and the homomorphism property of the exponential series. $\square$

THEOREM 5.4. *Let $X = X(A, P, \Sigma)$ be as in Construction 2.4 with Σ complete and $\kappa = (u, i_0, i_1, C)$ a horizontal Demazure P-root. For $s \in \mathbb{K}$ set*

$$\alpha(s, \nu, \iota) \ := \ \beta_\iota \binom{l_{i_1 c_{i_1}}}{\nu} s^\nu, \qquad \iota = 0, \ldots, r, \quad \iota \neq i_0, i_1, \quad \nu = 1, \ldots, l_{i_1 c_{i_1}},$$

where β is the unique vector in the row space of A with $\beta_{i_0} = 0$ and $\beta_{i_1} = 1$. Consider the linear forms $u_{\nu,\iota}$ from Construction 5.1 and the automorphisms

$$\varphi_u(s) \ := \ \prod_{\iota \neq i_0, i_1} \prod_{\nu=1}^{l_{i_1 c_{i_1}}} \lambda_{u_{\nu,\iota}}(\alpha(s, \nu, \iota)) \ \in \ \mathrm{Aut}(Z).$$

Then the automorphism $\lambda_\kappa(s)$ of X can be presented as the restriction of an automorphism of Z as follows:

$$\lambda_\kappa(s) \ = \ \lambda_u(s) \circ \varphi_u(s)|_X.$$

More explicitly, the comorphism satisfies $\bar{\lambda}_\kappa(s)^*(S_k) = S_k$ and $\bar{\lambda}_\kappa(s)^*(T_{ij}) = T_{ij}$, whenever $i = i_0$ or $j \neq c_i$ and in the remaining cases

$$\begin{aligned} \bar{\lambda}_\kappa(s)^*(T_{i_1 c_{i_1}}) \ &= \ T_{i_1 c_{i_1}} + s\delta_u(T_{i_1 c_{i_1}}), \\ \bar{\lambda}_\kappa(s)^*(T_{\iota c_\iota}) \ &= \ T_{\iota c_\iota} + \sum_{\nu=1}^{l_{i_1 c_{i_1}}} \alpha(s, \nu, \iota) \delta_{u_{\nu,\iota}}(T_{\iota c_\iota}). \end{aligned}$$

PROOF. By definition, the comorphism of $\bar{\lambda}_\kappa(s)$ equals $\exp(s\delta_\kappa)$. In a first step, we compute the powers δ_κ^ν occuring in the exponential series. We will make repeated use of the fact

$$\delta_{u_{\nu,\iota}}(T_{\iota c_\iota}) \ = \ T_{\iota c_\iota} h^{\nu u} \frac{T_{i_1}^{l_{i_1}}}{T_\iota^{l_\iota}} \ = \ f_{u,\nu,\iota} T_{i_1 c_{i_1}}^{l_{i_1 c_{i_1}} - \nu},$$

where $h^{\nu u}$ is as in Construction 3.6 and $f_{u,\nu,\iota}$ is a monomial in the variables T_{ij} and S_k but not depending on any T_{ic_i} with $i \neq i_0$; see Lemma 5.3 (ii). Now, recall from Construction 3.6 that, apart from the T_{ic_i}, all variables T_{ij} and S_k are annihilated by δ_κ. Moreover, we have

$$\delta_\kappa(T_{i_0 c_{i_0}}) = \beta_{i_0} = 0, \qquad \delta_\kappa(T_{i_1 c_{i_1}}) = \beta_{i_1} \frac{h^u}{h^\zeta} \prod_{i \neq i_0, i_1} \frac{\partial T_i^{l_i}}{\partial T_{ic_i}} = T_{i_1 c_{i_1}} h^u = \delta_u(T_{i_1 c_{i_1}}).$$

Since $\delta_u(T_{i_1c_{i_1}})$ does not depend on any T_{ic_i} with $i \neq i_0$, we conclude $\delta_\kappa^2(T_{i_1c_{i_1}}) = 0$. Finally, for $\iota \neq i_0, i_1$, Construction 3.6 and the above formula for $\nu = 1$ give us

$$\delta_\kappa(T_{\iota c_\iota}) \;=\; \beta_\iota \frac{h^u}{h^\zeta} \prod_{i \neq i_0, \iota} \frac{\partial T_i^{l_i}}{\partial T_{ic_i}} \;=\; \beta_\iota l_{i_1c_{i_1}} T_{\iota c_\iota} h^u \frac{T_{i_1}^{l_{i_1}}}{T_\iota^{l_\iota}} \;=\; \beta_\iota l_{i_1c_{i_1}} \delta_{u_{1,\iota}}(T_{\iota c_\iota}).$$

To evaluate higher powers of δ_κ, we use the representation $\delta_{u_{\nu,\iota}}(T_{\iota c_\iota}) = f_\nu T_{i_1c_{i_1}}^{l_{i_1c_{i_1}}-\nu}$ given above. Applying the Leibniz rule yields

$$\delta_\kappa(\delta_{u_{\nu,\iota}}(T_{\iota c_\iota})) \;=\; (l_{i_1c_{i_1}} - \nu)\delta_{u_{\nu+1,\iota}}(T_{\iota c_\iota}).$$

Putting things together, we arrive at

$$\delta_\kappa^\nu(T_{\iota c_\iota}) \;=\; \beta_\iota \frac{l_{i_1c_{i_1}}!}{(l_{i_1c_{i_1}} - \nu)!} \delta_{u_{\nu,\iota}}(T_{\iota, c_\iota}).$$

In the next step, we compute the values of $\bar{\lambda}_\kappa(s)^* = \exp(s\delta_\kappa)$ on the generators T_{ij} and S_k. From above, we infer $\bar{\lambda}_\kappa(s)^*(T_{ij}) = T_{ij}$, whenever $i = i_0$ or $j \neq c_i$. Moreover, we have

$$\bar{\lambda}_\kappa^*(s)(T_{i_1c_{i_1}}) \;=\; T_{i_1c_{i_1}} + s\delta_u(T_{i_1c_{i1}}).$$

Finally, for $\iota \neq i_0, i_1$, plugging the above representations of the $\delta_\kappa^\nu(T_{\iota c_\iota})$ into the exponentional series gives the remaining statements on comorphisms:

$$\bar{\lambda}_\kappa^*(s)(T_{\iota c_\iota}) \;=\; T_{\iota c_\iota} + \sum_{\nu=1}^{l_{i_1c_{i_1}}} \beta_\iota \binom{l_{i_1c_{i_1}}}{\nu} s^\nu \delta_{u_{\nu,\iota}}(T_{\iota c_\iota}) \;=\; T_{\iota c_\iota} + \sum_{\nu=1}^{l_{i_1c_{i_1}}} \alpha(s,\nu,\iota)\delta_{u_{\nu,\iota}}(T_{\iota c_\iota}).$$

We turn to $\lambda_\kappa(s) = \lambda_u(s) \circ \varphi_u(s)|_X$. We verify the corresponding identity on $\bar{\lambda}_\kappa$ and $\bar{\lambda}_u \circ \bar{\varphi}_u$ by comparing the comorphisms. First recall from Construction 3.2 that we have

$$\bar{\lambda}_u^*(s) \;=\; \mathrm{id} + s\delta_u,$$

where δ_u annihilates all variables T_{ij} and S_k except $T_{i_1c_{i1}}$. Next note that $\varphi_u(s)$ doesn't depend on the order of composition due to Lemma 5.2 (v). Moreover, Lemma 5.2 (iv) allows to compute

$$\bar{\varphi}_u(s)^* \;=\; \prod_{\iota \neq i_0, i_1} \prod_{\nu=1}^{l_{i_1c_{i_1}}} \bar{\lambda}_{u_{\nu,\iota}}(\alpha(s,\nu,\iota))^* \;=\; \mathrm{id} + \sum_{\iota \neq i_0, i_1} \sum_{\nu=1}^{l_{i_1c_{i_1}}} \alpha(s,\nu,\iota)\delta_{u_{\nu,\iota}}.$$

Now, we explicitly evaluate $\bar{\varphi}_u(s)^* \circ \bar{\lambda}_u(s)^*$ on the generators T_{ij} and S_k of the polynomial ring $\mathbb{K}[T_{ij}, S_k]$. Obviously, we have

$$\bar{\varphi}_u(s)^* \circ \bar{\lambda}_u(s)^*(S_k) \;=\; \bar{\varphi}_u(s)^*(S_k) \;=\; S_k, \quad k = 1, \ldots, m,$$

$$\bar{\varphi}_u(s)^* \circ \bar{\lambda}_u(s)^*(T_{ij}) \;=\; \bar{\varphi}_u(s)^*(T_{ij}) \;=\; T_{ij}, \quad i = i_0 \text{ or } j \neq c_i.$$

Using $\delta_u(T_{i_1c_{i_1}}) = h^u T_{i_1c_{i_1}}$, where the latter monomial doesn't depend on any T_{ic_i} with $i \neq i_0$, we compute

$$\bar{\varphi}_u(s)^* \circ \bar{\lambda}_u(s)^*(T_{i_1c_{i_1}}) = \bar{\varphi}_u(s)^*(T_{i_1c_{i_1}}) + \bar{\varphi}_u(s)^*(h^u T_{i_1c_{i_1}}) = T_{i_1}c_{i_1} + \delta_u(T_{i_1c_{i_1}}).$$

Finally, for any $\iota \neq i_0, i_1$, we obtain

$$\bar{\varphi}_u(s)^* \circ \bar{\lambda}_u(s)^*(T_{\iota c_\iota}) \; = \; \bar{\varphi}_u(s)^*(T_{\iota c_\iota}) \; = \; T_{\iota c_\iota} + \sum_{\nu=1}^{l_{i_1 c_{i_1}}} \alpha(s,\nu,\iota)\delta_{u_{\nu,\iota}}(T_{\iota c_\iota}).$$

Thus, comparing with the previously obtained values of $\bar{\lambda}_\kappa(s)^*$ on the generators, we arrive at the identity $\bar{\lambda}_\kappa(s)^* = \bar{\varphi}_u(s)^* \circ \bar{\lambda}_u(s)^*$ of comorphisms, which in turn induces the desired equation $\lambda_\kappa(s) = \lambda_u(s) \circ \varphi_u(s)$ on Z and hence X. $\square$

6. Rational projective $\mathbb{K}^*$-surfaces

We will use the approach provided by Constructions 2.4 and 4.1 producing all rational projective varieties with torus action of complexity one as $X = X(A,P,\Sigma)$. Recall that the defining $(r+s) \times (n+m)$ block matrix P is of the form

$$P \; = \; \left[\begin{array}{cc} L & 0 \\ d & d' \end{array} \right] \; = \; [v_{01},\ldots,v_{0n_0},\ldots,v_{r1},\ldots,v_{rn_r},v_1,\ldots,v_m],$$

where the columns v_{ij} and v_k are pairwise distinct primitive integral vectors generating $\mathbb{Q}^{r+s}$ as a vector space. In the case of a $\mathbb{K}^*$-surface X, several aspects simplify. First, we have $s = 1$. Thus, the lower part $[d,d']$ of P is just one row and $m \leq 2$ holds. Observe

$$v_{0j} = (-l_{0j},\ldots,-l_{0j},d_{0j}), \qquad v_{ij} = (0,\ldots,0,l_{ij},0\ldots,0,d_{ij}), \quad i = 1,\ldots,r,$$

where l_{ij} sits at the i-th place for $i = 1,\ldots,r$ and we always have $\gcd(l_{ij},d_{ij}) = 1$. Moreover, we arrange P to be *slope ordered*, that means that for each $0 \leq i \leq r$, we order the block $v_{i1},\ldots,v_{in_i}$ of columns in such a way that

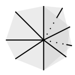
$$m_{i1} \; > \; \ldots \; > \; m_{ini}, \qquad \text{where } m_{ij} \; := \; \frac{d_{ij}}{l_{ij}}.$$

Finally, in the surface case the defining fan Σ of the ambient toric variety Z is basically unique and needs no extra specification. More precisely, the rays of Σ are the cones over the columns of P and we always have the maximal cones

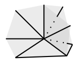
$$\tau_{ij} \; := \; \mathrm{cone}(v_{ij},v_{ij+1}) \; \in \; \Sigma, \qquad i = 0,\ldots,r, \; j = 1,\ldots,n_i - 1.$$

Writing $v^+ := v_1 = (0,\ldots,0,1)$ and $v^- := v_2 = (0,\ldots,0,-1)$ for the columns of P that arise for $m = 1,2$, the collection of maximal cones of Σ is complemented depending on the value of m as follows

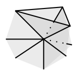
$$m=2: \quad \text{(p-p)} \quad \begin{aligned} \tau_i^+ &:= \operatorname{cone}(v^+, v_{i1}) \\ \tau_i^- &:= \operatorname{cone}(v_{in_i}, v^-) \end{aligned}$$



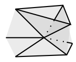
$$m=1: \quad \text{(p-e)} \quad \begin{aligned} \tau_i^+ &:= \operatorname{cone}(v^+, v_{i1}) \\ \sigma^- &:= \operatorname{cone}(v_{0n_0}, \dots, v_{rn_r}) \end{aligned}$$



$$\text{(e-p)} \quad \begin{aligned} \sigma^+ &:= \operatorname{cone}(v_{01}, \dots, v_{r1}) \\ \tau_i^- &:= \operatorname{cone}(v_{in_i}, v^-) \end{aligned}$$



$$m=0: \quad \text{(e-e)} \quad \begin{aligned} \sigma^+ &:= \operatorname{cone}(v_{01}, \dots, v_{r1}) \\ \sigma^- &:= \operatorname{cone}(v_{0n_0}, \dots, v_{rn_r}) \end{aligned}$$



In particular, the $\mathbb{K}^*$-surfaces delivered by Construction 2.5 only depend on the matrices A and P, which allows us to denote them as $X = X(A,P)$. The $\mathbb{K}^*$-action on X is given on the torus $\mathbb{T}^{r+1} \subseteq Z$ by $t \cdot z = (z_1, \dots, z_r, tz_{r+1})$.

Remark 6.1. Let $X = X(A,P)$ be a $\mathbb{K}^*$-surface as above. Then the fan Σ of the ambient toric variety Z of X reflects the geometry of the $\mathbb{K}^*$-action on X, as outlined in the introduction, in the following way.

(i) If P has a column v^+ or v^-, then the toric prime divisor on Z corresponding to $\varrho^+ = \operatorname{cone}(v^+)$, or $\varrho^- = \operatorname{cone}(v^-)$, cuts out a parabolic fixed point curve forming source or sink:

$$\begin{aligned} D^+ &= (B^+ \cap X) \cup \{x_0^+\} \cup \dots \cup \{x_r^+\}, \\ D^- &= (B^- \cap X) \cup \{x_0^-\} \cup \dots \cup \{x_r^-\}. \end{aligned}$$

Here $B^+, B^- \subseteq Z$ denote the toric orbits corresponding to $\varrho^+, \varrho^- \in \Sigma$ and $x_i^+ \in B_i^+ \cap X$, $x_i^- \in B_i^- \cap X$ are the unique points in the intersections with the toric orbits $B_i^+, B_i^- \subseteq Z$ corresponding to $\tau_i^+, \tau_i^- \in \Sigma$.

(ii) If we have a cone σ^+, resp. σ^-, in Σ, then the associated toric fixed point x^+, resp. x^-, of Z is an elliptic fixed point of the $\mathbb{K}^*$-action on X forming the source, resp. the sink.

(iii) The toric prime divisor of Z corresponding to the ray $\varrho_{ij} = \operatorname{cone}(v_{ij})$ of Σ cuts out the closure $D_{ij} \subseteq X$ of an orbit $\mathbb{K}^* \cdot x_{ij} \subseteq X$. If P is irredundant, then the arms of X are precisely

$$\mathcal{A}_i = D_{i1} \cup \dots \cup D_{in_i}, \qquad i = 0, \dots, r.$$

The order of the isotropy group $\mathbb{K}^*_{x_{ij}}$ equals l_{ij}. The hyperbolic fixed point forming $D_{ij} \cap D_{ij+1}$ is the point cut out from X by the toric orbit of Z corresponding to the cone $\tau_{ij} \in \Sigma$.

Definition 6.2. For any rational projective $\mathbb{K}^*$-surface $X = X(A,P)$, we define the numbers

$$\begin{aligned} l^+ &:= l_{01} \cdots l_{r1}, & m^+ &:= m_{01} + \dots + m_{r1}, \\ l^- &:= l_{0n_0} \cdots l_{rn_r}, & m^- &:= m_{0n_0} + \dots + m_{rn_r}. \end{aligned}$$

Remark 6.3. Let $X = X(A,P)$. We have $l^+m^+ \in \mathbb{Z}$ and if there is an elliptic fixed point $x^+ \in X$, then

$$\det(\sigma^+) := \det(v_{01}, \dots, v_{r1}) = l^+m^+ > 0.$$

Similarly, $l^-m^- \in \mathbb{Z}$ holds and if there is an elliptic fixed point $x^- \in X$, then we obtain

$$\det(\sigma^-) \ := \ \det(v_{0n_0}, \dots, v_{rn_r}) \ = \ l^-m^- \ < \ 0.$$

Rational projective $\mathbb{K}^*$-surfaces turn out to be always $\mathbb{Q}$-factorial. That means in particular that intersection numbers are well defined. Let us recall [**2**, Cor. 5.4.4.2].

Remark 6.4. For $X = X(A, P)$, the self intersection numbers of the orbit closures $D_{ij} \subseteq X$ and possible parabolic fixed point curves $D^+, D^- \subseteq X$ are given by

$$D_{i1}^2 \quad = \quad \begin{cases} \frac{1}{l_{i1}^2}\left(\frac{1}{m^+} - \frac{1}{m^-}\right), & \text{(e-e)}, \\ 0 & \text{(p-p)}, \\ \frac{1}{l_{i1}^2 m^+}, & \text{(e-p)}, \\ \frac{-1}{l_{i1}^2 m^-}, & \text{(p-e)}, \end{cases} \qquad \text{for } n_i = 1,$$

$$D_{ij}^2 \quad = \quad \begin{cases} \frac{1}{l_{ij}^2}\left(\frac{1}{m^+} - \frac{1}{m_{ij}-m_{ij+1}}\right), & j = 1 \text{ and (e-e) or (e-p)}, \\ \frac{-1}{l_{ij}^2(m_{ij}-m_{ij+1})}, & j = 1 \text{ and (p-p) or (p-e)}, \\ \frac{-(m_{ij-1}-m_{ij+1})}{l_{ij}^2(m_{ij-1}-m_{ij})(m_{ij}-m_{ij+1})}, & 1 < j < n_i, \\ \frac{-1}{l_{ij}^2(m_{ij-1}-m_{ij})}, & j = n_i \text{ and (p-p) or (e-p)}, \\ \frac{1}{l_{ij}^2}\left(-\frac{1}{m^-} + \frac{1}{m_{ij}-m_{ij-1}}\right), & j = n_i \text{ and (e-e) or (p-e)}, \end{cases} \qquad \text{for } n_i > 1,$$

$$(D^+)^2 \quad = \quad -m^+,$$

$$(D^-)^2 \quad = \quad m^-.$$

Recall that an irreducible curve D on a normal projective surface X is called *contractible* if there is a morphism $\pi \colon X \to X'$ onto a normal surface X' mapping D to a point $x' \in X'$ and inducing an isomorphism from $X \setminus D$ onto $X' \setminus \{x'\}$.

Remark 6.5. Consider $X = X(A, P)$. Then, provided that they are present, the vectors v^+ and v^- satisfy the identities

$$l_{01}^{-1}v_{01} + \ldots + l_{r1}^{-1}v_{r1} \ = \ m^+v^+, \qquad l_{0n_0}^{-1}v_{0n_0} + \ldots + l_{rn_r}^{-1}v_{rn_r} \ = \ -m^-v^-.$$

Combining this with Remark 6.4, we rediscover that D^+ and D^- are contractible if and only if they are of negative self intersection.

More generally, contractibility of invariant curves on rational normal projective $\mathbb{K}^*$-surfaces is characterized as follows.

Remark 6.6. Consider $X = X(A, P)$. Given a column v of P, let $D \subseteq X$ be the corresponding curve and P' the matix obtained from P by removing v. Then the following statements are equivalent.

(i) The curve $D \subseteq X$ is contractible.
(ii) The matrices A and P' define a $\mathbb{K}^*$-surface $X' = X(A, P')$.
(iii) We have $D^2 < 0$ for the self intersection number.

If one of these statements holds, then D is contracted by the $\mathbb{K}^*$-equivariant morphism $X \to X'$ induced by the map of fans $\Sigma \to \Sigma'$ and there is a unique cone $\sigma' \in \Sigma'$ containing v in its relative interior.

We turn to singularities of $\mathbb{K}^*$-surfaces $X = X(A, P)$. Note that due to normality of the surfaces, every singularity is a fixed point.

Definition 6.7. Let X be a rational projective $\mathbb{K}^*$-surface and $p\colon \hat{X} \to X$ its characteristic space. A point $x \in X$ is *quasismooth* if $x = p(z)$ holds for a smooth point $z \in \hat{X}$.

We characterize quasismoothness and smoothness of parabolic, hyperpbolic and elliptic fixed points in terms of the defining matrix P of $X = X(A, P)$. All the statements are direct consequences of the general (quasi-)smoothness criterion [**26**, Cor. 7.16].

Proposition 6.8. *Consider $X = X(A, P)$. Then we have the following statements on (quasi-)smoothness of possible parabolic fixed points.*

(i) *All points of $B^+ \subseteq D^+$ are smooth and all points $x_i^+ \in D^+$ are quasismooth. Moreover, $x_i^+ \in D^+$ is smooth if and only if $l_{i1} = 1$ holds.*
(ii) *All points of $B^- \subseteq D^-$ are smooth and all points $x_i^- \in D^-$ are quasismooth. Moreover, $x_i^- \in D^-$ is smooth if and only if $l_{in_i} = 1$ holds.*

Proposition 6.9. *Consider $X = X(A, P)$. Then every hyperbolic fixed point of X is quasismooth. Moreover, the hyperbolic fixed point corresponding to $\tau_{ij} \in \Sigma$ is smooth if and only if $l_{ij+1}d_{ij} - l_{ij}d_{ij+1} = 1$ holds.*

Proposition 6.10. *Assume that the $\mathbb{K}^*$-surface $X = X(A, P)$ has an elliptic fixed point $x \in X$.*

(i) *If $x = x^+$, then x is quasismooth if and only if there are $0 \le \iota_0, \iota_1 \le r$ with $l_{i1} = 1$ for every $i \neq \iota_0, \iota_1$.*
(ii) *If $x = x^+$, then x is smooth if and only if there are $0 \le \iota_0, \iota_1 \le r$ with $l_{i1} = 1$ for every $i \neq \iota_0, \iota_1$ and*
$$\det(\sigma^+) \ = \ l^+m^+ \ = \ l_{\iota_0 1}d_{\iota_1 1} + l_{\iota_1 1}d_{\iota_0 1} \ = \ 1.$$
(iii) *If $x = x^-$, then x is quasismooth if and only if there are $0 \le \iota_0, \iota_1 \le r$ with $l_{in_i} = 1$ for every $i \neq \iota_0, \iota_1$.*
(iv) *If $x = x^-$, then x is smooth if and only if there are $0 \le \iota_0, \iota_1 \le r$ with $l_{in_i} = 1$ for every $i \neq \iota_0, \iota_1$ and*
$$\det(\sigma^-) \ = \ l^-m^- \ = \ l_{\iota_0 n_{\iota_0}}d_{\iota_1 n_{\iota_1}} + l_{\iota_1 n_{\iota_1}}d_{\iota_0 n_{\iota_0}} \ = \ -1.$$

Definition 6.11. Given an elliptic fixed point $x \in X = X(A, P)$, we call the numbers $0 \le \iota_0, \iota_1 \le r$ from Proposition 6.10 *leading indices* for x.

As a consequence of Proposition 6.10, we obtain the following characterization of quasismoothness of $\mathbb{K}^*$-surface X. We say that a singularity $x \in X$ is a *toric surface singularity* if there is a $\mathbb{K}^*$-invariant open neighbourhood $x \in U \subseteq X$ such that U is a toric surface. Recall from [**12**] that toric surface singularities are quotients of $\mathbb{K}^2$ by finite cyclic groups.

Corollary 6.12. *A rational projective $\mathbb{K}^*$-surface is quasismooth if and only if it has at most toric surface singularities.*

Proof. We may assume that our $\mathbb{K}^*$-surface is given as $X = X(A, P)$. By normality, any singular point of X is a $\mathbb{K}^*$-fixed point. The parabolic and hyperbolic fixed point are toric surfaces singularities due to [**2**, Prop. 3.4.4.6].

Thus, we are left with discussing quasismooth elliptic fixed points. It suffices to consider $x^- \in X$. Let $0 \le \iota_0, \iota_1 \le r$ be leading indices for x^-. The cone $\sigma^- \in \Sigma$ defines affine open subsets

$$Z^- \subseteq Z, \qquad X^- := X \cap Z^- \subseteq X.$$

Recall that x^- is the toric fixed point of Z^-. Then X^- is the affine $\mathbb{K}^*$-surface given by the data A and $P^- = [v_{0n_0}, \ldots, v_{rn_r}]$ in the sense of [**30**, Constr. 1.6 and Cor. 1.9]. Consider the defining relations

$$g_{i_1,i_2,i_3} := \det \begin{bmatrix} T_{i_1 n_{i_1}}^{l_{i_1 n_{i_1}}} & T_{i_2 n_{i_2}}^{l_{i_2 n_{i_2}}} & T_{i_3 n_{i_3}}^{l_{i_3 n_{i_3}}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}$$

of the Cox ring $\mathcal{R}(X^-)$ of X^-. By Proposition 6.10 the point x^- is quasismooth if and only if $l_{in_i} = 1$ for all $i \ne \iota_0, \iota_1$. The latter is equivalent to the fact that $\mathcal{R}(X^-)$ is a polynomial ring. This in turn holds if and only if X^- is an affine toric surface. $\square$

Remark 6.13. Consider $X = X(A, P)$. The *canonical resolution* of singularities $X'' \to X$ from [**2**, Constr. 5.4.3.2] is obtained by the following two step procedure.

(i) Enlarge P to a matrix P' by adding e_{r+1} and $-e_{r+1}$, if not already present. Then the surface $X' := X(A, P')$ is quasismooth and there is a canonical morphism $X' \to X$.
(ii) Let P'' be the slope ordered matrix having the primitive generators of the regular subdivision of $\Sigma(P')$ as its columns. Then $X'' := X(A, P'')$ is smooth and there is a canonical morphism $X'' \to X'$.

Contracting all (-1)-curves inside the smooth locus that lie over singularities of X gives $X'' \to \tilde{X} \to X$, where $\tilde{X} = X(A, \tilde{P})$ is the *minimial resolution* of X.

7. Self intersection numbers and continued fractions

This section presents some variations on [**43**, Thm. 2.5] (iii) and (iv) on continued fractions over the numbers $-D_{i1}^2, \ldots, -D_{n_i}^2$ given by an arm of a smooth $\mathbb{K}^*$-surface with two parabolic fixed point curves. Proposition 7.5 shows how to express the entries l_{ij} and d_{ij} of P for smooth $X(A, P)$ of types (p-p), (p-e) and (e-e) via continued fractions over partial arms. For convenience, we present the full proofs.

Definition 7.1. Consider the defining matrix P of a smooth rational projective $\mathbb{K}^*$-surface $X(A, P)$. By our assumptions, P is irredundant and slope ordered.

(i) We call P *adapted to the source* if it satisfies
    (a) $-l_{i1} < d_{i1} \le 0$ for $i = 1, \ldots, r$,
    (b) $l_{01}, l_{11} \ge l_{21} \ge \ldots \ge l_{r1}$.

(ii) We call P *adapted to the sink* if it satisfies
(a) $0 \le d_{in_i} < l_{in_i}$ for $i = 1, \ldots, r$,
(b) $l_{01}, l_{11} \ge l_{21} \ge \ldots \ge l_{rn_r}$.

Definition 7.2. Consider a smooth rational projective $\mathbb{K}^*$-surface $X = X(A, P)$ and the entries l_{ij} and d_{ij} of the defining matrix P.

(i) Assume that X has a parabolic fixed point curve D^+ and let P be adapted to the source. Set

$$\begin{array}{lll} \text{for } i = 0, \ldots, r: & l_{i0} := 0, \quad d_{i0} := 1, & D_{i0}^2 := (D^+)^2, \\ \text{for } i = 0: & l_{i(-1)} := -l_{11}, \quad d_{i(-1)} := d_{11}, & \\ \text{for } i = 1, \ldots, r: & l_{i(-1)} := -l_{01}, \quad d_{i(-1)} := d_{01}. & \end{array}$$

(ii) Assume that X has an elliptic fixed point x^- and let P be adapted to the sink. Set

$$\begin{array}{lll} \text{for } i = 0: & l_{0n_0+1} := -l_{1n_1}, & d_{0n_0+1} := d_{1n_1}, \\ \text{for } i = 1: & l_{1n_1+1} := -l_{0n_0}, & d_{1n_1+1} := d_{0n_0}, \\ \text{for } i = 2, \ldots, r: & l_{in_i+1} := -l_{0n_0} l_{1n_1}, & d_{in_i+1} := -1. \end{array}$$

Lemma 7.3. *Consider a smooth rational projective $\mathbb{K}^*$-surface $X = X(A, P)$ and the curves D_{ij} in the arms of X.*

(i) *Assume that X has a parabolic fixed point curve D^+ and let P be adapted to the source. Then, for all $i = 0, \ldots, r$ and $j = 0, \ldots, n_i - 1$, we have*

$$-l_{ij} D_{ij}^2 \;=\; l_{ij-1} + l_{ij+1}, \qquad -d_{ij} D_{ij}^2 \;=\; d_{ij-1} + d_{ij+1}.$$

(ii) *Assume that X has an elliptic fixed point curve x^- and let P be adapted to the sink. Then, for all $i = 0, \ldots, r$ and $j = 2, \ldots, n_i$, we have*

$$-l_{ij} D_{ij}^2 \;=\; l_{ij-1} + l_{ij+1}, \qquad -d_{ij} D_{ij}^2 \;=\; d_{ij-1} + d_{ij+1}.$$

Proof. The statements follow directly from the computation of self intersection numbers of $X = X(A, P)$ in terms of the entries of P given in Remark 6.4. □

Reminder 7.4. Given any finite sequence $a_1, \ldots, a_k$ of rational numbers, consider the process

$$\mathrm{CF}_1(a_1) = a_1, \quad \mathrm{CF}_2(a_1, a_2) = a_1 - \frac{1}{a_2}, \quad \mathrm{CF}_3(a_1, a_2, a_3) = a_1 - \frac{1}{a_2 - \frac{1}{a_3}} \quad \ldots$$

Provided there is no division by zero, these numbers are called *continued fractions*. The formal definition runs inductively:

$$\mathrm{CF}_1(a_1) \;:=\; a_1, \qquad \mathrm{CF}_k(a_1, \ldots, a_k) \;:=\; a_1 - \frac{1}{\mathrm{CF}_{k-1}(a_2, \ldots, a_k)}.$$

Proposition 7.5. *Consider a smooth rational projective $\mathbb{K}^*$-surface $X = X(A, P)$ and the curves D_{ij} in the arms of X.*

(i) *Assume that X has a parabolic fixed point curve D^+ and P is adapted to the source. Fix $0 \le i \le r$ and $1 \le j \le n_i$ and for $k = 1, \ldots, j-1$ set*

$$f_{ijk} \;:=\; \mathrm{CF}_k(-D_{ij-k}^2, \ldots, -D_{ij-1}^2).$$

Then the entries l_{ij} and d_{ij} of the matrix P can be expressed in terms of the above continued fractions f_{ijk} as

$$l_{ij} \;=\; \prod_{k=1}^{j-1} f_{ijk}, \qquad d_{ij} \;=\; \begin{cases} -\Big((D^+)^2 f_{0jj-1}+1\Big)\displaystyle\prod_{k=1}^{j-2} f_{0jk} & i=0,\\[2ex] -\displaystyle\prod_{k=1}^{j-2} f_{ijk}, & i\neq 0.\end{cases}$$

(ii) *Assume that X has an elliptic fixed point x^- and P is adapted to the sink. Fix $0 \le i \le r$ and $1 \le j \le n_i$ and for $k = 1, \dots, j-1$ set*

$$h_{ijk} \;:=\; \mathrm{CF}_k(-D^2_{in_i-j+k}, \dots, -D^2_{in_i}).$$

Then the entries l_{in_i-j} and d_{in_i-j} of the matrix P can be expressed in terms of the above continued fractions as

$$l_{n_i-j} \;=\; \left(\prod_{k=1}^{j} h_{ijk}\right) l_{in_i} - \left(\prod_{k=1}^{j-1} h_{ijk}\right) l_{in_i+1},$$

$$d_{n_i-j} \;=\; \left(\prod_{k=1}^{j} h_{ijk}\right) d_{in_i} - \left(\prod_{k=1}^{j-1} h_{ijk}\right) d_{in_i+1}.$$

Proof. For both assertions, the proof relies on partially solving tridigonal systems of linear equations of the following form:

$$\begin{bmatrix} a_1 & -1 & & & 0\\ -1 & a_2 & -1 & & \\ & & \ddots & & \\ & & -1 & a_{n-1} & -1\\ 0 & & & -1 & a_n \end{bmatrix} \cdot \begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix} \;=\; \begin{bmatrix} b_1\\ 0\\ \vdots\\ 0\\ b_n\end{bmatrix}$$

In [**35**, Thm. 15], the solutions are explicitly computed via continued fractions in the entries. In particular, with $f_k \;:=\; \mathrm{CF}_k(a_k, \dots, a_1)$, it gives us

$$b_1 \;=\; \left(\prod_{k=1}^{n} f_k\right) x_n - \left(\prod_{k=1}^{n-1} f_k\right) b_n.$$

We verify (i). Due to smoothness, Proposition 6.8 (i) yields $l_{i1} = 1$. Now, the relations among the l_{ij} provided by Lemma 7.3 (i) can be written as follows

$$\begin{bmatrix} -D^2_{ij-1} & -1 & & & 0\\ -1 & -D^2_{ij-2} & -1 & & \\ & & \ddots & & \\ & & -1 & -D^2_{i2} & -1\\ 0 & & & -1 & -D^2_{i1} \end{bmatrix} \cdot \begin{bmatrix} l_{ij-1}\\ l_{ij-2}\\ \vdots\\ l_{i2}\\ l_{i1}\end{bmatrix} \;=\; \begin{bmatrix} l_{ij}\\ 0\\ \vdots\\ 0\\ l_{i0}\end{bmatrix}$$

Thus, the above formula for b_1 gives the desired presentation of l_{ij}. With the d_{ij}, we proceed analogously. In order to verify (ii), look at

$$\begin{bmatrix} -D^2_{in_i-j+1} & -1 & & & 0 \\ -1 & -D^2_{in_i-j+2} & -1 & & \\ & & \ddots & & \\ & & -1 & -D^2_{in_i-1} & -1 \\ 0 & & & -1 & -D^2_{in_i} \end{bmatrix} \cdot \begin{bmatrix} l_{in_i-j+1} \\ l_{in_i-j+2} \\ \vdots \\ l_{in_i-1} \\ l_{in_i} \end{bmatrix} = \begin{bmatrix} l_{in_i-j} \\ 0 \\ \vdots \\ 0 \\ l_{in_i+1} \end{bmatrix}$$

encoding the relations among the l_{ij} from Lemma 7.3 (ii) and apply the above presentation of b_1. Again the d_{ij} are settled analogously. □

COROLLARY 7.6. *Consider a smooth rational projective $\mathbb{K}^*$-surface $X = X(A,P)$.*

(i) *Assume that there is a fixed point curve $D^+ \subseteq X$ and that P is adapted to the source. Fix any choice of indices $1 \le j_i \le n_i$, where $i = 0,\dots,r$. Then we have*

$$(D^+)^2 \;=\; -\sum_{i=0}^{r} m_{ij_i} - \sum_{i=0}^{r} \mathrm{CF}_{j_i-1}(-D^2_{i1},\dots,-D^2_{ij_i-1})^{-1}.$$

(ii) *Assume that there is a fixed point $x^- \in X$ and that P is adapted to the sink. Fix $0 \le j \le n_0 - 1$ and set $\bar\sigma^- := \mathrm{cone}(v_{0n_0-j}, v_{1n_1},\dots,v_{rn_r})$. Then we have:*

$$\frac{l_{0n_0-j}}{\det(\bar\sigma^-)} \;=\; \mathrm{CF}_j(-D^2_{in_i},\dots,-D^2_{in_i-j+1})^{-1} l_{1n_1}.$$

PROOF. We prove (i). Proposition 7.5 (i) allows us to express the slopes m_{ij_i} in the following way:

$$m_{0j_0} \;=\; -(D^+)^2 - f^{-1}_{0j_0j_0-1}, \qquad m_{ij_i} \;=\; -f^{-1}_{ij_ij_i-1}, \quad i=1,\dots,r.$$

By the definition of the f_{ijk}, this directly leads to the desired representation of the self intersection number:

$$(D^+)^2 \;=\; -m_{0j_0} - f^{-1}_{0j_0j_0-1} \;=\; -\sum_{i=0}^{r} m_{ij_i} - \sum_{i=0}^{r} f^{-1}_{ij_ij_i-1}.$$

We turn to (i). Since X is smooth, Proposition 6.10 (iv) tells us $\det(\sigma^-) = -1$. Thus, setting $h_0 := h_{0j1}\cdots h_{0jj}$, we have

$$\begin{aligned} -1 &= \det(\sigma^-) \\ &= l_{0n_0}d_{1n_1} + l_{1n_1}d_{0n_0} \\ &= (h_0^{-1}d_{0n_0-j} + h_{0jj}^{-1}d_{1n_1})l_{1n_1} + d_{1n_1}(h_0^{-1}l_{0n_0-j} - h_{0jj}^{-1}l_{1n_1}) \\ &= h_0^{-1}(d_{0n_0-j}l_{1n_1} + d_{1n_1}l_{0n_0-j}) \\ &= h_0^{-1}\det(\bar\sigma^-), \end{aligned}$$

as is seen by a direct computation. Using the representation of l_{0n_0} provided by Proposition 7.5 (ii), we obtain

$$\frac{l_{0n_0-j}}{\det(\bar\sigma^-)} - l_{0n_0} \;=\; -h_0^{-1}l_{0n_0-j} - l_{0n_0} \;=\; h_{0jj}^{-1}l_{1n_1}.$$

□

8. Quasismooth simple elliptic fixed points

The aim of this section is to establish Theorem 8.4 which specifies obstructions to the existence of quasismooth simple elliptic fixed points. This is a first step towards the proof of Theorem 0.1, but also has some general applications to the geometry of rational projective $\mathbb{K}^*$-surfaces, see Corollaries 8.6 to 8.8. Let us recall our definition of a simple elliptic fixed point.

Definition 8.1. We say that an elliptic fixed point x of a rational projective $\mathbb{K}^*$-surface X is *simple* if for the minimal resolution $\pi\colon \tilde{X} \to X$ of singularities, the fiber $\pi^{-1}(x)$ is contained in an arm of $\tilde{X}$.

Note that a simple elliptic fixed point has in particular no parabolic fixed point curve in its fiber of the minimal resolution of singularities. We discuss two examples of simple elliptic fixed points, making use of the resolution of singularities provided by Remark 6.13 and the formulae for the self intersection numbers given in Remark 6.4.

Example 8.2. The following matrices P and $\tilde{P}$ define the minimal resolutions $\tilde{X} \to X$ of non-toric $\mathbb{K}^*$-surfaces X, each of them with a simple elliptic fixed point $x^- \in X$.

(i) Here x^- is quasismooth but singular and the fiber over x^- equals the curve $\tilde{D}_{04} \subseteq \tilde{X}$ in the 0-th arm:

$$P = \begin{bmatrix} -1 & -2 & -3 & 1 & 1 & 0 & 0 & 0 \\ -1 & -2 & -3 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\tilde{P} = \begin{bmatrix} -1 & -2 & -3 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -2 & -3 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The curve $\tilde{D}_{04}$ is isomorphic to a projective line and has self intersection number equal to -2. In other words, $x^- \in X$ is an A_2-singularity.

(ii) Here x^- is not quasismooth and the fiber over x^- equals the curve $\tilde{D}_{08} \subseteq \tilde{X}$ in the 0-th arm:

$$P = \begin{bmatrix} -1 & -2 & -3 & -4 & -5 & -6 & -7 & 1 & 2 & 3 & 0 & 0 & 0 \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & -3 & -4 & -5 & -6 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\tilde{P} = \begin{bmatrix} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -1 & 1 & 2 & 3 & 0 & 0 & 0 \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & -3 & -4 & -5 & -6 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The curve $\tilde{D}_{08}$ is of intersection number -1 and has a cusp singularity. The singularity $x^- \in X$ is isomorphic to the Brieskorn-Pham singularity

$$0 \in V(T_1^7 + T_2^3 + T_3^2) \subseteq \mathbb{K}^3,$$

where we gain this presentation by looking at $X \cap Z_{\sigma^-}$ for the (smooth) affine toric chart $Z_{\sigma^-} \subseteq Z$; compare also [**36**, No. 2.5 on p. 72].

Definition 8.3. We say that a parabolic fixed point curve $D \subseteq X$ is of a rational projective $\mathbb{K}^*$-surface is *gentle* if there is an arm $\mathcal{A}_i = D_{i1} \cup \ldots \cup D_{in_i}$ such that the (unique) point $x \in D \cap \mathcal{A}_i$ is a smooth point of X.

Theorem 8.4. *Let X be a non-toric rational projective $\mathbb{K}^*$-surface X with a quasismooth simple elliptic fixed point $x \in X$.*

(i) *There is no gentle non-negative parabolic fixed point curve in X.*
(ii) *There is no other quasismooth simple elliptic fixed point in X.*

Remark 8.5. The assumption that X is non-toric is essential in Theorem 8.4. A cheap smooth toric counterexample is given by the projective plane $\mathbb{P}_2$: Consider the two $\mathbb{K}^*$-actions given by

$$t \cdot [z] \ = \ [z_0, z_1, tz_2], \qquad\qquad t \cdot [z] \ = \ [z_0, tz_1, t^2 z_2].$$

For the first one, $[0, 0, 1]$ is an elliptic fixed point and $V(T_2)$ a parabolic fixed point curve of self intersection one. The second one has $[1, 0, 0]$ and $[0, 0, 1]$ as elliptic fixed points.

Corollary 8.6. *Every rational projective $\mathbb{K}^*$-surface with two quasismooth simple elliptic fixed points is a toric surface.*

Corollary 8.7. *Every quasismooth non-toric rational projective $\mathbb{K}^*$-surface with a simple elliptic fixed point has a fixed point curve.*

The latter says that, when considering quasismooth non-toric rational projective $\mathbb{K}^*$-surfaces X, we always may assume that there is a curve $D^+ \subseteq X$. For smooth $X = X(A, P)$, this allows us to complement [**43**, Thm. 2.5] by showing that the defining matrix P is basically determined by the self intersection numbers of invariant curves. More precisely, we obtain the following.

Corollary 8.8. *Let $X = X(A, P)$ be smooth, non-toric with P adapted to $D^+ \subseteq X$. Then all entries l_{ij} and d_{ij} of P can be expressed via self intersection numbers according to Corollary 7.5.*

Definition 8.9. Let $X = X(A, P)$ have a simple elliptic fixed point $x \in X$. We call $0 \le i \le r$ an *exceptional index* of x if $\pi^{-1}(x)$ is contained in the i-th arm of $\tilde{X} = X(A, \tilde{P})$, where $\pi \colon \tilde{X} \to X$ is the minimal resolution of singularities.

Note that for any singular simple elliptic fixed point the exceptional index is unique. The following characterization of simple quasismooth elliptic fixed points is an important ingredient for the proof of Theorem 8.4.

Proposition 8.10. *Let $x \in X = X(A, P)$ be a quasismooth elliptic fixed point with leading indices ι_0, ι_1.*

(i) *Assume $x = x^+$. Then x is simple with exceptional index ι_0 if and only if there exists a vector $u \in \mathbb{Z}^r \times \mathbb{Z}_{<0}$ such that*

$$\langle u, v_{\iota_1 1} \rangle = -1, \quad \langle u, v_{i1} \rangle = 0, \ i \neq \iota_0, \iota_1, \quad \langle u, v_{ij} \rangle \geq 0, \ i \neq \iota_1.$$

We have $l_{i1} = 1$ whenever $i \neq \iota_0, \iota_1$. Moreover, if $u \in \mathbb{Z}^r \times \mathbb{Z}_{<0}$ is a vector as above, then the following holds:

$$0 \ < \ m^+ \ \leq \ -u_{r+1} m^+ \ \leq \ \frac{1}{l_{\iota_1 1}}.$$

(ii) *Assume $x = x^-$. Then x is simple with exceptional index ι_0 if and only if there exists a vector $u \in \mathbb{Z}^r \times \mathbb{Z}_{<0}$ such that*

$$\langle u, v_{\iota_1 n_{\iota_1}} \rangle = -1, \quad \langle u, v_{in_i} \rangle = 0, \ i \neq \iota_0, \iota_1, \quad \langle u, v_{ij} \rangle \geq 0, \ i \neq \iota_1.$$

We have $l_{in_i} = 1$ whenever $i \neq \iota_0, \iota_1$. Moreover, if $u \in \mathbb{Z}^r \times \mathbb{Z}_{>0}$ is a vector as above, then the following holds:

$$0 \ > \ m^- \ \geq \ u_{r+1} m^- \ \geq \ -\frac{1}{l_{\iota_1 n_{\iota_1}}}.$$

Lemma 8.11. *Consider a sequence of vectors $v_0, \ldots, v_k \in \mathbb{Q}^2$ such that there are $c_1, \ldots, c_{k-1} \in \mathbb{Z}_{\geq 2}$ with*

$$v_{j+1} \ = \ c_j v_j - v_{j-1}, \qquad j = 1, \ldots, k-1.$$

Then, for $k \geq 2$, the difference $v_k - v_{k-1}$ lies in $\tau := \operatorname{cone}(v_1, v_1 - v_0)$ and the vector v_k lies in the shifted cone $\tau + v_1$.

Proof. Clearly, $v_1 \in \tau + v_1$ and $v_1 - v_0 \in \tau$. We proceed inductively. For $j \geq 1$, assume $v_j \in \tau + v_1$ and $v_j - v_{j-1} \in \tau$. Write $v_j = v_j' + v_1$ with $v_j' \in \tau$. Then, using $c_j \geq 2$ we see

$$v_{j+1} \ = \ c_j v_j - v_{j-1} \ = \ (c_j - 1) v_j' + (c_j - 2) v_1 + (v_j - v_{j-1}) + v_1 \ \in \ \tau + v_1.$$

$\square$

Lemma 8.12. *Consider $H := \{(x, y) \in \mathbb{Q}^2;\ x - y \geq 1\}$. Given $v_0 = (a, b)$ and $v_1 = (c, d)$ in H with $a < 0$ and $c > 0$, there is no $u \in \mathbb{Z} \times \mathbb{Z}_{>0}$ satisfying*

(i) $\langle u, v_0 \rangle \ = \ u_1 a + u_2 b \ \geq \ 0,$ (ii) $\langle u, v_1 \rangle \ = \ u_1 c + u_2 d \ = \ -1.$

Proof. Since $b < a < 0$ and $u_2 > 0$ hold, we infer $u_1 \leq -2$ from (i). Then (ii) tells us $d > 0$. Now, plugging $u_1 = -(u_2 d + 1)/c$ into (i) leads to a contradiction:

$$u_2 \ \leq \ \frac{a}{bc - ad} \ \leq \ \frac{b+1}{bc - ad} \ \leq \ \frac{b+1}{bc} \ \leq \ 1 + \frac{1}{bc} \ < \ 1.$$

$\square$

Lemma 8.13. *Consider four vectors ξ_1, ξ_2 and η_1, η_2 in $\mathbb{Z}^2$ satisfying the following conditions:*

$$\det(\xi_2, \xi_1) \ = \ \det(\xi_1, \eta_1) \ = \ \det(\eta_1, \eta_2) \ = \ 1, \qquad \det(\xi_2, \eta_2) \ \geq \ 1.$$

Then $\xi_1 = a\eta_1 - \eta_2$ and $\xi_2 = b\eta_1 - c\eta_2$, where $a, b, c > 0$ and $c = ab - 1$. In particular,

$$a = 1 \ \Rightarrow \ \det(\xi_2, \eta_2) = 1, \qquad a \geq 2 \ \Rightarrow \ c - b \geq 1.$$

Proof. In suitable linear coordinates, we have $\eta_1 = (0, -1)$ and $\eta_2 = (1, 0)$ and moreover $\xi_1, \xi_2 \in \mathbb{Z}^2_{<0}$. In this situation, the assertion can be directly verified. $\square$

Proof of Proposition 8.10. First observe that multiplying the last row of P by -1 interchanges source and sink and thus it suffices to prove Assertion (ii). For this we may assume that P is adapted to the sink. Consider the minimal resolution $\pi\colon \tilde{X} \to X$, where $\tilde{X} = X(A, \tilde{P})$, as provided by Remark 6.13. Then, for every $i = 0, \ldots, r$, the columns $v_{i1}, \ldots, v_{in_i}$ of

P occur among the columns $\tilde v_{i1}, \ldots, \tilde v_{i\tilde n_i}$ of $\tilde P$. Moreover, Proposition 6.10 yields

$$\tilde l_{i\tilde n_i} \ = \ l_{in_i} \ = \ 1, \qquad \tilde d_{i\tilde n_i} \ = \ d_{in_i} \ = \ 0 \qquad \text{for } i = 2, \ldots, r.$$

First suppose that $x^- \in X$ is simple. We may assume that the exceptional index of x^- is 0 that means that the divisors inside $\pi^-(x)$ are located in the 0-th arm of $\tilde X$. Then $\tilde l_{1\tilde n_1} = l_{1n_1}$ and $\tilde d_{1\tilde n_1} = d_{1n_1}$ hold. Define $u \in \mathbb{Z}^{r+1}$ by $u_1 := \tilde d_{0\tilde n_0}$ and $u_i := 0$ for $i = 1, \ldots, r$ and $u_{r+1} := \tilde l_{0\tilde n_0}$. Then Proposition 6.10 yields

$$\langle u, v_{1n_1} \rangle \ = \ \tilde d_{0n_0} l_{1n_1} + \tilde l_{0\tilde n_0} d_{1n_1} \ = \ \tilde l_{0\tilde n_0} \tilde d_{1\tilde n_1} + \tilde l_{1\tilde n_1} \tilde d_{0\tilde n_0} \ = \ -1.$$

According to the definition of u, we have $\langle u, v_{in_i} \rangle = 0$ for $i = 2, \ldots, r$. Moreover, for $j = 0, \ldots, \tilde n_0$, we use slope orderedness of $\tilde P$ to see

$$\langle u, \tilde v_{0j} \rangle \ = \ -\tilde d_{0\tilde n_0} \tilde l_{0j} + \tilde l_{0\tilde n_0} \tilde d_{0j} \ = \ \tilde l_{0\tilde n_0} \tilde l_{0j} \left(\frac{\tilde d_{0j}}{\tilde l_{0j}} - \frac{\tilde d_{0\tilde n_0}}{\tilde l_{0\tilde n_0}} \right) \ \geq \ 0.$$

This means in particular $\langle u, v_{0j} \rangle \geq 0$ for $j = 1, \ldots, n_0$. Moreover, since P is slope ordered and adapted to the sink, we have $d_{ij} \geq 0$ for all $i \geq 1$ and thus

$$\langle u, v_{ij} \rangle \ = \ \tilde l_{0\tilde n_0} d_{ij} \ \geq \ 0, \qquad i = 2, \ldots, r, \quad j = 1, \ldots, n_i.$$

Let us care about the estimate for the slope sum m^-. Evaluating u at the vectors v_{0n_0} and v_{1n_1} gives us

$$-u_1 l_{0n_0} + u_{r+1} d_{0n_0} \ \geq \ 0, \qquad\qquad u_1 l_{1n_1} + u_{r+1} d_{1n_1} \ = \ -1.$$

Solving the second condition for u_1 and plugging the result into the first one, gives us the estimate

$$u_{r+1}(l_{0n_0} d_{1n_1} + l_{1n_1} d_{0n_0}) \ \geq \ -l_{0n_0}.$$

The expression $l_{0n_0} d_{1n_1} + l_{1n_1} d_{0n_0}$ equals $l_{0n_0} l_{1n_1} m^-$ and is negative due to Proposition 6.10. We conclude

$$1 \ \leq \ u_{r+1} \ \leq \ -\frac{1}{l_{1n_1} m^-}.$$

This directly yields the desired lower bound for $u_{r+1} m^-$. The upper bound $0 > m^-$ is guaranteed by Remark 6.3.

Now suppose that there is a vector $u \in \mathbb{Z}^r \times \mathbb{Z}_{>0}$ as in the proposition. Suitably arranging P and adapting u, we achieve $\iota_0 = 0$ and $\iota_1 = 1$. Note that for each $i = 2, \ldots, r$, we have $u_i = 0$ due to $d_{in_i} = 0$. We assume that x^- is not simple and show that this leads to a contradiction. For this, it suffices to verify

$$(-l_{0n_0}, d_{0n_0}),\, (l_{1n_1}, d_{1n_1}) \ \in \ H \ := \ \{(x, y) \in \mathbb{Q}^2;\ x - y \geq 1\},$$

because then Lemma 8.12 implies that u cannot evaluate non-negatively on v_{0n_0} and to -1 on v_{1n_1}. Since P is slope ordered, (l_{1n_1}, d_{1n_1}) lies in H. In order to see that also $(-l_{0n_0}, d_{0n_0})$ belongs to H, we use the assumption that x^- is not simple and thus we are in one of the following two cases.

Case 1: The fiber $\pi^{-1}(x) \subseteq \tilde{X}$ contains a parabolic fixed point curve $\tilde{D}^-$. Consider the sequence of divisors $\tilde{D}^-, \tilde{D}_{0\tilde{n}_0}, \dots, \tilde{D}_{0\tilde{n}_0-k}$ connecting $\tilde{D}^-$ with the proper transform $\tilde{D}_{0\tilde{n}_0-k}$ of $D_{0n_0} \subseteq X$. This give us a sequence of pairs

$$(0,-1),\ (-1,\tilde{d}_{0\tilde{n}_0}),\ \dots,\ (-\tilde{l}_{0\tilde{n}_0-k},\tilde{d}_{0\tilde{n}_0-k}) = (-l_{0n_0},d_{0n_0}),$$

where $\tilde{l}_{0\tilde{n}_0} = 1$ due to Proposition 6.8. Since $\tilde{X} \to X$ is the minimal resolution, we have $(\tilde{D}^-)^2 \le -2$ and $\tilde{D}_{0j}^2 \le -2$ whenever $j > \tilde{n}_0 - k$. Thus, Remark 6.4 shows $\tilde{d}_{0\tilde{n}_0} \le -2$. According to Lemma 7.3, the above sequence of pairs satisfies the assumptions of Lemma 8.11. Applying the latter yields $(-l_{0n_0}, d_{0n_0}) \in H$.

Case 2: The fiber $\pi^{-1}(x) \subseteq \tilde{X}$ contains an elliptic fixed point $\tilde{x}^-$ and curves from the arms 0 and 1 of $\tilde{X}$. The curves in $\pi^{-1}(x)$ are $\tilde{D}_{i\tilde{n}_i}, \dots, \tilde{D}_{i\tilde{n}_i-k_i}$ where $i = 0,1$ and $\tilde{D}_{i\tilde{n}_i-k_i-1}$ is the proper transform of $D_{0n_0} \subseteq X$. Consider the associated sequences of pairs

$$(\tilde{l}_{0\tilde{n}_0},\tilde{d}_{0\tilde{n}_0}),\ \dots,\ (-\tilde{l}_{0\tilde{n}_0-k_0},\tilde{d}_{0\tilde{n}_0-k_0}) = (-l_{0n_0},d_{0n_0}),$$

$$(\tilde{l}_{1\tilde{n}_1},\tilde{d}_{1\tilde{n}_1}),\ \dots,\ (\tilde{l}_{1\tilde{n}_1-k_1},\tilde{d}_{1\tilde{n}_1-k_1}) = (l_{1n_1},d_{1n_1}).$$

By slope orderedness of $\tilde{P}$, all members of the second sequence lie in H. Now, write ξ_1, ξ_2 for the first two pairs of the first sequence and η_1, η_2 for the first two pairs of the second one. Then Lemma 8.13 provides us with integers $a, b, c > 1$, where $c = ab - 1$ such that

$$\xi_1 \;=\; a\eta_1 - \eta_2, \qquad \xi_2 \;=\; b\eta_1 - c\eta_2.$$

Here $a \ge 2$ holds as otherwise Proposition 6.10 shows that $\tilde{D}_{0\tilde{n}_0}$ and $\tilde{D}_{1\tilde{n}_1}$ contract smoothly which contradicts to minimality of the resolution. Thus, $\xi_1, \xi_2 \in H$. Again by minimality of the resolution, all $\tilde{D}_{0j} \subseteq \pi^{-1}(x)$ are of self intersection at most -2. Using Lemmas 7.3 and 8.11, we arrive at $(l_{1n_1}, d_{1n_1}) \in H$. □

Lemma 8.14. *Consider the defining matrix P of a rational projective $\mathbb{K}^*$-surface $X(A,P)$.*

(i) *If $m_{ij} = 0$ holds for $0 \le i \le r$ and $1 \le j \le n_i$, then $d_{ij} = 0$ and $l_{ij} = 1$.*
(ii) *If P is adapted to the sink, then $0 \le m_{in_i} < m_{in_i} + l_{in_i}^{-1} \le 1$ for $i = 1, \dots, r$.*
(iii) *If P is irredundant and adapted to the sink, then $m_{i1} > 0$ for $i = 1, \dots, r$.*

Proof. We verify (i). If $m_{ij} = 0$ holds, then we must have $d_{ij} = 0$ and thus primitivity of the column v_{ij} yields $l_{ij} = 1$. We turn to (ii). As P is adapted to the sink, we have $0 \le d_{in_i} < l_{in_i}$ whenever $i \ge 1$ and the desired estimate follows. We prove (iii). By slope orderedness of P and (ii), we have $m_{i1} \ge m_{in_1} \ge 0$. We exclude $m_{i1} = 0$. Otherwise, $m_{in_i} = 0$ holds. Thus, $d_{i1} = d_{in_i} = 0$ and (i) yields $l_{i1} = l_{in_i} = 1$ and $n_i = 1$. This is a contradiction to irredundance of P. □

Proof of Theorem 8.4. We may assume $X = X(A,P)$ and that the quasismooth simple elliptic fixed point is $x^- \in X$. Moreover, we may assume that P is irredundant, adapted to the sink and that the two leading indices from Proposition 8.10 are 0 and 1. Then we have

$$m_{01} \ \geq \ m_{0n_0}, \qquad m_{11} \ \geq \ m_{1n_1} \ \geq \ 0, \qquad m_{i1} \ > \ m_{in_i} \ = \ 0, \ i = 2, \ldots, r,$$

by slope orderedness, Proposition 6.10 and Lemma 8.14. In particular, m^- equals $m_{0n_0} + m_{n_1}$. Using the estimate on m^- from Proposition 8.10 and Lemma 8.14 (ii), we see

$$0 \ \geq \ -m_{1n_1} \ > \ m_{0n_0} \ \geq \ -m_{1n_1} - \frac{1}{l_{1n_1}} \ \geq \ -1.$$

We prove (i). Let $D^+ \subseteq X$ be a non-negative parabolic fixed point curve. We have to show that D^+ is not gentle. According to Proposition 6.8, this means to verify $l_{i1} \geq 2$ for $i = 0, \ldots, r$. Remark 6.4 yields

$$0 \ \geq \ m^+ \ = \ m_{01} + m_{11} + m_{21} + \ldots + m_{r1},$$

where $m_{11}, \ldots, m_{r1} \ > \ 0$ and $r \ \geq \ 2$, hence $0 \ > \ m_{01}$. Moreover, $m_{01} \ \geq \ m_{0n_0} \ \geq \ -1$ yields $0 \ < \ m_{i1} \ < \ 1$ for $i \ = \ 1, \ldots, r$. This implies $l_{i1} \ \geq \ 2$ for $i = 1, \ldots, r$. We show $l_{01} \geq 2$. Otherwise, $l_{01} = 1$ and thus $m_{01} = -1$. Then $m_{0n_0} = -1$. Consequently $n_0 = 1$ and $l_{01} = -d_{01} = 1$ by primitivity of v_{01}. This is a contradiction to irredundance of P.

We prove (ii). Suppose that there is also a quasismooth simple elliptic fixed point $x^+ \in X$. Let $0 \leq \iota_0, \iota_1 \leq r$ be the leading indices as in Proposition 8.10 (i). Then $l_{i1} = 1$ holds whenever $i \neq \iota_0, \iota_1$. This allows us to assume $\iota_0, \iota_1 \leq 3$. The estimates from Proposition 8.10 yield

$$m^+ - m^- \ = \ m_{01} - m_{0n_0} + m_{11} - m_{1n_1} + m_{21} + m_{31} + \sum_{i=4}^{r} d_{i1} \ \leq \ \frac{1}{l_{1n_1}} + \frac{1}{l_{\iota_1 1}} \ \leq \ 2.$$

Case $\iota_0 \leq 1$ *and* $\iota_1 \leq 1$. Then we have $1 \leq d_{21} = m_{21}$. From the above estimate, we infer

$$0 \ \leq \ m^+ - m^- - 1 \ \leq \ \frac{1}{l_{1n_1}} + \frac{1}{l_{\iota_1 1}} - 1 \ = \ \frac{l_{1n_1} + l_{\iota_1 1} - l_{1n_1} l_{\iota_1 1}}{l_{1n_1} l_{\iota_1 1}}.$$

This leaves us with the following possibilities: first $l_{1n_1} = l_{\iota_1 1} = 2$, second $l_{1n_1} = 1$ and third $l_{\iota_1 1} = 1$. We go through these cases.

Let $l_{1n_1} = l_{\iota_1 1} = 2$. Then $m_{01} = m_{0n_0}$ and $m_{11} = m_{1n_1}$ as well as $d_{21} = 1$ hold. Thus, $n_0 = n_1 = 1$. Moreover, $l_{11} = 2$ implies $d_{11} = 1$. Proposition 8.10 tells us

$$-2\left(m_{01} + \frac{1}{2}\right) \ = \ -l_{1n_1} m^- \ \leq \ 1, \qquad 2\left(m_{01} + \frac{1}{2} + 1\right) \ \leq \ l_{\iota_1 1} m^+ \ \leq \ 1.$$

Thus $m_{01} \geq -1$ and $m_{01} \leq -1$. As v_{01} is primitive, we arrive at $l_{01} = -d_{01} = 1$, which is a contradiction to the irredundance of P.

Let $l_{1n_1} = 1$. As P is adapted to the sink, $d_{1n_1} = 0$ holds. Irredundance yields $n_1 \geq 2$ and $d_{11} > 0$. As noted before, $m_{01} \geq m_{0n_0} \geq -1$. If $\iota_1 = 1$, then

$$0 \ \leq \ d_{11} + l_{11}(m_{01} + 1) = l_{11}(m_{01} + m_{11} + 1) \ \leq \ l_{11} m^+ \ \leq \ 1$$

due to Proposition 8.10. This implies $d_{11} = 1$ and $m_{01} = m_{0n_0} = -1$. Thus, $n_0 = 1$ and $-d_{01} = l_{01} = 1$ holds; a contradiction. If $\iota_1 = 0$, then we have

$$0 \ \le \ d_{01} + l_{01} \ < \ l_{01}(m_{01} + m_{11} + 1) \ \le \ l_{01}m^+ \ \le \ 1.$$

We conclude $-d_{01} = l_{01} = 1$, using primitivity of v_{01}. Then $m_{01} \ge m_{0n_0} \ge -1$ implies $n_{01} = 1$. A contradiction to irredundance of P.

Let $l_{\iota_1 1} = 1$. This case transforms into the preceding one by switching source and sink via multiplying the last row of P by -1 and adapting to the new sink.

Case $\iota_0 \le 1$ *and* $\iota_1 \ge 2$. Then we may assume $\iota_1 = 2$. If $\iota_0 = 0$, then we have $l_{11} = 1$. Thus, $n_1 \ge 2$ and $d_{11} > 0$. Proposition 8.10 and Lemma 8.14 (ii) show

$$d_{11} + \frac{d_{21}}{l_{21}} \ = \ m_{11} + m_{21} \ \le \ m_{1n_1} + m^+ - m^- \ \le \ m_{1n_1} + \frac{1}{l_{1n_1}} + \frac{1}{l_{21}} \ \le \ 2.$$

Now, $d_{11} = 2$ yields $m_{21} = 0 = m_{2n_2}$, which is impossible by irredundance of P. Thus, $d_{11} = 1$. The above inequality gives $d_{21} = 1$ and $d_{1n_1} = l_{1n_1} - 1$. Hence,

$$m_{01} \ \le \ m^+ - m_{11} - m_{21} \ \le \ \frac{1}{l_{21}} - 1 - \frac{1}{l_{21}} \ = \ -1.$$

So, $m_{01} \ge m_{0n_0} \ge -1$ yields $m_{0n_0} = m_{01} = -1$. Thus, $n_0 = 1$ and $l_{01} = 1$; a contradiction. If $\iota_0 = 1$, then $l_{01} = 1$. Hence $m_{01} \in \mathbb{Z}$ and $n_0 \ge 2$. Now

$$-1 \ \le \ m_{0n_0} \ < \ m_{01} \ \le \ m^+ - m_{21} - m_{11} \ \le \ \frac{1}{l_{21}} - \frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}}$$

and $d_{21} > 0$ imply $d_{11} = 0$. Then $0 = m_{11} \ge m_{1n_1} \ge 0$ yields $n_1 = 1$ and $l_{11} = 1$. A contradiction to irredundance of P.

Case $\iota_0 \ge 2$ *and* $\iota_1 \le 1$. We may assume $\iota_0 = 2$. If $\iota_1 = 1$, then $l_{01} = 1$. Hence $m_{01} \in \mathbb{Z}$ and $n_0 \ge 2$. We derive $m_{1n_1} = m_{11} = d_{11} = 0$ and thus $l_{11} = 1$ from

$$-1 \ \le \ m_{0n_0} \ < \ m_{01} \ \le \ m^+ - m_{11} - m_{21} \ \le \ \frac{1}{l_{11}} - \frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}}.$$

A contradiction to irredundance of P. The case $\iota_1 = 0$ transforms to the case $\iota_0 = 1$ and $\iota_2 = 1$ settled before by switching sink and source.

Case $\iota_0 \ge 2$ *and* $\iota_1 \ge 2$. We may assume $\iota_0 = 2$ and $\iota_1 = 3$. Then $l_{01} = l_{11} = 1$ and $n_0, n_1 \ge 2$ hold. In particular, $d_{01} = m_{01} > m_{0n_0} \ge -1$. Moreover, d_{11}, d_{21} and d_{31} are all strictly positive. This contradicts to the estimate

$$d_{01} + d_{11} + \frac{d_{21}}{l_{21}} + \frac{d_{31}}{l_{31}} \ \le \ m^+ \ \le \ \frac{1}{l_{31}}.$$

$\square$

Example 8.15. We present $\mathbb{K}^*$-surfaces $X = X(A, P)$ with a smooth elliptic fixed point x^- and a positive parabolic fixed point curve D^+. Consider

$$P \ = \ \begin{bmatrix} -l_{01} & l_{11} & 0 & 0 & 0 \\ -l_{01} & 0 & l_{22} & 1 & 0 \\ d_{01} & d_{11} & 1 & 0 & 1 \end{bmatrix}.$$

Now choose the entries l_{ij} and d_{ij} in such a way that P is adapted to the sink and we have

$$l_{01}d_{11} + l_{11}d_{01} \;=\; -1, \qquad l_{21} \;>\; l_{01}l_{11}$$

Then x^- is smooth and D^+ has self intersection number $l_{21} - l_{01}l_{11}$. For example we can take

$$P \;=\; \begin{bmatrix} -2 & 3 & 0 & 0 & 0 \\ -2 & 0 & 7 & 1 & 0 \\ -1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Then x^- is smooth and we have $m^+ = -1/42$. Consequently, D^+ has self intersection number $-m^+ = 1/42$. Observe that D^+ is not gentle.

9. Horizontal and vertical P-roots

In section, we introduce horizontal and vertical P-roots as adapted versions of the general Demazure P-roots to the special case of rational projective $\mathbb{K}^*$-surfaces $X = X(A,P)$. This allows a less technical treatment. The main results of this section are Propositions 9.6, 9.17 and 9.18 showing geometric constraints to the existence of P-roots and Propositions 9.11, 9.15 which identify the P-roots in terms of the defining matrix P.

Definition 9.1. Consider a rational projective $\mathbb{K}^*$-surface $X = X(A,P)$ and assume that P is irredundant.

(i) Let $x^+ \in X$ be an elliptic fixed point and $0 \le i_0, i_1 \le r$. A *horizontal P-root* at (x^+, i_0, i_1) is a vector $u \in \mathbb{Z}^r \times \mathbb{Z}_{<0}$ such that

$$\langle u, v_{i_1 1}\rangle = -1, \quad \langle u, v_{i1}\rangle = 0,\, i \ne i_0, i_1, \quad l_{i1} = 1,\, i \ne i_0, i_1,$$

$$\langle u, v_{i_0 1}\rangle \ge 0,\ \langle u, v_{i_1 2}\rangle \ge 0,\ n_{i_1} > 1,\ \langle u, v_{i2}\rangle \ge l_{i2},\ i \ne i_0, i_1.$$

(ii) Let $x^- \in X$ be an elliptic fixed point and $0 \le i_0, i_1 \le r$. A *horizontal P-root* at (x^-, i_0, i_1) is a vector $u \in \mathbb{Z}^r \times \mathbb{Z}_{>0}$ such that

$$\langle u, v_{i_1 n_{i_1}}\rangle = -1, \quad \langle u, v_{i n_i}\rangle = 0,\, i \ne i_0, i_1, \quad l_{i n_i} = 1,\, i \ne i_0, i_1,$$

$$\langle u, v_{i_0 n_{i_0}}\rangle \ge 0,\ \langle u, v_{i_1 n_{i_1}-1}\rangle \ge 0,\ n_{i_1} > 1,\ \langle u, v_{i n_i -1}\rangle \ge l_{i n_i -1},\ i \ne i_0, i_1.$$

We say that an elliptic fixed point $x \in X$ *admits* a horizontal P-root if there is a vector u as in (i) if $x = x^+$, respectively a vector u as in (ii) if $x = x^-$.

Remark 9.2. Given $u = (u_1, \ldots, u_{r+1}) \in \mathbb{Q}^{r+1}$, set $u_0 := -u_1 - \ldots - u_r$. For $i = 0, \ldots, r$, the linear form u evaluates at the colums v_{ij} of a defining matrix P as

$$\langle u, v_{ij}\rangle \;=\; u_i l_{ij} + u_{r+1} d_{ij}.$$

This allows a unified treatment of the cases $i = 0$ and $i \ne 0$ and will be used frequently in the sequel.

The subsequent Propositions 9.3 and 9.4 together with Remark 9.5 give the precise relations between the horizontal P-roots just defined and the horizontal Demazure P-roots recalled in Definition 3.4.

Proposition 9.3. *Let $X = X(A,P)$ be non-toric, P irredundant and (u, i_0, i_1, C) a horizontal Demazure P-root. Then precisely one of the following statements holds:*

(i) *We have* $u_{r+1} < 0$*, there is an elliptic fixed point* $x^+ \in X$*, the vector* u *is a horizontal* P*-root at* (x^+, i_0, i_1) *and* $c_i = 1$ *holds for all* $i \neq i_0$*.*
(ii) *We have* $u_{r+1} > 0$*, there is an elliptic fixed point* $x^- \in X$*, the vector* u *is a horizontal* P*-root at* (x^-, i_0, i_1) *and* $c_i = n_i$ *holds for all* $i \neq i_0$*.*

Proof. We show $u_{r+1} \neq 0$. Otherwise, as X is non-toric, we find $0 \leq i \leq r$ different from i_0, i_1. Then $\langle u, v_{ic_i} \rangle = 0$ implies $u_i = 0$. Since P is irredundant and $l_{ic_i} = 1$ holds, there is a $1 \leq j \leq n_i$ different from c_i and we have $\langle u, v_{ij} \rangle \geq l_{ij} > 0$. This is impossible due to $u_i = u_{r+1} = 0$.

Next we claim $u_{r+1} m_{ic_i} \leq u_{r+1} m_{ij}$ holds for every $0 \leq i \leq r$ with $i \neq i_0$ and every $1 \leq j \leq n_i$. Indeed, we infer

$$u_i \;=\; \begin{cases} -u_{r+1} m_{ic_i}, & i \neq i_1, \\ -\frac{1}{l_{ic_i}} - u_{r+1} m_{ic_i}, & i = i_1, \end{cases} \qquad u_i l_{ic_i} + u_{r+1} d_{ic_i} \;\leq\; u_i l_{ij} + u_{r+1} d_{ij}$$

from the conditions on $\langle u, v_{ij} \rangle$ for $i \neq i_0$ stated in Definition 3.4. Eliminating u_i in the above inequalities then directly yields the claim.

We show that for $u_{r+1} < 0$, we arrive at (i). First, P has no column v^+ by Definition 3.4 and $u_{r+1} < 0$. Thus, there is an elliptic fixed point $x^+ \in X$. We have $m_{ic_i} \geq m_{ij}$ for $i \neq i_0$. Hence, slope orderedness of P forces $c_i = 1$ for all $i \neq i_0$. Clearly, u fulfills the conditions of a horizontal P-root at (x^+, i_0, i_1). Similarly, we see that $u_{r+1} > 0$ leads to (ii). □

Proposition 9.4. *Consider* $X = X(A, P)$ *and assume that* P *is irredundant. Define* $(r+1)$*-tuples* $C^+ := (1, \ldots, 1)$ *and* $C^- := (n_0, \ldots, n_r)$*.*

(i) *Let* $x^+ \in X$ *be an elliptic fixed point and* u *a horizontal* P*-root at* (x^+, i_0, i_1)*. Then* (u, i_0, i_1, C^+) *is a Demazure* P*-root.*
(ii) *Let* $x^- \in X$ *be an elliptic fixed point and* u *a horizontal* P*-root at* (x^-, i_0, i_1)*. Then* (u, i_0, i_1, C^-) *is a Demazure* P*-root.*

Proof. We exemplarily prove (i). Recall that here we have $u_{r+1} < 0$. Using the inequalities from Definition 9.1 (i) and slope orderedness of P, we obtain

$$\begin{aligned} \frac{u_{i_0}}{u_{r+1}} \;&\geq\; m_{i_0 1} \;\geq\; m_{i_0 j}, & j &= 1, \ldots, n_{i_0 1}, \\ -\frac{u_{i_1}}{u_{r+1}} \;&\geq\; m_{i_1 2} \;\geq\; m_{i_1 j}, & j &= 2, \ldots, n_{i_1 1}, \\ -\frac{u_i}{u_{r+1}} \;\geq\; m_{i2} - \frac{1}{u_{r+1}} \;&\geq\; m_{ij} - \frac{1}{u_{r+1}}, \quad i \neq i_0, i_1, & j &= 2, \ldots, n_i. \end{aligned}$$

Together with the two equations of Definition 9.1 (i), this directly leads to the conditions of a Demazure P-root for (u, i_0, i_1, C^+). □

Remark 9.5. Let $X = X(A, P)$ be non-toric and P irredundant. By Propositions 9.3 and 9.4, the horizontal Demazure P-roots map surjectively to the horizontal P-roots. Here κ and κ' have the same image if and only if $\kappa = (u, i_0, i_1, C)$ and $\kappa' = (u, i_0, i_1, C')$, where C and C' differ at most in the i_0-th entry. In this case, the locally nilpotent derivations δ_κ and $\delta_{\kappa'}$ on $\mathcal{R}(X) = R(A, P)$ coincide.

PROPOSITION 9.6. *Consider a rational projective $\mathbb{K}^*$-surface $X = X(A,P)$, assume P to be irredundant and let $0 \le i_0, i_1 \le r$.*

(i) *Let X have an elliptic fixed point x^+. If there is a horizontal P-root u at (x^+, i_0, i_1), then x^+ is simple, quasismooth with leading indices i_0, i_1 and*

$$0 \;<\; m^+ \;\le\; -u_{r+1}m^+ \;\le\; \frac{1}{l_{i_1 1}}.$$

Additionally, the presence of a horizontal P-root u at (x^+, i_0, i_1) forces $D_{i1}^2 \ge 0$ for all $i = 0, \ldots, r$ with $i \ne i_0$.

(ii) *Let X have an elliptic fixed point x^-. If there is a horizontal P-root at (x^-, i_0, i_1), then x^- is simple, quasismooth with leading indices i_0, i_1 and*

$$0 \;>\; m^- \;\ge\; u_{r+1}m^- \;\ge\; -\frac{1}{l_{i_1 n_{i_1}}}.$$

Additionally, the presence of a horizontal P-root u at (x^-, i_0, i_1) forces $D_{in_i}^2 \ge 0$ for all $i = 0, \ldots, r$ with $i \ne i_0$.

Moreover, there exists at most one elliptic fixed point in X admitting a horizontal P-root.

PROOF. The estimates can be treated at once, writing $x = x^+, x^-$. Proposition 6.10 tells us that x is quasismooth with leading indices i_0, i_1. Moreover, according to Proposition 9.4, the horizontal P-root u satisfies the assumptions of Proposition 8.10. Thus, x is simple and we obtain the desired estimates. For the self intersection numbers, we exemplarily look at x^-. For $n_i = 1$, the claim directly follows from Remark 6.4. For $n_i > 1$, assume $D_{in_i}^2 < 0$. Then we infer from Remark 6.6 that v_{in_i} is a positive combination over v_{in_i-1} and the v_{kn_k} with $k \ne i$. This contradicts to the definition of a horizontal P-root u at (x^-, i_0, i_1). The supplement is a consequence of Theorem 8.4. $\square$

Our next step is to identify the horizontal P-roots as certain integers contained in intervals $\Delta(\iota, \kappa) \subseteq \mathbb{Q}_{\ge 0}$, which in turn are extracted in the following way from the defining matrix P.

CONSTRUCTION 9.7. Consider the defining matrix P of $X(A,P)$. For $0 \le i, k \le r$, define rational numbers

$$\eta_k \;:=\; -\frac{1}{l_{kn_k}m^-}, \qquad\qquad \xi_i \;:=\; \begin{cases} 0, & n_i = 1, \\ \frac{1}{l_{in_i}(m_{in_i-1} - m_{in_i})}, & n_i \ge 2. \end{cases}$$

Then all ξ_i and η_k are non-negative. Moreover, for $0 \le i, k \le r$ with $i \ne k$, consider the sets

$$[\xi_i, \eta_k] \;=\; \{t \in \mathbb{Q};\; \xi_i \le t \le \eta_k\} \;\subseteq\; \mathbb{Q}_{\ge 0}.$$

Note that $[\xi_i, \eta_k]$ may be empty. Finally, for any two $0 \le \iota, \kappa \le r$, we have the intersections

$$\Delta(\iota, \kappa) \;=\; \bigcap_{i \ne \iota} [\xi_i, \eta_\kappa] \;\subseteq\; \mathbb{Q}_{\ge 0}.$$

Remark 9.8. Using Remark 6.4, we can express the length of the intervals $[\xi_i, \eta_k]$ from Construction 9.7 via intersection numbers:

$$\eta_k - \xi_i \;=\; l_{in_i} D_{in_i}^2 + (l_{in_i} - l_{kn_k}) D_{in_i} D_{kn_k}.$$

Moreover, for any two $0 \le \iota, \kappa \le r$, the (possibly empty) set $\Delta(\iota,\kappa)$ is explicitly given as

$$\Delta(\iota,\kappa) \;=\; \left[\max\left(0,\; \frac{1}{l_{in_i}(m_{in_i-1} - m_{in_i})},\ \text{where}\ \ i \ne \iota,\ n_i \ge 2\right),\ -\frac{1}{l_{\kappa n_\kappa} m^-}\right].$$

Definition 9.9. Consider the defining matrix P of $X(A,P)$ and let $0 \le i_0, i_1 \le r$. Set $e_0' := 0 \in \mathbb{Z}^{r+1}$ and $e_i' := e_i \in \mathbb{Z}^{r+1}$ for $i = 1, \ldots, r+1$. Given $\gamma \in \mathbb{Q}$, define

$$u(i_0,i_1,\gamma) \;:=\; \gamma e_{r+1}' - \frac{1}{l_{i_1 n_{i_1}}}(e_{i_1}' - e_{i_0}') - \gamma \sum_{i \ne i_0, r+1} m_{in_i}(e_i' - e_{i_0}') \;\in\; \mathbb{Q}^{r+1}.$$

Lemma 9.10. *Let $u := u(i_0,i_1,\gamma)$ be as in Definition 9.9. Then the evaluation of u at a column v_{ij} of the matrix P is given by*

$$\langle u, v_{i_0 j}\rangle \;=\; l_{i_0 j}\left(\gamma m_{i_0 j} + \frac{1}{l_{i_1 n_{i_1}}} + \gamma \sum_{i \ne i_0} m_{in_i}\right),$$

$$\langle u, v_{i_1 j}\rangle \;=\; l_{i_1 j}\left(\gamma m_{i_1 j} - \frac{1}{l_{i_1 n_{i_1}}} - \gamma m_{i_1 n_{i_1}}\right),$$

$$\langle u, v_{ij}\rangle \;=\; l_{ij}\left(\gamma m_{ij} - \gamma m_{in_i}\right), \quad i \ne i_0, i_1.$$

Proof. Set $e_0 := -e_1 - \ldots - e_r \in \mathbb{Z}^{r+1}$. Then the evaluations of $e_0' = 0 \in \mathbb{Z}^{r+1}$ and $e_i' = e_i \in \mathbb{Z}^{r+1}$, where $i = 0, \ldots, r+1$, at $e_0, e_1, \ldots, e_{r+1}$ are

$$\langle e_i', e_k\rangle \;=\; \begin{cases} 0, & i = 0, \\ -1, & 1 \le i \le r,\ k = 0, \\ 1, & 1 \le i \le r,\ k = i, \\ 0, & 1 \le i \le r,\ k \ne i, \end{cases} \qquad \langle e_{r+1}', e_k\rangle \;=\; \delta_{r+1\,k}.$$

Here, as usual, we define $\delta_{ik} := 1$ if $i = k$ and $\delta_{ik} := 0$ if $i \ne k$. Consequently, for all $0 \le i, k \le r$ with $i \ne i_0$ and $0 \le i, k \le r+1$ we obtain

$$\langle e_i' - e_{i_0}', e_k\rangle \;=\; \begin{cases} \delta_{ik}, & k \ne i_0, \\ -1, & k = i_0. \end{cases}$$

This enables us to verify the assertion by explicitly evaluating $u = u(i_0,i_1,\gamma)$ at the vectors $v_{ij} = l_{ij} e_i + d_{ij} e_{r+1}$. $\square$

Proposition 9.11. *Assume that $X = X(A,P)$ has an elliptic fixed point $x^- \in X$ and let $0 \le i_0, i_1 \le r$. Then we have mutually inverse bijections:*

$$\left\{ \begin{array}{l} \textit{horizontal } P\textit{-roots} \\ u \textit{ at } (x^-, i_0, i_1) \end{array} \right\} \;\longleftrightarrow\; \left\{ \begin{array}{l} \textit{integers } \gamma \in \Delta(i_0,i_1) \textit{ such} \\ \textit{that } \gamma d_{i_1 n_{i_1}} \equiv -1 \bmod l_{i_1 n_{i_1}} \end{array} \right\}$$

$$u \;\mapsto\; u_{r+1}$$

$$u(i_0,i_1,\gamma) \;\leftarrow\!\shortmid\; \gamma$$

Proof. Given a horizontal P-root u at (x^-, i_0, i_1), we use Lemma 9.10 to see $u = u(i_0, i_1, \gamma)$ for $\gamma := u_{r+1}$ by comparing the values of u and $u(i_0, i_1, \gamma)$ at the vectors v_{in_i} for $i \neq i_0$. Now consider $0 \le i_0, i_1 \le r$ and any vector $u \in \mathbb{Z}^r \times \mathbb{Z}_{>0}$. Then we have

$$\langle u, v_{i_0 n_{i_0}} \rangle \ge 0 \;\Leftrightarrow\; u_{r+1} \le \eta_{i_0}.$$

Moreover, if $n_{i_1} > 1$, then

$$\langle u, v_{i_1 n_{i_1}-1} \rangle \ge 0 \;\Leftrightarrow\; u_{r+1} \ge \xi_{i_1}.$$

Finally, if $i \neq i_0, i_1$, then

$$\langle u, v_{in_i-1} \rangle \ge l_{in_i-1} \;\Leftrightarrow\; u_{r+1} \ge \xi_i.$$

So, the inequalities of Definition 9.1 (ii) are satisfied if and only if $u_{r+1} \in \Delta(i_0, i_1)$ holds. Thus, if u is a horizontal P-root u at (x^-, i_0, i_1), then $u_{r+1} \in \Delta(i_0, i_1)$ and $u_{r+1} d_{i_1 n_{i_1}} \equiv -1 \bmod l_{i_1 n_{i_1}}$ holds due to

$$-1 \;=\; \langle u, v_{i_1 n_{i_1}} \rangle \;=\; u_{i_1} l_{i_1 n_{i_1}} + u_{r+1} d_{i_1 n_{i_1}}.$$

Conversely, given any $\gamma \in \Delta(i_0, i_1)$ with $\gamma d_{i_1 n_{i_1}} \equiv -1 \bmod l_{i_1 n_{i_1}}$, we directly verify that the vector $u(i_0, i_1, \gamma)$ is a horizontal P-root at (x^-, i_0, i_1) having γ as its last coordinate. □

Proposition 9.12. *Let $X = X(A, P)$ have an elliptic fixed point $x^- \in X$ and let u be a horizontal P-root at (x^-, i_0, i_1). Then $x^- \in X$ is quasismooth simple with leading indices i_0, i_1.*

(i) *If $l_{i_0 n_{i_0}} \le l_{i_1 n_{i_1}}$ holds, then x^- is smooth, we have $\langle u, v_{i_0 n_{i_0}} \rangle = 0$ and u is the only horizontal P-root at (x^-, i_0, i_1).*

(ii) *If the point $x^- \in X$ is singular, then $l_{i_0 n_{i_0}} > l_{i_1 n_{i_1}}$ holds and i_0 is the exceptional index of $x^- \in X$.*

Proof. Proposition 9.6 tells us that $x^- \in X$ is quasismooth simple with leading indices i_0, i_1. Moreover, the second assertion is an immediate consequence of the first one and Proposition 8.10. Thus, we only have to prove the first assertion.

Suppose that there are two distinct horizontal P-roots at (x^-, i_0, i_1). Then, by Proposition 9.11 they are given as $u(i_0, i_1, \gamma)$ and $u(i_0, i_1, \gamma')$ with positive integers $\gamma, \gamma' \in \Delta(i_0, i_1)$ differing by a non-zero integral mutiple of $l_{i_1 n_{i_1}}$. We conclude

$$\eta_{i_1} \;=\; -\frac{1}{l_{i_1 n_{i_1}} m^-} \;>\; l_{i_1 n_{i_1}} \;\ge\; l_{i_0 n_{i_0}}.$$

This implies $l^- m^- = l_{i_1 n_{i_1}} l_{i_0 n_{i_0}} m^- > -1$ which contradicts to Remark 6.3. Thus, there exists only one horizontal P-root $u = u(i_0, i_1, \gamma)$ at (x^-, i_0, i_1). We show $\langle u, v_{i_0 n_{i_0}} \rangle = 0$. Otherwise Lemma 9.10 yields

$$\langle u, v_{i_0 n_{i_0}} \rangle \;=\; l_{i_0 n_{i_0}} \left(\gamma m^- + \frac{1}{l_{i_1 n_{i_1}}} \right) \;\ge\; 1.$$

This implies

$$\gamma m^- \;\ge\; \frac{1}{l_{i_0 n_{i_0}}} - \frac{1}{l_{i_1 n_{i_1}}} \;\ge\; 0.$$

Again we arrive at a contradiction to Remark 6.3, telling us $m^- < 0$. Thus $\langle u, v_{i_0 n_{i_0}} \rangle = 0$ holds. According to Lemma 9.10 this forces

$$\gamma \;=\; -\frac{1}{l_{i_1 n_{i_1}} m^-} \;=\; -\frac{l_{i_0 n_{i_0}}}{l^- m^-}.$$

In particular, as γ is an integer, $l^- m^-$ divides $l_{i_0 n_{i_0}}$. Moreover, making use of $\gamma d_{i_1 n_{i_1}} \equiv -1 \mod l_{i_1 n_{i_1}}$, we obtain an integer

$$a \;:=\; \frac{1}{l_{i_1 n_{i_1}}} + \gamma m_{i_1 n_{i_1}} \;=\; \frac{m^- - m_{i_1 n_{i_1}}}{l_{i_1 n_{i_1}} m^-} \;=\; \frac{d_{i_0 n_{i_0}}}{l^- m^-} + \frac{l_{i_0 n_{i_0}}}{l^- m^-} \left(\sum_{i \neq i_0, i_1} d_{i n_i} \right).$$

Thus, $l^- m^-$ also divides $d_{i_0 n_{i_0}}$. Since $l_{i_0 n_{i_0}}$ and $d_{i_0 n_{i_0}}$ are coprime, we arrive at $l^- m^- = -1$. Proposition 6.10 yields that x^- is smooth. $\square$

Corollary 9.13. *Consider $X = X(A, P)$ with a smooth elliptic fixed point $x^- \in X$ and fix $0 \leq i_0, i_1 \leq r$ such that $l_{i n_i} = 1$ for all $i \neq i_0, i_1$. Let $\varepsilon \in \mathbb{Z}$ be maximal with*

$$l_{i_0 n_{i_0}} - \varepsilon l_{i_1 n_{i_1}} \;\geq\; \frac{1}{l_{i n_i}(m_{i n_i - 1} - m_{i n_i})} \quad \text{whenever } i \neq i_0 \text{ and } n_i \geq 2.$$

For every integer μ with $0 \leq \mu \leq \varepsilon$, set $u_\mu := u(i_0, i_1, l_{i_0 n_{i_0}} - \mu l_{i_1 n_{i_1}})$ according to Definition 9.9. Then the following holds.

(i) *For every $0 \leq \mu \leq \varepsilon$ the linear form u_μ is a horizontal P-root at (x^-, i_0, i_1) and we have $\langle u_\mu, v_{i_0} \rangle = \mu$.*

(ii) *There exist horizontal P-roots at (x^-, i_0, i_1) if and only if $\varepsilon \geq 0$ holds. Moreover, $u_0, \ldots, u_\varepsilon$ are the only horizontal P-roots at (x^-, i_0, i_1).*

(iii) *If u is a horizontal P-root at (x^-, i_0, i_1) then, for any two $0 \leq \mu \leq \alpha \leq \varepsilon$, we have $u_\mu = u_\alpha - (\alpha - \mu) u$.*

Proof. We check that $\gamma := l_{i_0 n_{i_0}} - \mu l_{i_1 n_{i_1}}$ is as in Proposition 9.11 By the definition of ε, we have $\gamma \geq \xi_i$ for all $i \neq i_0$. Moreover, $l_{i_0 n_{i_0}} l_{i_1 n_{i_1}} m^- = -1$ by smoothness of $x^- \in X$ and Proposition 6.10 (iv). Thus,

$$\gamma \;=\; l_{i_0 n_{i_0}} - \mu l_{i_1 n_{i_1}} \;\leq\; l_{i_0 n_{i_0}} \;\leq\; -\frac{1}{l_{i_1 n_{i_1}} m^-} \;=\; \eta_{i_1}.$$

Consequently, $\gamma \in \Delta(i_0, i_1)$. Finally, also $l_{i_0 n_{i_0}} d_{i_1 n_{i_1}} \equiv -1 \mod l_{i_1 n_{i_1}}$ holds due to Proposition 6.10 (iv). So, Proposition 9.11 shows that u_μ is a horizontal P-root at (x^-, i_0, i_1) and, in addition yields (ii). Lemma 9.10 gives us

$$\begin{aligned}
\langle u_\mu, v_{i_0 n_{i_0}} \rangle &= l_{i_0 n_{i_0}} \left((l_{i_0 n_{i_0}} - \mu l_{i_1 n_{i_1}}) \sum_{i=0}^{r} m_{i n_i} + \frac{1}{l_{i_1 n_{i_1}}} \right) \\
&= l_{i_0 n_{i_0}} \left((l_{i_0 n_{i_0}} - \mu l_{i_1 n_{i_1}}) \frac{-1}{l_{i_0 n_{i_0}} l_{i_1 n_{i_1}}} + \frac{1}{l_{i_1 n_{i_1}}} \right) \\
&= \mu.
\end{aligned}$$

Concerning (iii), there is only something to show for $\varepsilon \geq 1$. Then $l_{i_0 n_{i_0}} \geq l_{i_1 n_{i_1}}$ holds and Proposition 9.12 shows that u is only horizontal P-root at

(x^-, i_1, i_0). Assertions (i) and (ii) just verified yield $u = u(i_1, i_0, l_{i_1 n_{i_1}})$. Now the desired identity is directly checked via Definition 9.9:

$$\begin{aligned} u_\mu - u_\alpha &= u(i_0, i_1, l_{i_0 n_{i_0}} - \mu l_{i_1 n_{i_1}}) - u(i_0, i_1, l_{i_0 n_{i_0}} - \alpha l_{i_1 n_{i_1}}) \\ &= (\alpha - \mu) l_{i_1 n_{i_1}} e'_{r+1} - (\alpha - \mu) l_{i_1 n_{i_1}} \sum_{i \neq i_0, r+1} m_{i n_i} (e'_i - e'_{i_0}) \\ &= (\alpha - \mu) \left(l_{i_1 n_{i_1}} e'_{r+1} + l_{i_1 n_{i_1}} \sum_{i \neq r+1} m_{i n_i} (e'_i - e'_{i_0}) \right) \\ &= (\alpha - \mu)\, u. \end{aligned}$$

□

Definition 9.14. Consider a rational projective $\mathbb{K}^*$-surface $X = X(A, P)$ where P is irredundant.

(i) Assume that there is a parabolic fixed point curve $D^+ \subseteq X$. A *vertical P-root* at D^+ is a vector $u \in \mathbb{Z}^r \times \mathbb{Z}_{<0}$ such that

$$\langle u, v^+ \rangle \ = \ -1, \qquad \langle u, v_{i1} \rangle \ \geq \ 0, \quad i = 0, \ldots, r.$$

(ii) Assume that there is a parabolic fixed point curve $D^- \subseteq X$. A *vertical P-root* at D^- is a vector $u \in \mathbb{Z}^r \times \mathbb{Z}_{>0}$ such that

$$\langle u, v^- \rangle \ = \ -1, \qquad \langle u, v_{i n_i} \rangle \ \geq \ 0, \quad i = 0, \ldots, r.$$

Proposition 9.15. *Consider a rational projective $\mathbb{K}^*$-surface $X(A, P)$, assume P to be irredundant and let $u \in \mathbb{Z}^{r+1}$.*

(i) *If there is a curve $D^+ \subseteq X$, then the following statements are equivalent:*
   (a) *u is is a vertical P-root at D^+,*
   (b) *$u_{r+1} = -1$ and $u_i \geq m_{i1}$ for all $i = 0, \ldots, r$,*
   (c) *$u_{r+1} = -1$ and $u_i \geq m_{ij}$ for all $i = 0, \ldots, r$, $j = 1, \ldots, n_i$.*
   If one of the statements (a), (b) or (c) holds, then we have $(D^+)^2 \geq 0$.

(ii) *If there is a curve $D^- \subseteq X$, then the following statements are equivalent:*
   (a) *u is is a vertical P-root at D^-,*
   (b) *$u_{r+1} = 1$ and $u_i \leq m_{i n_i}$ for all $i = 0, \ldots, r$,*
   (c) *$u_{r+1} = 1$ and $u_i \leq m_{ij}$ for all $i = 0, \ldots r$, $j = 1, \ldots, n_i$.*
   If one of the statements (a), (b) or (c) holds, then we have $(D^-)^2 \geq 0$.

In particular, $(u, k) \mapsto u$ defines a one-to-one correspondence between the vertical Demazure P-roots and the vertical P-roots.

Proof. In each of the items, the equivalence of (a) and (b) is clear by Remark 9.2 and the equivalence of (b) and (c) holds due to slope orderedness. The assertions on the self intersection numbers are clear by the definition of vertical P-roots and Remarks 6.4 and 6.5. □

Corollary 9.16. *Let $X(A, P)$ be a $\mathbb{K}^*$-surface, assume P to be irredundant, let $u \in \mathbb{Z}^{r+1}$ and fix $0 \leq i_0 \leq r$.*

(i) *Assume that there is a curve $D^+ \subseteq X$. Then u is a vertical P-root at D^+ if and only if*

$$u_i \ \geq \ m_{i1} \text{ for all } i \neq i_0, r+1, \qquad \sum_{i \neq i_0, r+1} u_i \ \leq \ -m_{i_0 1}.$$

(ii) *Assume that there is a curve $D^- \subseteq X$. Then u is a vertical P-root at D^- if and only if*

$$u_i \ \geq \ -m_{in_i} \text{ for all } i \neq i_0, r+1, \qquad \sum_{i \neq i_0, r+1} u_i \ \leq \ m_{i_0 n_{i_0}}.$$

Proposition 9.17. *Let $X = X(A,P)$ be non-toric and P irredundant. If there is a quasismooth simple elliptic fixed point $x \in X$, then there are no vertical P-roots.*

Proof. We may assume $x = x^-$ having leading indices $0, 1$, exceptional index 0 and that P is adapted to the sink. Suppose that $D^+ \subseteq X$ admits a vertical P-root $u \in \mathbb{Z}^{r+1}$. Then Proposition 9.15 yields

$$u_i \geq m_{i1} \text{ for } 1 \leq i \leq r, \qquad -u_0 = u_1 + \ldots + u_r \leq -m_{01}.$$

For $i = 2, \ldots, r$, we have $l_{in_i} = 1$ and thus $m_{in_i} = 0$, as P is adapted to the sink. Irredundance of P implies $m_{i1} > 0$ and hence $u_i \geq 1$ for $i = 2, \ldots, r$. Using Proposition 8.10 (ii) for the inequality, we obtain

$$m_{11} + (r-1) \ \leq \ u_1 + \ldots + u_r \ \leq \ -m_{01} \ \leq \ -m_{0n_0} \ \leq \ m_{1n_1} + \frac{1}{l_{1n_1}}.$$

We claim $r \leq 1$. For $l_{1n_1} \geq 2$ this follows from $m_{11} \geq m_{1n_1}$. If $l_{1n_1} = 1$, then $m_{1n_1} = 0$, hence $m_{11} > 0$ by irredundance and the claim follows. Now, $r \leq 1$ means that X is toric, which contradicts to our assumptions. □

Proposition 9.18. *Let $X = X(A,P)$ with P irredundant and assume that X has fixed point curves D^+ and D^-. If D^+ admits a vertical P-root, then there is no vertical P-root at D^-.*

Proof. We may assume that P is adapted to the source. Let $u^+ \in \mathbb{Z}^{r+1}$ be a vertical root at D^+. Proposition 9.15 yields

$$u_i^+ \geq m_{i1} > -1 \text{ for } 1 \leq i \leq r, \qquad u_0^+ = -u_1^+ - \ldots - u_r^+ \geq m_{01}.$$

We conclude $u_i^+ \geq 0$ for $i = 1, \ldots, r$ and hence $m_{01} \leq 0$. Now suppose that there is a vertical P-root u^- at D^-. Then

$$u_i^- \geq -m_{in_i} \geq -m_{i1} \geq 0 \text{ for } 1 \leq i \leq r, \qquad 0 \leq -u_0^- \leq m_{0n_0} \leq m_{01} \leq 0.$$

Consequently $m_{01} = m_{0n_0} = 0$, which in turn implies $n_0 = 1$ and $l_{01} = 1$. This contradicts to the assumption that P is irredundant. □

10. Generating root groups

In this section, we provide suitable generators for the unipotent part of the automorphism group of a non-toric rational projective $\mathbb{K}^*$-surface.

Definition 10.1. Consider $X = X(A,P)$. We denote by $U(X) \subseteq \mathrm{Aut}(X)$ the subgroup generated by all root groups of X.

Note that $U(X) \subseteq \mathrm{Aut}(X)^0$ holds. We have two cases. The first one is that $U(X)$ is generated by the root groups stemming from horizontal P-roots. In this situation, we prove the following.

Proposition 10.2. *Let $X = X(A,P)$ be non-toric with horizontal P-roots. Then there are a quasismooth simple elliptic fixed point $x \in X$ and $0 \le i_0, i_1 \le r$ such that $U(X)$ is generated by the root groups arising from horizontal P-roots at (x, i_0, i_1) or (x, i_1, i_0).*

According to Proposition 9.17, the remaining case is that $U(X)$ is generated by the root groups given by the vertical roots. Here we obtain the following.

Proposition 10.3. *Let $X = X(A,P)$ be non-toric with vertical P-roots at D^+ and let $0 \le i_0, i_1 \le r$. Then $U(X)$ is generated by the root groups arising from vertical P-roots u at D^+ with $0 \le \langle u, v_{i1}\rangle < l_{i1}$ for all $0 \le i \le r$ different from i_0, i_1.*

We begin with discussing the horizontal case. First, we summarize the necessary background. By Proposition 9.11, all horizontal P-roots at x^- are of the form $u(i_0, i_1, \gamma)$. According to Proposition 9.4, each such $u(i_0, i_1, \gamma)$ defines a Demazure P-root in the sense of Definition 3.4:

$$\tau(i_0, i_1, \gamma) \ := \ (u, i_0, i_1, C^-), \quad u \ := \ u(i_0, i_1, \gamma), \quad C^- \ := \ (n_0, \ldots, n_r).$$

Construction 3.6 associates with $\tau(i_0, i_1, \gamma)$ a locally nilpotent derivation on $R(A,P)$ which in turn gives rise to a root group

$$\lambda_{\tau(i_0,i_1,\gamma)} \colon \mathbb{K} \ \to \ \mathrm{Aut}(X).$$

Our statement involves the unique vectors $\beta = \beta(A, i_0, i_1)$ in the row space of the defining matrix A having i_0-th coordinate zero and i_1-th coordinate one as introduced in Construction 3.6. Moreover, the following will be frequently used.

Definition 10.4. For the defining matrix P of $X(A,P)$, we denote by $I(P) \subseteq \{0, \ldots, r\}$ the set of all indices i with $l_{in_i} = 1$.

Proposition 10.5. *Let $X = X(A,P)$ be non-toric with an elliptic fixed point $x^- \in X$. Then we obtain the following relations among the root subgroups associated with horizontal P-roots at x^-.*

(i) *Let $i_1, \iota_1 \in I(P)$ and $0 \le i_0 \le r$. If there are horizontal P-roots $u(i_0, i_1, \gamma)$ and $u(i_0, \iota_1, \gamma)$, then, for every $s \in \mathbb{K}$, we have*

$$\lambda_{\tau(i_0,i_1,\gamma)}(s) \ = \ \lambda_{\tau(i_0,\iota_1,\gamma)}(\beta(A, i_0, \iota_1)_{i_1}^{-1} s).$$

(ii) *Let $i_0, \iota_0 \in I(P)$ and $0 \le i_1 \le r$. If there are horizontal P-roots $u(i_0, i_1, 1)$, $u(\iota_0, i_1, 1)$ and $u(i_1, \iota_0, \nu)$, $\nu = 1, \ldots, l_{i_1 n_{i_1}}$, then, for every $s \in \mathbb{K}$, we have*

$$\lambda_{\tau(i_0,i_1,1)}(s) \ = \ \lambda_{\tau(\iota_0,i_1,1)}(s) \prod_{\nu=1}^{l_{i_1 n_{i_1}}} \lambda_{\tau(i_1,\iota_0,\nu)}\left(\beta(A, i_0, i_1)_{\iota_0} \binom{l_{i_1 n_{i_1}}}{\nu} s^\nu\right).$$

LEMMA 10.6. *Consider the defining matrix A of $X = X(A,P)$. Then the vectors $\beta \in \mathbb{K}^{r+1}$ introduced in Construction 3.6 satisfy*

$$\begin{aligned}\beta(A,i_0,i_1) &= \beta(A,i_0,i_1)_{i_1}^{-1}\beta(A,i_0,\iota_1),\\ \beta(A,i_1,\iota_0) &= \beta(A,i_0,i_1)_{\iota_0}^{-1}\left(\beta(A,i_0,i_1)-\beta(A,\iota_0,i_1)\right).\end{aligned}$$

PROOF. The identities follow from the fact that $\beta(A,i_0,i_1)$ is the unique vector in the row space of A having i_0-th coordinate zero and i_1-th coordinate one. □

LEMMA 10.7. *Consider the defining matrix P of $X(A,P)$ and the linear forms $u(i_0,i_1,\gamma)$ from Definition 9.9. Define*

$$u(i_0,i_1,\gamma)_{\nu,\iota} \ := \ \nu u + e'_{i_1} - e'_{\iota} \ \in \mathbb{Z}^{r+1}, \qquad \nu = 1,\ldots,l_{i_1 n_{i_1}}, \quad \iota \neq i_0, i_1,$$

as we did in Construction 5.1 in the case of Demazure P-roots. Then, for any two indices $i_1,\iota_1 \in I(P)$, we have

$$u(i_0,i_1,\gamma)_{1,\iota_1} \ = \ u(i_0,\iota_1,\gamma).$$

Moreover, if $l^- m^- = -1$ and there is an i_1 with $\iota \in I(P)$ for all $\iota \neq i_1$, then, for any two $i_0,\iota_1 \in I(P)$, we have

$$u(i_0,i_1,1)_{\nu,\iota_1} \ = \ u(i_1,\iota_1,\nu).$$

PROOF. For the first identity, observe that we have $\nu = 1$. Now, using the definition of $u(i_0,i_1,\gamma)$ and $l_{i_1 n_{i_1}} = l_{\iota_1 n_{\iota_1}} = 1$, we compute

$$\begin{aligned}u(i_0,i_1,\gamma)_{1,\iota_1} &= \gamma e'_{r+1} - (e'_{i_1} - e'_{i_0}) - \sum_{i\neq i_0} \gamma m_{i n_i}(e'_i - e'_{i_0}) + e'_{i_1} - e'_{\iota_1}\\ &= \gamma e'_{r+1} - (e'_{\iota_1} - e'_{i_0}) - \sum_{i\neq i_0} \gamma m_{i n_i}(e'_i - e'_{i_0})\\ &= u(i_0,\iota_1,\gamma).\end{aligned}$$

We prove the second identity. Due to the assumptions, we have $l_{i_1 n_{i_1}}^{-1} = -m^-$. Consequently, we obtain

$$\begin{aligned}u(i_0,i_1,1)_{\nu,\iota_1} &= \nu\left(e'_{r+1} + m^-(e'_{i_1} - e'_{i_0}) - \sum_{i\neq i_0} m_{i n_i}(e'_i - e'_{i_0})\right) + e_{i_1} - e_{\iota_1}\\ &= \nu e'_{r+1} - (e'_{\iota_1} - e'_{i_1}) - \sum_{i\neq i_1} \nu m_{i n_i}(e'_i - e'_{i_1})\\ &= u(i_1,\iota_1,\nu).\end{aligned}$$

□

LEMMA 10.8. *Consider the defining matrix P of $X(A,P)$, let $1 \leq i_0,i_1,\iota_1 \leq r$ with $i_1,\iota_1 \in I(P)$ and set $C^- := (n_0,\ldots,n_r)$. Then the monomials h^u and h^ζ from Construction 3.6 satisfy*

$$\frac{h^{u(i_0,i_1,\gamma)}}{h^{\zeta(i_0,i_1,C^-)}} \ = \ \frac{h^{u(i_0,\iota_1,\gamma)}}{h^{\zeta(i_0,\iota_1,C^-)}}.$$

Proof. By Lemma 10.7, the linear form $u(i_0, i_1, \gamma)$ equals $u(i_0, \iota_1, \gamma)_{1,\iota_1}$. From Construction 5.1 we infer how the latter evaluates and conclude

$$\frac{h^{u(i_0,i_1,\gamma)}}{h^{u(i_0,\iota_1,\gamma)}} = \frac{T_{\iota_1}^{l_{\iota_1}}}{T_{i_1}^{l_{i_1}}} = \frac{\prod_{\iota\neq i_0,i_1,\iota_1} T_\iota^{l_\iota} \prod_{i\neq i_0} T_{in_i}^{-1} T_{\iota_1}^{l_{\iota_1}}}{\prod_{\iota\neq i_0,i_1,\iota_1} T_\iota^{l_\iota} \prod_{i\neq i_0} T_{in_i}^{-1} T_{i_1}^{l_{i_1}}} = \frac{h^{\zeta(i_0,i_1,C^-)}}{h^{\zeta(i_0,\iota_1,C^-)}}.$$

□

Proof of Proposition 10.5. We prove (i). It suffices to verify the corresponding relation for the locally nilpotent derivations associated with $\tau(i_0, i_1, \gamma)$ and $\tau(i_0, \iota_1, \gamma)$; see Construction 3.6. Lemmas 10.6 and 10.8 yield

$$\beta(A, i_0, \iota_1)_{i_1} \delta_{\tau(i_0,i_1,\gamma)} = \delta_{\tau(i_0,\iota_1,\gamma)}.$$

We turn to (ii). First consider the map $\varphi_{u(i_0,i_1,1)}(s)$ as given in Theorem 5.4. For the $\alpha(s, \nu, \iota)$ defined there, we write

$$\alpha_{i_0,i_1,\iota} := \alpha_{i_0,i_1,\iota}(\nu, s) := \beta(A, i_0, i_1)_\iota \binom{l_{i_1 n_{i_1}}}{\nu} s^\nu,$$

which allows to specify in the case of varying i_0 and i_1. Now, using Lemma 10.7 and $\iota, i_1 \in I(P)$, we obtain

$$\begin{aligned}
\varphi_{u(i_0,i_1,1)}(s) &= \prod_{\iota\neq i_0,i_1} \prod_{\nu=1}^{l_{i_1 n_{i_1}}} \lambda_{u(i_0,i_1,1)_{\nu,\iota}}(\alpha_{i_0,i_1,\iota}) \\
&= \prod_{\nu=1}^{l_{i_1 n_{i_1}}} \prod_{\iota\neq i_0,i_1} \lambda_{u(i_1,\iota,\nu)}(\alpha_{i_0,i_1,\iota}), \\
\varphi_{u(\iota_0,i_1,1)}(s)^{-1} &= \prod_{\nu=1}^{l_{i_1 n_{i_1}}} \prod_{\iota\neq \iota_0,i_1} \lambda_{u(i_1,\iota,\nu)}(-\alpha_{\iota_0,i_1,\iota}),
\end{aligned}$$

where for the last equation, we used $\varphi_{u(1,\iota_0,i_1)}(s)^{-1} = \varphi_{u(1,\iota_0,i_1)}(-s)$. Next we observe

$$\alpha_{i_0,i_1,i_0} = 0, \qquad \alpha_{i_0,i_1,\iota} - \alpha_{\iota_0,i_1,\iota} = \beta(A, i_1, \iota_0)_\iota \beta(A, i_0, i_1)_{\iota_0} \binom{l_{i_1 n_{i_1}}}{\nu} s^\nu,$$

where the second identity follows from Lemma 10.6. With the aid of these considerations, we compute

$$
\begin{aligned}
\psi \ := \ & \varphi_{u(\iota_0,i_1,1)}(s)^{-1}\varphi_{u(i_0,i_1,1)}(s) \\
= \ & \prod_{\nu=1}^{l_{i_1 n_{i_1}}} \lambda_{u(i_1,\iota_0,\nu)}(\alpha_{i_0,i_1,\iota_0})\lambda_{u(i_1,i_0,\nu)}(-\alpha_{\iota_0,i_1,i_0}) \\
& \prod_{\iota\neq i_0,\iota_0,i_1} \lambda_{u(i_1,\iota,\nu)}(\alpha_{i_0,i_1,\iota}-\alpha_{\iota_0,i_1,\iota}) \\
= \ & \prod_{\nu=1}^{l_{i_1 n_{i_1}}} \lambda_{u(i_1,\iota_0,\nu)}(\alpha_{i_0,i_1,\iota_0}) \\
& \prod_{\iota\neq \iota_0,i_1} \lambda_{u(i_1,\iota,\nu)}(\alpha_{i_0,i_1,\iota}-\alpha_{\iota_0,i_1,\iota}) \\
= \ & \prod_{\nu=1}^{l_{i_1 n_{i_1}}} \lambda_{u(i_1,\iota_0,\nu)}\left(\beta(A,i_0,i_1)_{\iota_0}\binom{l_{i_1 n_{i_1}}}{\nu}s^{\nu}\right) \\
& \prod_{\iota\neq \iota_0,i_1} \lambda_{u(i_1,\iota_0,\nu)_{1,\iota}}\left(\beta(A,i_1,\iota_0)_{\iota}\beta(A,i_0,i_1)_{\iota_0}\binom{l_{i_1 n_{i_1}}}{\nu}s^{\nu}\right) \\
= \ & \prod_{\nu=1}^{l_{i_1 n_{i_1}}} \lambda_{u(i_1,\iota_0,\nu)}\left(\beta(A,i_0,i_1)_{\iota_0}\binom{l_{i_1 n_{i_1}}}{\nu}s^{\nu}\right) \\
& \prod_{\iota\neq \iota_0,i_1} \lambda_{u(i_1,\iota_0,\nu)_{1,\iota}}\left(\alpha\left(\beta(A,i_1,\iota_0)_{\iota}\binom{l_{i_1 n_{i_1}}}{\nu}s^{\nu},1,\iota\right)\right) \\
= \ & \prod_{\nu=1}^{l_{i_1 n_{i_1}}} \lambda_{\tau(i_1,\iota_0,\nu)}\left(\beta(A,i_0,i_1)_{\iota_0}\binom{l_{i_1 n_{i_1}}}{\nu}s^{\nu}\right).
\end{aligned}
$$

Note that we used Theorem 5.4 for the last equality. Thus, the right hand side of our equation is given as $\lambda_{\tau(\iota_0,i_1,1)}(s)\circ\psi$. Using Theorem 5.4 we obtain

$$
\begin{aligned}
\lambda_{\tau(i_0,i_1,1)}(s) \ &= \ \lambda_{u(\iota_0,i_1,1)}(s)\varphi_{u(i_0,i_1,1)}(s) \\
&= \ \lambda_{u(\iota_0,i_1,1)}(s)\varphi_{u(\iota_0,i_1,1)}(s)\psi \\
&= \ \lambda_{\tau(\iota_0,i_1,1)}(s)\psi.
\end{aligned}
$$

□

Definition 10.9. Provided that the defining matrix P of $X = X(A,P)$ is adapted to the sink in the sense of Definition 7.1 (ii), we say that P is *normalized* if $l_{0n_0} \geq \ldots \geq l_{rn_r}$ and for all $i < j$ with $l_{in_i} = l_{jn_j}$ and $n_i, n_j \geq 2$, we have

$$m_{in_i-1} - m_{in_i} \ \leq \ m_{jn_j-1} - m_{jn_j}.$$

Remark 10.10. The above definition of normalized coincides with the one given in the introduction as Remark 6.4 ensures

$$m_{in_i-1} - m_{in_i} \leq m_{jn_j-1} - m_{jn_j} \ \Leftrightarrow \ D_{in_i}^2 \leq D_{jn_j}^2.$$

LEMMA 10.11. *Let the defining matrix P of $X = X(A,P)$ be adapted to the sink and normalized. Consider the intervals $[\xi_i, \eta_k]$ and $\Delta(\iota,\kappa)$ from Construction 9.7.*

(i) *If $i, \iota \in I(P)$ satisfy $i \le \iota$, then we have*
$$[\xi_i, \eta_k] \ \subseteq \ [\xi_\iota, \eta_k], \qquad \Delta(\iota, k) \ \subseteq \ \Delta(i,k).$$
(ii) *For any two $k, \kappa \in I(P)$ and every $i = 0, \ldots, r$, we have*
$$[\xi_i, \eta_k] \ = \ [\xi_i, \eta_\kappa], \qquad \Delta(i, k) \ = \ \Delta(i,\kappa).$$
(iii) *Assume $l^- m^- = -1$. Let i with $\iota \in I(P)$ for all $\iota \ne i$ and $k \ge 2$. Then*
$$1 \in \Delta(k,i) \quad \Rightarrow \quad \Delta(i,k) \cap \mathbb{Z} \ = \ [1, l_{in_i}] \cap \mathbb{Z}.$$

PROOF. Irredundance of P implies $n_i \ge 2$ for all $i \in I(P)$. Consider $\iota, i, k \in I(P)$ with $i \le \iota$. Then Construction 9.7 and the fact that P is normalized yield
$$\xi_\iota \ = \ \frac{1}{m_{\iota n_\iota - 1} - m_{\iota n_\iota}} \ \le \ \frac{1}{m_{i n_i - 1} - m_{i n_\iota}} \ = \ \xi_i, \qquad \eta_k \ = \ -\frac{1}{l_{k n_k} m^-}.$$
This gives the first assertion. The second one is obvious. For the third one, observe $l_{1n_1} = \ldots = l_{rn_r} = 1$. Thus $l^- = l_{0n_0}$ and $l_{0n_0} m^- = -1$. We conclude
$$\eta_\kappa \ = \ -\frac{1}{l_{\iota n_\kappa} m^-} \ = \ \frac{l_{0n_0}}{l_{\kappa n_\kappa}} \ = \ \begin{cases} l_{0n_0}, & \kappa \ge 1, \\ 1, & \kappa = 0. \end{cases}$$
Now, let $1 \in \Delta(k,i)$. Then, by the definition of $\Delta(k,i)$, we have $\xi_\iota \le 1$ for all $\iota \ne k$. Moreover, there is a $\kappa \in \{0,1\} \cap I(P)$ with $\kappa \ne i$. We claim
$$\Delta(i,k) \cap \mathbb{Z} \ = \ \Delta(i,\kappa) \cap \mathbb{Z} \ = \ \bigcap_{\iota \ne i} [\xi_\iota, \eta_\kappa] \cap \mathbb{Z} \ = \ [1, \eta_\kappa] \cap \mathbb{Z} \ = \ [1, l_{in_i}] \cap \mathbb{Z}.$$
The first equality is due to (ii). The second one holds by definition. For the third one, use $k \ge 2$ to see $\xi_k \le \xi_\kappa \le 1$. For the last equality, use $l_{in_i} = 1$ for $\kappa = 0$ and
$$\kappa = 1, \, i = 0 \ \Rightarrow \ l_{in_i} = l_{0n_0}, \qquad\qquad \kappa = 1, \, i \ge 2 \ \Rightarrow \ l_{in_i} = 1 = l_{0n_0}.$$
□

PROOF OF PROPOSITION 10.2. Proposition 9.6 tells us that there is a unique elliptic fixed point $x \in X$ admitting horizontal P-roots. We may assume that $x = x^-$ holds and that P is adapted to the sink and normalized. We claim that then $i_0 = 0$ and $i_1 = 1$ are as wanted. So, given any horizontal P-root $u(\iota_0, \iota_1, \gamma)$ at (x, ι_0, ι_1), the task is to show that the associated root group maps into the subgroup generated by all the root subgroups arising from horizontal P-roots at $(x, 0, 1)$ and $(x, 1, 0)$.

Let $\iota_0, \iota_1 \ne 0$. Then we have $l_{0n_0} = \ldots = l_{rn_r} = 1$. Proposition 9.12 (i) says that x^- is smooth and $\langle u, v_{\iota_0 n_{\iota_0}} \rangle = 0$ holds. Thus, Proposition 6.10 yields $m^- = -1$ and Lemma 9.10 shows $\gamma = 1$. By Lemma 10.11 (ii) we have $1 \in \Delta(\iota_0, \iota_1) = \Delta(\iota_0, 0)$. Hence, there is a horizontal P-root $u(\iota_0, 0, 1)$. Proposition 10.5 (i) implies
$$\lambda_{\tau(\iota_0, \iota_1, 1)}(\mathbb{K}) \ = \ \lambda_{\tau(\iota_0, 0, 1)}(\mathbb{K}).$$

Let $\iota_0 = 0$ *and* $\iota_1 \neq 1$. Then we have $l_{1n_1} = l_{\iota_1 n_{\iota_1}} = 1$. Moreover, using Lemma 10.11 (i), we see $\gamma \in \Delta(0, \iota_1) = \Delta(0, 1)$. Thus, Proposition 10.5 (i) applies and we obtain

$$\lambda_{\tau(0,\iota_1,\gamma)}(\mathbb{K}) \;=\; \lambda_{\tau(0,1,\gamma)}(\mathbb{K}).$$

Let $\iota_1 = 0$ *and* $\iota_0 \neq 1$. Then we have $l_{1n_1} = 1$. Proposition 9.12 (i) says that x^- is smooth and that $\langle u, v_{\iota_0 n_{\iota_0}} \rangle = 0$ holds. According to Lemma 9.10, the latter means

$$0 \;=\; \gamma m^- + \frac{1}{l_{\iota_1 n_{\iota_1}}} \;=\; \gamma m^- + \frac{1}{l_{0n_0}} \;=\; -\frac{\gamma}{l_{0n_0}} + \frac{1}{l_{0n_0}}.$$

where the last equality is due to Proposition 6.10, showing $l_{0n_0} m^- = l^- m^- = -1$. We conclude $\gamma = 1$. Proposition 9.11 gives $1 \in \Delta(\iota_0, 0)$. Now, Lemma 10.11 (i) shows that $1 \in \Delta(\iota_0, 0) = \Delta(1, 0)$, hence there is a horizontal P-root $u(1, 0, 1)$. Furthermore, Lemma 10.11 (iii) shows

$$\Delta(0, 1) \cap \mathbb{Z} \;=\; [1, l_{0n_0}] \cap \mathbb{Z}.$$

Consequently, there is a a horizontal P-root $u(0, 1, \nu)$ for every $1 \leq \nu \leq l_{0n_0}$. Now Proposition 10.5 tells us

$$\lambda_{\tau(\iota_0,0,1)}(\mathbb{K}) \;\subseteq\; \lambda_{\tau(1,0,1)}(\mathbb{K}) \prod_{\nu=1}^{l_{0n_0}} \lambda_{\tau(0,1,\nu)}(\mathbb{K}).$$

$\square$

We enter the vertical case. According to Proposition 9.15, every vertical P-root u corresponds to a vertical Demazure P-root $\kappa = (u, i)$ and via the associated locally nilpotent derivation of $R(A, P)$ we obtain the root group

$$\lambda_\kappa = \lambda_u \colon \mathbb{K} \;\to\; \mathrm{Aut}(X).$$

Lemma 10.12. *Let* $A \in \mathrm{Mat}(2, r+1; \mathbb{K})$ *and* g_{i_1,i_2,i_3} *be as in Construction 2.4 and* $\beta(i_1, i_2, A)$, $\beta(i_2, i_1, A)$ *as in Construction 3.6. Then there is a* $b_{i_1,i_2,i_3} \in \mathbb{K}^*$ *with*

$$b_{i_1,i_2,i_3} g_{i_1,i_2,i_3} \;=\; T_{i_3}^{l_{i_3}} - \beta(i_1, i_2, A)_{i_3} T_{i_1}^{l_{i_1}} - \beta(i_2, i_1, A)_{i_3} T_{i_2}^{l_{i_2}}.$$

Proof. Consider $A' = [a_{i_1}, a_{i_2}, a_{i_3}] \in \mathrm{Mat}(2, 3, \mathbb{K})$. As a direct computation shows, we have

$$B \cdot A' \;=\; \begin{bmatrix} 1 & 0 & \beta(A, i_2, i_1)_{i_3} \\ 0 & 1 & \beta(A, i_1, i_2)_{i_3} \end{bmatrix}$$

with a unique matrix $B \in \mathrm{GL}(2, \mathbb{K})$. Setting $b_{i_1,i_2,i_3} := \det(B)^{-1}$, we infer the assertion from

$$g_{i_1,i_2,i_3} \;=\; b_{i_1,i_2,i_3} \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_2}^{l_{i_2}} \\ 1 & 0 & \beta(A, i_2, i_1)_{i_3} \\ 0 & 1 & \beta(A, i_1, i_2)_{i_3} \end{bmatrix}.$$

$\square$

LEMMA 10.13. *Consider an $X = X(A,P)$ with a curve $D^+ \subseteq X$. Fix $u \in \mathbb{Z}^{r+1}$ and $0 \le \iota \le r$. For any $0 \le i_0 \le r$ with $i_0 \ne \iota$ set*

$$u_{\iota,i_0} \ := \ u + e'_{i_0} - e'_\iota, \qquad e'_0 \ := \ 0, \qquad e'_i \ := \ e_i, \quad i = 0, \ldots, r.$$

Now assume that u is a vertical P-root at D^+ with $\langle u, v_{\iota 1}\rangle \ge l_{\iota 1}$ and let $0 \le i_0, i_1 \le r$ with $i_0 \ne i_1$. Then u_{ι,i_0} and u_{ι,i_1} are vertical P-roots at D^+ and we have

$$\lambda_u(s) \ = \ \lambda_{u_{\iota,i_0}}(\beta(A,i_1,i_0)_\iota s)\lambda_{u_{\iota,i_1}}(\beta(A,i_0,i_1)_\iota s).$$

PROOF. First let us see how u_{ι,i_0} evaluates on the vectors v_{ij}, v^+ and v^-, if present. We have

$$\langle u_{\iota,i_0}, v_{ij}\rangle \ = \ \begin{cases} \langle u, v_{ij}\rangle, & i \ne \iota, i_0, \\ \langle u, v_{\iota j}\rangle - l_{\iota j}, & i = \iota, \\ \langle u, v_{i_0 j}\rangle + l_{i_0 j}, & i = i_0, \end{cases} \qquad \langle u_{\iota,i_0}, v^\pm\rangle \ = \ \mp 1.$$

In particular, we see that u_{ι,i_0} is a vertical P-root at D^+: using Remark 9.2 and slope-orderedness of P, we infer $\langle u_{\iota,i_0}, v_{\iota j}\rangle \ge 0$ for any $j = 1, \ldots, n_\iota$ from

$$\langle u, v_{\iota 1}\rangle \ge l_{\iota 1} \ \Rightarrow \ u_\iota \ge 1 + m_{\iota 1} \ \Rightarrow \ u_\iota \ge 1 + m_{\iota j} \ \Rightarrow \ \langle u, v_{\iota j}\rangle \ge l_{\iota j}.$$

Moreover, we observe that the locally nilpotent derivation δ_u provided by Construction 3.6 gives us a polynomial

$$f \ := \ \delta_u(S_1)T_\iota^{-l_\iota} \ = \ S_1 T_\iota^{-l_\iota} \prod_{i,j} T_{ij}^{\langle u, v_{ij}\rangle} \ \in \ \mathbb{K}[T_{ij}, S_k].$$

Next we claim that the locally nilpotent derivation δ_u on $R(A,P)$ coincides with $\beta(A,i_1,i_0)_\iota \delta_{u_{\iota,i_0}} + \beta(A,i_0,i_1)_\iota \delta_{u_{\iota,i_1}}$. Indeed, we compute

$$\begin{aligned} \delta_u(S_1) - f b_{i_0,i_1,\iota} g_{i_0,i_1,\iota} \ &= \ \delta_u(S_1) - f\left(T_\iota^{l_\iota} - \beta(A,i_1,i_0)_\iota T_{i_0}^{l_{i_0}} - \beta(A,i_0,i_1)_\iota T_{i_1}^{l_{i_1}}\right) \\ &= \ \beta(A,i_1,i_0)_\iota f T_{i_0}^{l_{i_0}} + \beta(A,i_0,i_1)_\iota f T_{i_1}^{l_{i_1}} \\ &= \ \beta(A,i_1,i_0)_\iota \delta_{u_{\iota,i_0}}(S_1) + \beta(A,i_0,i_1)_\iota \delta_{u_{\iota,i_1}}(S_1), \end{aligned}$$

using Lemma 10.12. Computing the associated root groups according to Proposition 4.2 (i) gives the assertion. ☐

PROOF OF PROPOSITION 10.3. According to Propositions 9.17 and 9.18, the group $U(X)$ is generated by the rout groups arising from vertical P-roots at D^+. Given a vertical P-root u with $\langle u, v_{\iota 1}\rangle \ge l_{\iota 1}$, take any two distinct $0 \le i_0, i_1 \le r$ differing from ι. Then Lemma 10.13 tells us

$$\lambda_u(\mathbb{K}) \ \subseteq \ \lambda_{u_{\iota,i_0}}(\mathbb{K})\lambda_{u_{\iota,i_1}}(\mathbb{K}).$$

Recall that the evaluations of the linear forms u_{ι,i_0} and u_{ι,i_1} at the vectors v_{ij} are given by

$$\langle u_{\iota,i_0}, v_{ij}\rangle = \begin{cases} \langle u, v_{ij}\rangle, & i \ne \iota, i_0, \\ \langle u, v_{\iota j}\rangle - l_{\iota j}, & i = \iota, \\ \langle u, v_{i_0 j}\rangle + l_{i_0 j}, & i = i_0, \end{cases} \qquad \langle u_{\iota,i_1}, v_{ij}\rangle = \begin{cases} \langle u, v_{ij}\rangle, & i \ne \iota, i_1, \\ \langle u, v_{\iota j}\rangle - l_{\iota j}, & i = \iota, \\ \langle u, v_{i_1 j}\rangle + l_{i_0 j}, & i = i_1. \end{cases}$$

Thus, the automorphism $\lambda_u(s)$ can be expressed as a composition of automorphisms stemming from vertical P-roots evaluating strictly smaller at $v_{\iota 1}$

and equal to u at all other v_{i1} with $i \neq i_0, i_1$. Suitably iterating this process, we arrive at the assertion. $\square$

DEFINITION 10.14. Consider the defining matrix P of $X(A,P)$ and let $0 \le i_0, i_1 \le r$. Define an interval

$$\Gamma(i_0,i_1) \ := \ \Big[m_{i_1 1}, \ -m_{i_0 1} - \sum_{i \neq i_0, i_1} \lceil m_{i1} \rceil\Big] \ \subseteq \ \mathbb{Q}.$$

Moreover, denote $e_0' := 0 \in \mathbb{Z}^{r+1}$ and $e_i' := e_i \in \mathbb{Z}^{r+1}$ for $i = 1, \ldots, r+1$. Given $\alpha \in \mathbb{Q}$, define

$$u(i_0,i_1,\alpha) \ := \ -e_{r+1}' + \alpha(e_{i_1}' - e_{i_0}') + \sum_{i \neq i_0, i_1, r+1} \lceil m_{i1} \rceil (e_i' - e_{i_0}') \ \in \ \mathbb{Q}^{r+1}.$$

PROPOSITION 10.15. *Assume that $X = X(A,P)$ has a parabolic fixed point curve $D^+ \subseteq X$. Then we have mutually inverse bijections*

$$\begin{array}{ccc} \left\{ \begin{array}{l} \textit{vertical } P\textit{-roots } u \textit{ at } D^+ \textit{ such that} \\ 0 \le \langle u, v_{i1} \rangle < l_{i_1} \textit{ for all } i \neq i_0, i_1 \end{array} \right\} & \longleftrightarrow & \Gamma(i_0,i_1) \cap \mathbb{Z} \\ u & \mapsto & u_{i_1} \\ u(i_0,i_1,\alpha) & \leftarrow\!\shortmid & \alpha \end{array}$$

PROOF. First we consider any vertical P-root u at D^+. Let $u_0 = -u_1 - \ldots - u_r$ as in Remark 9.2 and set $\varepsilon_i := u_i - m_{i1}$. Using Proposition 9.15 we obtain

$$m_{i1} \ \le \ \langle u, v_{i1} \rangle \ = \ u_i l_{i1} - d_{i1} \ = \ \varepsilon_i l_{i1}.$$

Now let u stem from the left hand side set above. Then we must have $0 \le \varepsilon_i < 1$ and hence $u_i = \lceil m_{i1} \rceil$ for all $i \neq i_0, i_1$. Corollary 9.16 yields

$$m_{i_1 1} \ \le \ u_{i_1} \ \le \ -m_{i_0 1} - \sum_{i \neq i_0, i_1} u_i \ = \ -m_{i_0 1} - \sum_{i \neq i_0, i_1} \lceil m_{i1} \rceil.$$

One directly checks that any $\alpha \in \Gamma(i_0,i_1) \cap \mathbb{Z}$ delivers an $u(i_0, i_1 \alpha)$ in the left hand side set and the assignments are inverse to each other. $\square$

11. Root groups and resolution of singularities

In this section, we show how to lift the root groups arising from the horizontal or vertical P-roots of $X = X(A,P)$ with respect to the minimal resolution of singularities $\pi\colon \tilde{X} \to X$. The following theorem gathers the essential results; observe that items (iii) and (iv) are as well direct consequences of the general existence of a functorial resolution in characteristic zero, whereas (i) and (ii), used later, are more specific.

THEOREM 11.1. *Consider $X = X(A,P)$ and its minimal resolution $\pi\colon \tilde{X} \to X$, where $\tilde{X} = X(A, \tilde{P})$.*

(i) *There is a natural bijection $\lambda \mapsto \tilde{\lambda}$ between the root groups of X and those of $\tilde{X}$, made concrete in terms of defining data in Propositions 11.5 and 11.7.*

(ii) *For every root group* $\lambda\colon \mathbb{K} \to \mathrm{Aut}(X)$ *and every* $s \in \mathbb{K}$ *we have a commutative diagram*

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\lambda}(s)"] \arrow[d, "\pi"'] & \tilde{X} \arrow[d, "\pi"] \\ X \arrow[r, "\lambda(s)"'] & X \end{tikzcd}$$

(iii) *The isomorphism* $\pi\colon \pi^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}}$ *gives rise to a canonical isomorphism of groups* $\mathrm{Aut}(X)^0 \cong \mathrm{Aut}(\tilde{X})^0$.

(iv) *Every action* $G \times X \to X$ *of a connected algebraic group* G *lifts to an action* $G \times \tilde{X} \to \tilde{X}$.

The proof of Theorem 11.1 essentially relies on the preceding results showing that we either have only horizontal roots at a common simple quasismooth elliptic fixed point or there are only vertical roots at a common parabolic fixed point curve. This allows us to relate the resolution of singularities closely to resolving toric surface singularities. We begin the preparing discussion with a brief reminder on Hilbert bases of two-dimensional cones and then enter the horizontal case.

Remark 11.2. Consider two primitive vectors v_0 and v_1 in $\mathbb{Z}^2$. Assume $\det(v_0, v_1)$ to be positive. Set $v_0' := v_0$ and let $v_1' \in \mathbb{Z}^2$ be the unique vector with

$$v_1' \in \mathrm{cone}(v_0', v_1), \qquad \det(v_0', v_1') = 1, \qquad 0 \le \det(v_1', v_1) < \det(v_0', v_1).$$

Iterating gives us a finite sequence $v_0 = v_0', v_1', \ldots, v_q', v_{q+1}' = v_2$, the *Hilbert basis* $\mathcal{H}(\sigma)$ of $\sigma = \mathrm{cone}(v_0, v_1)$ in $\mathbb{Z}^2$. We have

$$\det(v_i', v_{i+1}') \;=\; 1, \qquad\qquad v_{i-1}' + v_{i+1}' \;=\; c_i v_i'$$

with unique integers $c_1, \ldots, c_q \in \mathbb{Z}_{\ge 2}$. Subdividing σ along the Hilbert basis gives us the fan of the minimal resolution of the affine toric surface Z_σ.

Construction 11.3. Assume that $X = X(A, P)$ has a quasismooth elliptic fixed point $x^- \in X$ with leading indices i_0, i_1. Consider

$$v_0 \;:=\; (-l_{i_0 n_0}, d_{i_0 n_0}), \qquad v_1 \;:=\; (l_{i_1 n_{i_1}}, d_{i_1 n_{i_1}}), \qquad \sigma \;:=\; \mathrm{cone}(v_0, v_1).$$

As earlier, let $e_0 = -e_1 - \ldots - e_r$, where $e_i \in \mathbb{Z}^{r+1}$ are the canonical basis vectors. Then, with every $v' = (l', d') \in \mathcal{H}(\sigma)$, we associate $\tilde{v} \in \mathbb{Z}^{r+1}$ by

$$\tilde{v} \;:=\; \begin{cases} -l' e_{i_0 n_{i_0}} + d' e_{r+1}, & l' < 0, \\ l' e_{i_1 n_{i_1}} + d' e_{r+1}, & l' > 0, \\ -e_{r+1}, & l' = 0. \end{cases}$$

Inserting the columns $\tilde{v}$, where $v_0, v_1 \neq v' \in \mathcal{H}(\sigma)$, at suitable places of P produces a slope ordered defining matrix P'.

Proposition 11.4. *Let* A, P *and* P' *be as in 11.3. Consider* $X' := X(A, P')$ *and the natural morphism* $\pi'\colon X' \to X$. *Then* π' *is an isomorphism* *over $X \setminus \{x^-\}$, each $x' \in X'$ over $x^- \in X$ is smooth and the minimal resolution $\tilde{X} \to X$ factors as*

$$\begin{tikzcd} \tilde{X} \arrow[rr] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

Proof. We may assume that P' is adapted to the sink. Remark 6.3 ensures $\det(v_0, v_1) > 0$. Let $v'_0, \ldots, v'_{q+1}$ be the members of $\mathcal{H}(\sigma)$, constucted as in Remark 11.2. Write $v'_i = (l'_i, d'_i)$. There are unique integers $1 \le k^- < k^+ \le q$ with

$$l'_i < 0 \text{ for } i = 0, \ldots, k^-, \qquad\qquad l'_i > 0 \text{ for } i = k^+, \ldots, q+1,$$

where $k^+ = k^-+1$ if all l'_i differ from zero and otherwise we have $k^+ = k^-+2$ and $l'_{k^0} = 0$ for $k^0 = k^- + 1$. The curve of X' corresponding to a column $\tilde{v}_i$ of P' lies in $\mathcal{A}_{i_0}$ if $l'_i < 0$, in $\mathcal{A}_{i_1}$ if $l'_i > 0$ and equals D^- if $i = k^0$.

We verify smoothness of the points $x' \in X'$ lying over $x^- \in X$. First consider the case that $x' \in X'$ is a parabolic or a hyperbolic fixed point. According to Remark 11.2 we have

$$\det(v'_i, v'_{i+1}) \ = \ 1.$$

This gives us precisely the smoothness conditions from Propositions 6.8 (ii) and 6.9. If $x' \in X'$ is an elliptic fixed point, then we can apply Proposition 6.10 to obtain smoothness of x' in terms of P':

$$\tilde{l}^- \tilde{m}^- = \tilde{l}_{i_0 \tilde{n}_{i_0}} \tilde{d}_{i_1 \tilde{n}_{i1}} + \tilde{l}_{i_1 \tilde{n}_{i_1}} \tilde{d}_{i_0 \tilde{n}_{i0}} = -l'_{k^-} d'_{k^+} + l'_{k^+} d'_{k^-} = -\det(v_{k^-}, v_{k^+}) = -1.$$

We show minimality of $X' \to X$. The conditions $v'_{i-1} + v'_{i+1} = c_i v'_i$ from Remark 11.2 translate to the equations Lemma 7.3 for the exceptional curves E_i of $X' \to X$ corresponding to v'_i. We conclude $E_i^2 \le -2$. $\square$

Proposition 11.5. *Consider the minimal resolution $\pi \colon \tilde{X} \to X$ of $\mathbb{K}^*$-surfaces defined by (A, P) and $(A, \tilde{P})$. Assume that there is a quasismooth simple elliptic fixed point $x^- \in X$ and let $\tilde{x}^- \in \pi^{-1}(x^-)$ be the corresponding elliptic fixed point. Given $0 \le i_0, i_1 \le r$ and $u \in \mathbb{Z}^{r+1}$, the following statements are equivalent.*

(i) *The linear form $u \in \mathbb{Z}^{r+1}$ is a horizontal P-root at (x^-, i_0, i_1).*
(ii) *The linear form $u \in \mathbb{Z}^{r+1}$ is a horizontal $\tilde{P}$-root at $(\tilde{x}^-, i_0, i_1)$.*

Proof. As $x^- \in X$ is a simple elliptic fixed point with exceptional index i_0, we have $\tilde{v}_{i\tilde{n}_i} = v_{in_i}$ for all $i \ne i_0$. The fiber of $\pi \colon \tilde{X} \to X$ over $x^- \in X$ is of the form

$$\pi^{-1}(x^-) \ = \ \tilde{D}_{i_0 \tilde{n}_{i_0} - q - 1} \cup \ldots \cup \tilde{D}_{i_0 \tilde{n}_{i_0}}$$

By Proposition 11.4, the corresponding columns $\tilde{v}_{i_0 \tilde{n}_{i_0} - q - 1}, \ldots, \tilde{v}_{i_0 \tilde{n}_{i_0}}$ of $\tilde{P}$ are obtained by running Construction 11.3 with the initial data

$$v_0 \ := \ (-l_{i_0 n_{i_0}}, d_{i_0 n_{i_0}}), \qquad\qquad v_1 \ := \ (l_{i_1 n_{i_1}}, d_{i_1 n_{i_1}}).$$

In the notation of Remark 11.2, the vector $\tilde{v}_{i_0 \tilde{n}_{i_0}}$ stems from the penultimate Hilbert basis member $v'_q \in \mathcal{H}(\sigma)$ of $\sigma = \operatorname{cone}(v_0, v_1)$ which is determined by the conditions

$$\det(v'_q, v_1) \;=\; 1, \qquad\qquad 0 \;\le\; \det(v_0, v'_q) \;<\; \det(v_0, v_1).$$

In terms of $\tilde{P}$, we have $v'_q = (-\tilde{l}_{i_0\tilde{n}_{i_0}}, \tilde{d}_{i_0\tilde{n}_{i_0}})$. Together with the definitions of v_0 and v_1 this gives us

$$\tilde{l}_{i_0\tilde{n}_{i_0}} d_{i_1 n_{i_1}} \;\equiv\; -1 \bmod l_{i_1 n_{i_1}}, \qquad -\frac{1}{l_{i_0 n_{i_0}} m^-} - l_{i_1 n_{i_1}} \;<\; \tilde{l}_{i_0\tilde{n}_{i_0}} \;\le\; -\frac{1}{l_{i_0 n_{i_0}} m^-},$$

where the estimate is obtained by resolving the first characterizing condition of v'_q for $\tilde{d}_{i_0\tilde{n}_{i_0}}$ and plugging the result into the second one. Next look at

$$v_0 \;:=\; (l_{in_i}, d_{in_i}), \qquad\qquad v_1 \;:=\; (l_{in_i-1}, d_{in_i-1})$$

in case $n_i \ge 1$. Then, according to Remark 11.2, the Hilbert basis member $v'_1 \in \mathcal{H}(\sigma)$ of $\sigma = \operatorname{cone}(v_0, v_1)$ is characterized by the conditions

$$\det(v_0, v'_1) \;=\; 1, \qquad\qquad 0 \;\le\; \det(v'_1, v_1) \;<\; \det(v_0, v_1).$$

Similarly as before, we have $v'_1 = (\tilde{l}_{\tilde{n}_i-1}, \tilde{d}_{\tilde{n}_i-1})$ and making the above conditions explicit, we arrive at

$$\tilde{l}_{\tilde{n}_i-1} d_{in_i} \equiv -1 \bmod l_{in_i}, \quad \frac{1}{l_{in_i}(m_{n_i-1} - m_{n_i})} \le \tilde{l}_{\tilde{n}_i-1} < \frac{1}{l_{in_i}(m_{n_i-1} - m_{n_i})} + l_{in_i}.$$

Now consider a horizontal P-root $u \in \mathbb{Z}^{r+1}$ at (i_0, i_1, x^-). Proposition 9.11 yields $u = u(i_0, i_1, \gamma)$ with a non-negative integer γ satisfying $\gamma d_{i_1 n_{i_1}} \equiv -1 \bmod l_{i_1 n_{i_1}}$ and

$$\frac{1}{l_{in_i}(m_{n_i-1} - m_{n_i})} \;=\; \xi_i \;\le\; \gamma \;\le\; \eta_{i_0 n_{i_0}} \;=\; -\frac{1}{l_{i_0 n_{i_0}} m^-},$$

for all $i = 0, \ldots, r$ with $i \ne i_0$ and $n_i > 1$. We compare γ with $\tilde{l}_{i_0\tilde{n}_{i_0}}$ and $\tilde{l}_{\tilde{n}_i-1}$. First, using the modular identities, we observe

$$(\tilde{l}_{i_0\tilde{n}_{i_0}} - \gamma) d_{i_1 n_{i_1}} \;\in\; l_{i_1 n_{i_1}}\mathbb{Z}, \qquad (\gamma - \tilde{l}_{i_1\tilde{n}_{i_1}-1}) d_{i_1 n_{i_1}} \;\in\; l_{i_1 n_{i_1}}\mathbb{Z}.$$

As $l_{i_1 n_{i_1}}$ and $d_{i_1 n_{i_1}}$ are coprime, $\tilde{l}_{i_0\tilde{n}_{i_0}} - \gamma$ as well as $\gamma - \tilde{l}_{i_1\tilde{n}_{i_1}-1}$ are multiples of $l_{i_1 n_{i_1}}$. Thus, the previous estimates and $l_{in_i} = 1$ for $i \ne i_0, i_1$ give us

$$\tilde{l}_{\tilde{n}_i-1} \;\le\; \gamma \;\le\; \tilde{l}_{i_0\tilde{n}_{i_0}}, \qquad i = 0, \ldots, r,\; i \ne i_0,\; n_i > 1.$$

Now we can directly check the defining conditions of a horizontal $\tilde{P}$-root at $(\tilde{x}, i_0, i_1)$ for $u = u(i_0, i_1, \gamma)$: Lemma 9.10 together with Propositions 6.9 and 6.10 yields

$$\begin{array}{rcccl}
\langle u, \tilde{v}_{i_0\tilde{n}_{i_0}} \rangle & = & \frac{\tilde{l}_{i_0\tilde{n}_{i_0}} - \gamma}{\tilde{l}_{i_1\tilde{n}_{i_1}}} & \ge & 0, \\
\langle u, \tilde{v}_{i_1\tilde{n}_{i_1}-1} \rangle & = & \frac{\gamma - \tilde{l}_{i_1\tilde{n}_{i_1}-1}}{\tilde{l}_{i_1\tilde{n}_{i_1}-1}} & \ge & 0, \\
\langle u, v_{i\tilde{n}_i-1} \rangle & = & \gamma & \ge & l_{i\tilde{n}_i-1},
\end{array}$$

where $i \ne i_0, i_1$ with $n_i > 1$ in the last case. This, verifies "(i)⇒(ii)". The reverse implication is a direct consequence of Proposition 9.4. □

Remark 11.6. From the proof of Proposition 11.5 we infer that $\gamma = \tilde{l}_{i_0\tilde{n}_{i_0}}$ is the maximal integer such that $u(i_0, i_1, \gamma)$ is a horizontal P-root x^-.

Proposition 11.7. *Consider the minimal resolution* $\pi\colon \tilde{X} \to X$ *of* $\mathbb{K}^*$*-surfaces defined by* (A, P) *and* $(A, \tilde{P})$*. Assume that there is a parabolic fixed point curve* $D^+ \subseteq X$ *and let* $\tilde{D}^+ \subseteq \tilde{X}$ *be the proper transform. Given* $u \in \mathbb{Z}^{r+1}$*, the following statements are equivalent.*

(i) *The linear form* $u \in \mathbb{Z}^{r+1}$ *is a vertical* P*-root at* D^+*.*
(ii) *The linear form* $u \in \mathbb{Z}^{r+1}$ *is a vertical* $\tilde{P}$*-root at* $\tilde{D}^+$*.*

Proof. The implication "(ii)⇒(i)" is clear due to Proposition 9.15. We care about "(i)⇒(ii)". By Remark 6.13, the columns $\tilde{v}_{i1}, \ldots, \tilde{v}_{q_i} = v_{i1}$ of $\tilde{P}$, where $i = 0, \ldots, r$, arise from subdividing $\operatorname{cone}(v_{i1}, v^+)$ along the Hilbert basis. Consider

$$v_0 \ := \ (0, -1), \qquad v_1 \ := \ (l_1, -d_{i1}),$$

where $i = 0, \ldots, r$. In the setting of Remark 11.2, we have $v_1' = (\tilde{l}_{i1}, -\tilde{d}_{i1})$. Moreover, the conditions on the determinants lead to

$$1 \ = \ \det(v_0, v_1') \ = \ \tilde{l}_{i1}, \qquad l_{i1}\tilde{d}_{i1} - d_{i1} \ = \ \det(v_1', v_1) \ < \ \det(v_0, v_1) \ = \ l_{i1}.$$

Using also slope orderedness of $\tilde{P}$, we see $m_{i1} \le \tilde{d}_{i1} < m_{i1} + 1$. Now, let $u \in \mathbb{Z}^{r+1}$ be a vertical P-root at D^+. Proposition 9.15 ensures $u_i \ge m_{i1}$. This implies $u_i \ge \tilde{d}_{i1}$. Using Proposition 9.15 again, we obtain that u is a vertical $\tilde{P}$-root at $\tilde{D}^+$. □

Remark 11.8. From the proof of Proposition 11.7 we infer that $m_{i1} \le \tilde{d}_{i1} < m_{i1} + 1$ holds for $i = 0, \ldots, r$. In particular, if P is adpated to the source, then also $\tilde{P}$ is adapted to the source.

Proposition 11.9. *Consider* $\tilde{X} = X(A, \tilde{P})$ *and* $X = X(A, P)$*, where each column of* P *also occurs as a column of* $\tilde{P}$*.*

(i) *There is a proper birational morphism* $\pi\colon \tilde{X} \to X$ *contracting precisely the curves* $\tilde{D}_{ij}$ *and* $\tilde{D}^{\pm}$*, where* $\tilde{v}_{ij}$ *and* $\tilde{v}^{\pm}$ *is not a column of* P*.*
(ii) *If there is an elliptic fixed point* $\tilde{x}^- \in \tilde{X}$ *and* u *is a horizontal* P*-root at* $(\tilde{x}^-, i_0, i_1)$*. Then* $x^- = \pi(\tilde{x}^-) \in \tilde{X}$ *is an elliptic fixed point forming the sink and* u *is a horizontal* P*-root at* (x^-, i_0, i_1)*.*
(iii) *If we have a parabolic source* $\tilde{D}^+ \subseteq \tilde{X}$ *and* u *is a vertical* $\tilde{P}$*-root at* $\tilde{D}^+$*, then* $D^+ = \pi(\tilde{D}^+) \subseteq X$ *is a curve forming the source and* u *is a vertical* P*-root at* D^+*.*

Moreover, the root groups $\tilde{\lambda}\colon \mathbb{K} \to \operatorname{Aut}(\tilde{X})$ *and* $\lambda\colon \mathbb{K} \to \operatorname{Aut}(X)$ *arising from a common root* u *fit for every* $s \in \mathbb{K}$ *into the commutative diagram*

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\lambda}(s)"] \arrow[d, "\pi"'] & \tilde{X} \arrow[d, "\pi"] \\ X \arrow[r, "\lambda(s)"'] & X \end{tikzcd}$$

Proof. For (i) observe that each cone of the fan $\tilde{\Sigma}$ of the ambient toric variety $\tilde{Z}$ of $\tilde{X}$ is contained in a cone of the fan Σ of the ambient toric variety Z of X. The corresponding toric morphism $\tilde{Z} \to Z$ restricts to the morphism $\pi\colon \tilde{X} \to X$. Assertion (ii) is clear by Proposition 9.4 and (iii) follows from Proposition 9.15.

We prove the supplement. Consider the Cox rings $R(A,P)$ of X and $R(A,\tilde{P})$ of $\tilde{X}$. Recall that these are given as factor algebras

$$R(A,P) \;=\; \mathbb{K}[T_{ij},S^{\pm}]/\langle g_I;\; I\in\mathfrak{I}\rangle, \qquad R(A,\tilde{P}) \;=\; \mathbb{K}[\tilde{T}_{ij},\tilde{S}^{\pm}]/\langle \tilde{g}_I;\; I\in\tilde{\mathfrak{I}}\rangle,$$

where $R(A,P)$ is graded by $K = \mathrm{Cl}(X)$ and $R(A,\tilde{P})$ by $\tilde{K} = \mathrm{Cl}(\tilde{X})$ see Construction 2.4 for the details. Define a homomorphism of the graded polynomial algebras

$$\Psi\colon \mathbb{K}[\tilde{T}_{ij},\tilde{S}^{\pm}] \;\to\; \mathbb{K}[T_{ij},S^{\pm}]$$

by sending $\tilde{T}_{ij}$ and $\tilde{S}^{\pm}$ to the variables of $\mathbb{K}[T_{ij},S^{\pm}]$ corresponding to $\pi(\tilde{D}_{ij})$ and $\pi(\tilde{D}^{\pm})$ in case these divisors are not exceptional and to $1\in\mathbb{K}[T_{ij},S^{\pm}]$ otherwise. Then Ψ descends to a homomorphism

$$\psi\colon R(A,\tilde{P}) \;\to\; R(A,P).$$

Now note that any Demazure root δ_u on the fan $\tilde{\Sigma}$ having $\tilde{P}$ as generator matrix is as well a Demazure root on the fan Σ having P as generator matrix. Moreover, we have a commutative diagram

$$\begin{tikzcd}
\mathbb{K}[\tilde{T}_{ij},\tilde{S}^{\pm}] \arrow[r, "\delta_u"] \arrow[d, "\Psi"'] & \mathbb{K}[\tilde{T}_{ij},\tilde{S}^{\pm}] \arrow[d, "\Psi"] \\
\mathbb{K}[T_{ij},S^{\pm}] \arrow[r, "\delta_u"'] & \mathbb{K}[T_{ij},S^{\pm}]
\end{tikzcd}$$

Next, given a horizontal or vertical P-root and its corresponding $\tilde{P}$-root, we look at the associated Demazure P-root κ and Demazure $\tilde{P}$-root $\tilde{\kappa}$. Presenting $\bar{\lambda}_\kappa(s)^*$ and $\bar{\lambda}_{\tilde{\kappa}}(s)^*$ as in Theorem 5.4 and using commutativity of the previous diagram we see that the following diagram commutes as well:

$$\begin{tikzcd}
R(A,\tilde{P}) \arrow[r, "\bar{\lambda}_{\tilde{\kappa}}(s)"] \arrow[d, "\psi"'] & R(A,\tilde{P}) \arrow[d, "\psi"] \\
R(A,P) \arrow[r, "\bar{\lambda}_{\kappa}(s)"'] & R(A,P)
\end{tikzcd}$$

Cover $\pi^{-1}(X_{\mathrm{reg}})$ by affine open subsets of the form $\tilde{X}_{[\tilde{D}],\tilde{f}}$, where $\tilde{D}$ is a Weil divisor of $\tilde{X}$ having $\tilde{f}\in R(A,\tilde{P})$ as a section and $\tilde{X}_{[\tilde{D}],\tilde{f}}$ is obtained by removing the support of $\tilde{D}+\mathrm{div}(\tilde{f})$ from $\tilde{X}$. Set $D=\pi_*\tilde{D}$ and $f=\psi(\tilde{f})$. Then we have commutative diagrams

$$\begin{tikzcd}
 & R(A,\tilde{P})_{\tilde{f}} \arrow[r, "\psi"] & R(A,P)_f & \\
\Gamma(\tilde{X}_{[\tilde{D}],\tilde{f}},\mathcal{O}_{\tilde{X}}) \arrow[r, equal] & R(A,\tilde{P})_{(\tilde{f})} \arrow[u] \arrow[r, shift left, "\psi_0"] & R(A,P)_{(f)} \arrow[u] \arrow[l, shift left, "\pi^*"] \arrow[r, equal] & \Gamma(X_{[D],f},\mathcal{O}_X)
\end{tikzcd}$$

where the lower row represents the degree zero part of the upper one. The homomorphisms ψ_0 and π^* in the lower row are directly seen to be inverse to each other; see [**2**, Prop. 1.5.2.4]. Passing to the spectra and gluing gives us a commutative diagram

$$\begin{tikzcd} p^{-1}(X_{\mathrm{reg}}) \arrow[r, "\varphi"] \arrow[d, "p"'] & \tilde{p}^{-1}(\pi^{-1}(X_{\mathrm{reg}})) \arrow[d, "\tilde{p}"] \\ X_{\mathrm{reg}} \arrow[r, leftrightarrow, "\varphi_0", "\pi"'] & \pi^{-1}(X_{\mathrm{reg}}) \end{tikzcd}$$

where p and $\tilde{p}$ denote the quotients of characteristic spaces of X and $\tilde{X}$ by the respective characteristic quasitori; use again [**2**, Prop. 1.5.2.4]. By construction, the morphisms φ arising from ψ and φ_0 arising from ψ_0 satisfy

$$\varphi \circ \bar{\lambda}_\kappa(s) \ = \ \bar{\lambda}_{\tilde{\kappa}}(s) \circ \varphi, \qquad\qquad \varphi_0 \circ \lambda_\kappa(s) \ = \ \lambda_{\tilde{\kappa}}(s) \circ \varphi_0.$$

□

Proof of Theorem 11.1. Propositions 11.5 and 11.7 provide us with a bijection between the P-roots and the $\tilde{P}$-roots. Applyying Proposition 11.9, we obtain proves the second assertion of the Theorem. Assertions (iii) and (iv) are then direct consequences. □

12. Structure of the automorphism group

Here we prove Theorem 0.1. In a first step, we express the number of necessary P-roots to generate the unipotent part of the automorphism group of a $\mathbb{K}^*$-surface $X = X(A,P)$ in terms of intersection numbers of invariant curves of X. We make use of the numbers defined in the introduction:

$$c_i(D^+) \ = \ \mathrm{CF}_{q_i}(-E_{i1}^2, \ldots, -E_{iq_i}^2)^{-1}, \qquad c(x^-) \ = \ \mathrm{CF}_q(-E_q^2, \ldots, -E_1^{-1})^{-1},$$

where the $E_{ij} \subseteq \tilde{X}$ are the exceptional curves lying over $D^+ \subseteq X$ in and the $E_i \subseteq \tilde{X}$ over $x^- \in X$ with respect to the minimal resolution of singularities $\tilde{X} \to X$.

Definition 12.1. Let $X = X(A,P)$ be non-toric with a fixed point curve $D^+ \subseteq X$. Given $0 \le i_0, i_1 \le r$, we call a vertical P-root u at D^+ *essential* with respect to i_0, i_1, if $0 \le \langle u, v_{i1}\rangle < l_{i1}$ for all $i \ne i_0, i_1$.

Proposition 12.2. *Consider a non-toric $\mathbb{K}^*$-surface $X = X(A,P)$.*

(i) *Assume that there is a curve $D^+ \subseteq X$ and let $0 \le i_0, i_1 \le r$. Then the number of vertical P-roots at D^+ essential to i_0, i_1 is given by*

$$\max(0, \ (D^+)^2 + 1 - \sum_{i=0}^{r} c_i(D^+)).$$

(ii) *Assume that there is a quasismooth simple $x^- \in X$ and that P is normalized. Then the number of horizontal P-roots at $(x^-, 0, 1)$ is given by*

$$\max\left(0, \ \left\lfloor l_{1n_1}^{-1} \min_{i \ne 0}(l_{in_i} D_{in_i}^2 + (l_{in_i} - l_{1n_1}) D_{in_i} D_{1n_1}) - c(x^-) \right\rfloor + 1\right).$$

(iii) *Assume that there is a quasismooth simple $x^- \in X$ and that P is normalized. Then there is a horizontal P-root at $(x^-, 1, 0)$ if and only if*

$$l_{in_i} D_{in_i}^2 \ \ge \ (l_{0n_0} - l_{in_i}) D_{in_i} D_{0n_0}, \qquad \textit{for all } i \ne 1.$$

Moreover, if these conditions hold, then $x^- \in X$ is smooth and there exists precisely one horizontal P-root at $(x^-, 1, 0)$.

PROOF. Let $\tilde{X} = X(A, \tilde{P})$ be the minimal resolution. We verify (i). By Proposition 11.7, the numbers ρ of vertical P-roots and $\tilde{\rho}$ of vertical $\tilde{P}$-roots essential to i_0, i_1 coincide. By Proposition 10.15, the number $\tilde{\rho}$ equals the number of integers in the interval

$$\tilde{\Gamma}(i_0, i_1) \;=\; \Big[\tilde{m}_{i_1 1},\; -\tilde{m}_{i_0 1} - \sum_{i \neq i_0, i_1} \lceil \tilde{m}_{i1} \rceil\Big].$$

Since $\tilde{X}$ is smooth, the slopes $\tilde{m}_{i0}, \ldots, \tilde{m}_{ir}$ are all integral numbers; see Proposition 6.8. Thus, we see that $\rho = \tilde{\rho}$ equals the maximum of zero and the number

$$-\tilde{m}_{i_0 1} - \tilde{m}_{i_1 1} - \sum_{i \neq i_0, i_1} \tilde{m}_{i1} + 1 \;=\; -\tilde{m}^+ + 1 \;=\; (\tilde{D}^+)^2 + 1,$$

where the last equality holds by Remark 6.4. We have $\tilde{D}_{ij} = E_i$ for $j = 1, \ldots, q_i$. Moreover, $\tilde{D}_{iq_i+1} = D_{i1}$ and $\tilde{m}_{iq_i+1} = m_{i1}$. Thus, applying Corollary 7.6 (i) with $j_i = q_i + 1$, we see that $\rho = \tilde{\rho}$ equals the maximum of zero and

$$(\tilde{D}^+)^2 + 1 \;=\; (D^+)^2 - \sum_{i=0}^{r} c_i(D^+) + 1.$$

We verify (ii). According to Proposition 9.11, the number ρ of horizontal P-roots at $(x^-, 0, 1)$ equals the number of integers γ satisfying

$$\gamma \;\in\; \Delta(0,1) \;=\; \bigcap_{i \neq 0} [\xi_i, \eta_1], \qquad \gamma d_{1n_1} \equiv -1 \mod l_{1n_1},$$

see Construction 9.7 for the notation. By Remark 11.6, the maximal integer γ satisfying these conditions is $\tilde{l}_{0\tilde{n}_0}$. Thus, we can replace $[\xi_i, \eta_1]$ with $[\xi_i, \tilde{l}_{0\tilde{n}_0}]$. So, the number of integers γ in $[\xi_i, \tilde{l}_{0\tilde{n}_0}]$ with $\gamma d_{1n_1} \equiv -1 \mod l_{1n_1}$ is the maximum of zero and the round down $\vartheta(i) \in \mathbb{Z}$ of

$$\begin{aligned}
\frac{\tilde{l}_{0\tilde{n}_0} - \xi_i}{l_{1n_1}} + 1 \;&=\; \frac{(\eta_1 - \xi_i) - (\eta_1 - \tilde{l}_{0\tilde{n}_0})}{l_{1n_1}} + 1 \\
&=\; l_{1n_1}^{-1}\Big(l_{in_i} D_{in_i}^2 + (l_{in_i} - l_{1n_1}) D_{in_i} D_{1n_1} - c(x^-)\Big) + 1.
\end{aligned}$$

Here, the second equality needs explanation. First, we express $\eta_1 - \xi_i$ in terms of intersections numbers according to Remark 9.8. Moreover, the definition of η_1, Remark 6.3, quasismoothness of x^- and Proposition 6.10 yield

$$\eta_1 = \frac{1}{l_{1n_1} m^-}, \qquad l^- m^- = \det(\sigma^-), \qquad l^- = l_{0n_0} l_{1n_1}.$$

Proposition 9.12 (ii) says that 0 is the exceptional index of $x_i \in X$ and thus $E_j = \tilde{D}_{0n_0 - q + j}$ holds for $j = 1, \ldots, q$. Using Corollary 7.6 (ii), we obtain

$$\eta_1 - \tilde{l}_{0\tilde{n}_0} \;=\; \frac{l_{0n_0}}{\det(\sigma^-)} - \tilde{l}_{0\tilde{n}_0} \;=\; c(x^-).$$

Since $\Delta(0,1)$ is the intersection over the intervals $[\xi, \eta_1]$, where $i \neq i_0$, we see that the number of all the wanted γ we have to take the minimum of the above round downs $\vartheta(i)$ as an upper bound.

We care about (iii). By Proposition 9.11, there exists a horizontal P-root at $(x^-,1,0)$ if and only if $\Delta(1,0)$ is non-empty. The latter precisely means $\eta_0 - \xi_i \geq 0$ for all $i \neq 1$. This in turn is equivalent to

$$l_{in_i} D_{in_i}^2 \ \geq \ (l_{0n_0} - l_{in_i}) D_{in_i} D_{0n_0}, \qquad \text{for all } i \neq 1,$$

see Remark 9.8. Now, if there is a horizontal P-root at $(x^-,1,0)$, then Proposition 9.12 (i) yields that there is no further one and that $x^- \in X$ is smooth. $\square$

The final puzzle piece for the proof of Theorem 0.1 is the following controled contraction $\tilde{X} \to X'$ of the minimal resolution $\tilde{X}$ of X onto a toric surface that allows to keep track of the relevant roots.

Proposition 12.3. *Consider a minimal resolution $\tilde{X} \to X$, where $X = X(A,P)$ is non-toric and $\tilde{X} = X(A,\tilde{P})$.*

(i) *Let $x^- \in X$ be a quasismooth simple elliptic fixed point and let P be adapted to the sink. Then there is a $\mathbb{K}^*$-equivariant morphism $\pi\colon \tilde{X} \to X'$ onto the toric smooth projective $\mathbb{K}^*$-surface X' defined the matrix*

$$P' \ = \ \begin{bmatrix} -\tilde{l}_{01} & \dots & -\tilde{l}_{0\tilde{n}_0} & \tilde{l}_{11} & \dots & \tilde{l}_{1\tilde{n}_1} & 0 \\ \tilde{d}_{01} & \dots & \tilde{d}_{0\tilde{n}_0} & \tilde{d}_{11} & \dots & \tilde{d}_{1\tilde{n}_1} & 1 \end{bmatrix}.$$

The horizontal $\tilde{P}$-roots at $(\tilde{x}^-,0,1)$ map injectively to the horizontal P'-roots at $(\pi(\tilde{x}^-),0,1)$ via

$$\mathbb{Z}^{r+1} \ni u(0,1,\gamma) \ \mapsto \ u(0,1,\gamma) \in \mathbb{Z}^2.$$

Similarly, the horizontal $\tilde{P}$-roots at $(\tilde{x}^-,1,0)$ map injectively to the horizontal P'-roots at $(\pi(\tilde{x}^-),1,0)$ via

$$\mathbb{Z}^{r+1} \ni u(1,0,\gamma) \ \mapsto \ u(1,0,\gamma) \in \mathbb{Z}^2.$$

(ii) *Assume that we have a curve $D^+ \subseteq X$ admitting vertical P-roots and let P be adapted to the source. Then we obtain a $\mathbb{K}^*$-equivariant morphism $\pi\colon \tilde{X} \to X'$ onto the smooth toric $\mathbb{K}^*$-surface X' defined by the matrix*

$$P' \ = \ \begin{bmatrix} -\tilde{l}_{01} & \dots & -\tilde{l}_{0\tilde{n}_0} & \tilde{l}_{11} & \dots & \tilde{l}_{1\tilde{n}_1} & 0 & 0 \\ \tilde{d}_{01} & \dots & \tilde{d}_{0n_0} & \tilde{d}_{11} & \dots & \tilde{d}_{1\tilde{n}_1} & 1 & -1 \end{bmatrix}.$$

The image $\pi(\tilde{D}^+) \subseteq X'$ is a curve forming the source and the vertical $\tilde{P}$-roots at $\tilde{D}^+$ essential with respect to $0,1$ map injectively to the vertical P'-roots at $\pi(\tilde{D}^+)$ essential with respect to $0,1$ given by

$$\mathbb{Z}^{r+1} \ni u(0,1,\alpha) \ \mapsto \ u(0,1,\alpha) \in \mathbb{Z}^2.$$

Moreover, the root groups $\tilde{\lambda}\colon \mathbb{K} \to \operatorname{Aut}(\tilde{X})$ and $\lambda'\colon \mathbb{K} \to \operatorname{Aut}(X')$ arising from a common root u fit for every $s \in \mathbb{K}$ into the commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\lambda}(s)"] \arrow[d, "\pi"'] & \tilde{X} \arrow[d, "\pi"] \\ X' \arrow[r, "\lambda'(s)"'] & X' \end{tikzcd}$$

Finally, each Demazure P'-root is also a Demazure root on the complete fan Σ' with generator matrix P' and the respective root groups in the sense of Constructions 3.6 and 3.2 coincide.

Proof. First we convince ourselves that in setting (i) we have $\tilde{l}_{i\tilde{n}_i} = 1$ as well as $\tilde{d}_{i\tilde{n}_i} = 0$ for all $i \geq 2$ and, moreover, that there is a curve $\tilde{D}^+ \subseteq \tilde{X}$. Note that $x^- \in X$ has exceptional index 0 or 1 by Propositions 8.10 and 11.9. Thus, for any $i \geq 2$, Proposition 6.10 yields $\tilde{l}_{i\tilde{n}_i} = l_{in_i} = 1$ and the fact that P is adapted to the sink ensures $\tilde{d}_{i\tilde{n}_i} = d_{in_i} = 0$. The existence of $\tilde{D}^+ \subseteq \tilde{X}$ is guaranteed by Corollary 8.7. In the situation of (ii), the existence of $\tilde{D}^+ \subseteq \tilde{X}$ is clear. We ensure $\tilde{l}_{i1} = 1$ and $\tilde{d}_{i1} = 0$ for all $i \geq 1$. Moreover, by Proposition 9.17, there must be a curve $\tilde{D}^- \subseteq \tilde{X}$. Now, as P is adapted to the source, Remark 11.8 yields that also $\tilde{P}$ is adapted to the source. Thus, we have $\tilde{l}_{i1} = 1$ and $\tilde{d}_{i1} = 0$ for $i = 1, \ldots, r$. From now, we treat Settings (i) and (ii) together. Consider the data

$$n''_0 = n_0,\ v''_{0j} = v_{0j}, \quad n''_1 = n_1,\ v''_{1j} = v_{1j}, \quad n''_i = 1,\ v''_{i1} = v_{in_i},\ i \geq 2.$$

These, together with v^+ in the setting of (i) and v^+, v^- in the setting of (ii) are the columns of a matrix P''. It defines a $\mathbb{K}^*$-surface $X'' = X(A, P'')$ which is smooth due to Propositions 6.8 and 6.10. Proposition 11.9 gives us a morphism $\tilde{X} \to X''$ having the desired properties concerning the roots and the associated root groups.

Now, the matrix P'' is highly redundant. Removing all these redundancies, that means erasing the column v_{in_i} and the i-th rows for $i = 2, \ldots, r$ turns P'' into P'. The $\mathbb{K}^*$-surface X'' is isomorphic to the toric $\mathbb{K}^*$-surface $X' = X(A', P')$, where A' is the 2×2 unit matrix. Using $\tilde{l}_{i\tilde{n}_i} = 1$ as well as $\tilde{d}_{i\tilde{n}_i} = 0$ in (i) as well as $\tilde{l}_{i1} = 1$ and $\tilde{d}_{i1} = 0$ in (ii) as seen before, one checks that the P''-roots turn into P'-roots as claimed. The supplement is directly verified. □

Proof of Theorem 0.1. Let $X := X(A, P)$ be a non-toric $\mathbb{K}^*$-surface. Theorem 3.8 on the automorphism group of a rational projective variety with torus action of complexity one says that $\operatorname{Aut}(X)^0$ is generated by the acting torus and the additive one-parameter groups associated with the Demazure P-roots. In the surface case, the latter ones are given by horizontal and vertical P-roots; see Propositions 9.11 and 9.15. Thus, $\operatorname{Aut}(X)^0 = \mathbb{K}^*$ if and only if there are neither horizontal nor vertical P-roots. Horizontal P-roots only exist if X admits a quasismooth simple elliptic fixed point, and in this case there is no other such fixed point; see Proposition 9.6 and Theorem 8.4. This setting is Case (ii) of Theorem 0.1. Moreover, existence of vertical P-roots requires a non-negative parabolic fixed point curve and excludes quasismooth simple elliptic fixed points; see Propositions 9.15 and 9.17. This setting restitutes Case (i) of Theorem 0.1. Recall that $U(X) \subseteq \operatorname{Aut}(X)^0$ denotes the subgroup generated by all root subgroups.

We determine $\operatorname{Aut}(X)^0$ in Case (i) of Theorem 0.1. Thus, we have to deal with a non-negative parabolic fixed point curve hosting vertical P-roots, if present, and which we may assume to be $D^+ \subseteq X$. Moreover, we may assume that the defining matrix $\tilde{P}$ of the minimal resolution $\tilde{X}$ of X is adapted to the source. Proposition 9.18 yields that D^+ is the only fixed point curve admitting vertical roots. Fix any two distinct $0 \leq i_0, i_1 \leq r$. Then

Proposition 10.3 says that $U(X)$ is generated by all root groups arising from vertical P-roots being essential at i_0, i_1. Proposition 12.2 (i) shows that ρ from Theorem 0.1 (i) equals the number the number of vertical P-roots essential to i_0, i_1. Theorem 11.1 and Proposition 12.3 realize $U(X)$ as the subgroup generated by Demazure roots at a common primitive ray generator of the automorphism group of a suitable toric surface X'. Moreover the original $\mathbb{K}^*$-action of X is given on X' by the one parameter group $\mathbb{K}^* \to \mathbb{T}^2$ sending t to $(1, t)$. Applying Proposition 4.2 yields the desired isomorphism $\mathbb{K}^\rho \rtimes_\psi \mathbb{K}^* \cong \mathrm{Aut}(X)^0$.

We enter Case (ii) of Theorem 0.1. The pattern of arguments is similar to that of the preceding case. Now we have a unique quasismooth simple elliptic fixed point which we can assume to be $x^- \in X$. Moreover, we can assume P to be normalized. By Proposition 10.2, the group $U(X)$ is generated by the root groups stemming from the horizontal P-roots at (i_0, i_1, x^-) and (i_1, i_0, x^-). Due to Proposition 9.12, we may assume $i_0 = 0$ and $i_1 = 1$, and, moreover, that $i_0 = 0$ is the exceptional index of $x^- \in X$. Proposition 12.2 (ii) shows that ρ and ζ from Theorem 0.1 (ii) equal the numbers of horizontal P-roots at $(0, 1, x^-)$ and $(1, 0, x^-)$, respectively. Theorem 11.1 and Proposition 12.3 realize $U(X)$ as the subgroup generated by Demazure roots at two common primitive ray generators of the automorphism group of a suitable toric surface X'. Here, using Proposition 9.12 (i) and Corollary 9.13, we see that the Demazure roots of X' corresponding to horizontal P-roots at $(0, 1, x^-)$ and $(1, 0, x^-)$ are as in the setting of Proposition 4.4. Moreover, $\mathbb{K}^*$ acts X' via the one parameter group $\mathbb{K}^* \to \mathbb{T}^2$ sending t to $(1, t)$. Thus, Proposition 4.4, yields the desired isomorphism $(\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}) \rtimes_\psi \mathbb{K}^* \cong \mathrm{Aut}(X)^0$. □

13. Almost homogeneous $\mathbb{K}^*$-surfaces

Here, we investigate almost transitive algebraic group actions on rational projective $\mathbb{K}^*$-surfaces $X = X(A, P)$. Our considerations fit together to the proof of Theorem 0.2, given at the end of the section. Moreover, in Propositions 13.12 and 13.17 we specify the two-dimensional subgroups of $\mathrm{Aut}(X)^0$ that act almost transitively on X. The first observation of the section says in particular that almost transitive actions can only exist in the presence of horizontal P-roots.

PROPOSITION 13.1. *Consider a non-toric $\mathbb{K}^*$-surface $X = X(A, P)$.*

- (i) *If $\lambda\colon \mathbb{K} \to \mathrm{Aut}(X)$ is a root group defined by a vertical P-root, then each orbit of $\lambda(\mathbb{K})$ is contained in the closure of a $\mathbb{K}^*$-orbit.*
- (ii) *If X admits vertical P-roots, then $\mathrm{Aut}(X)$ acts with orbits of dimension at most one.*

PROOF. The first statement is a consequence of [**3**, Cor. 5.11 (ii)]. Alternatively, it directly follows from Remark 3.9, Construction 3.2 and the definition of the $\mathbb{K}^*$-action on $X = X(A, P)$. For the second statement, recall from Proposition 9.17 that the presence of a vertical P-roots excludes quasismooth simple elliptic fixed points and hence also excludes horizontal P-roots. □

From now on, we assume presence of horizontal P-roots and we work in the setting of Theorem 0.1 (ii).

Construction 13.2. Let $X = X(A, P)$ be non-toric with $x^- \in X$ and P normalized. Consider the unit component of its automorphism group

$$\mathrm{Aut}(X)^0 \;=\; \left(\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta\right) \rtimes_\psi \mathbb{K}^*,$$

where the numbers ρ and ζ as well as the twisting homomorphisms φ and ψ are specified in Theorem 0.1. Moreover, define lines in $\mathbb{K}^{\rho+\zeta}$ by

$$U_k \;:=\; \mathbb{K}e_k, \qquad k = 1, \ldots, \rho + \zeta.$$

Then each of the following semidirect products G_k is a two-dimensional subgroup of $\mathrm{Aut}(X)^0$ containing $\mathbb{K}^*$:

$$G_k \;=\; U_k \rtimes_{\psi_k} \mathbb{K}^*, \qquad \psi_k(t)(s) \;=\; \begin{cases} t^{\tilde{l}_{0\tilde{n}_0} - (k-1) l_{1n_1}} s, & k = 1, \ldots, \rho, \\ t^{l_{1n_1}} s, & \zeta = 1,\ k = \rho + 1. \end{cases}$$

Remark 13.3. Let $X = X(A, P)$ be as in Construction 13.2. Then each of the lines $U_1, \ldots, U_{\rho+\zeta}$ is a root group. More precisely, the following holds.

(i) For $k = 1, \ldots, \rho$, the lines $U_k \subseteq \mathrm{Aut}(X)^0$ are precisely the root groups defined by the horizontal P-roots $u(0, 1, \gamma_k)$ with $\gamma_k = \tilde{l}_{0\tilde{n}_0} - (k-1) l_{1n_1}$.

(ii) For $\zeta = 1$ and $k = \rho + \zeta$, the line $U_k \subseteq \mathrm{Aut}(X)^0$ is precisely the root group defined by the horizontal P-root $u(1, 0, \gamma_k)$ with $\gamma_k = l_{1n_1}$.

By a $\mathbb{K}^*$-*general point* of a rational projective $\mathbb{K}^*$-surface $X = X(A, P)$, we mean an $x \in X$ which is not a fixed point and not contained in any arm of X. Observe that a point $x \in X$ is $\mathbb{K}^*$-general if and only if each of its Cox coordinates is non-zero.

Lemma 13.4. *Let $X = X(A, P)$ admit a horizontal P-root u at (x^-, i_0, i_1). Then, for every $\mathbb{K}^*$-general $x \in X$, the root group $\lambda\colon \mathbb{K} \to \mathrm{Aut}(X)$ given by u satisfies*

$$D_{in_i} \cap \lambda(\mathbb{K}) \cdot x \;\neq\; \emptyset, \quad i \neq i_0, \qquad D_{ij} \cap \lambda(\mathbb{K}) \cdot x \;=\; \emptyset, \quad i = i_0 \text{ or } j \neq n_i.$$

Moreover, if there is a fixed point curve $D^+ \subseteq X$, then we have $\lambda(\mathbb{K}) \cdot x \cap D^+ = \emptyset$. Finally, we have $\lambda(\mathbb{K}) \cdot D_{i_0 n_{i_0}} = D_{i_0 n_{i_0}}$.

Proof. Theorem 5.4 yields $\bar{\lambda}(s)^*(S^+) = S^+$ in the case that there is a fixed point curve $D^+ \subseteq X$. Moreover it shows

$$\begin{aligned} i = i_0 :\quad \bar{\lambda}(s)^*(T_{i_0 n_{i_0}}) \;&=\; T_{i_0 n_{i_0}}, \\ i = i_1 :\quad \bar{\lambda}(s)^*(T_{i_1 n_{i_1}}) \;&=\; T_{i_1 n_{i_1}} + s\delta_u(T_{i_1 n_{i_1}}), \\ i \neq i_0, i_1 :\quad \bar{\lambda}(s)^*(T_{i n_i}) \;&=\; T_{i n_i} + \textstyle\sum_{\nu=1}^{l_{i_1 n_{i_1}}} \alpha(s, \nu, i)\delta_{u_{\nu,i}}(T_{i n_i}), \\ j \neq n_i :\quad \bar{\lambda}(s)^*(T_{ij}) \;&=\; T_{ij}. \end{aligned}$$

Thus, $\lambda(\mathbb{K}) \cdot x \cap D^+ = \emptyset$ and $D_{ij} \cap \lambda(\mathbb{K}) \cdot x = \emptyset$ provided $i = i_0$ or $j \neq n_i$. For $i \neq i_0$, a suitable choice of s yields $\bar{\lambda}(s)^*(T_{in_i}) = 0$ and hence $\lambda(s) \cdot x \in D_{in_i}$. $\square$

Proposition 13.5. *Let $X = X(A, P)$ be as in Construction 13.2. Then we have the following statements on the actions of the subgroups $G_1, \ldots, G_{\rho+\zeta} \subseteq \mathrm{Aut}(X)^0$.*

(i) *For each $k = 1, \ldots, \rho$, the group G_k acts almost transitively on X and at any point of its open orbit, G_k has cyclic isotropy group of order l_{1n_1}.*
(ii) *For $k = 1, \ldots, \rho$, the G_k-action turns X into an equivariant G_k-compactification if and only if $l_{1n_1} = 1$ holds.*
(iii) *For $\zeta = 1$ and $k = \rho + \zeta$, the group G_k acts almost transitively on X and at any point of its open orbit, G_k has cyclic isotropy group of order l_{0n_0}.*
(iv) *For $\zeta = 1$ and $k = \rho+\zeta$, the G_k-action turns X into an equivariant G_k-compactification if and only if $l_{0n_0} = 1$ holds.*

Proof. We prove (i). Lemma 13.4 shows that each of the groups $G_k = U_k \rtimes_{\psi_k} \mathbb{K}^*$ acts almost transitively; see also [**3**, Cor. 5.11]. Now, fix k and set $G := G_k$. For the $\mathbb{K}^*$-general point $x \in X$, the isotropy group G_x projects via $G \to \mathbb{K}^*$ isomorphically onto a finite cyclic group. Take a generator $g \in G_x$. As g is semisimple, we have $sgs^{-1} \in \mathbb{K}^*$ for suitable $s \in U_k$. Thus $sG_xs^{-1} = G_{s\cdot x} = \mathbb{K}^*_{s\cdot x}$. Remark 13.3 and Lemma 13.4 yield the assertion. Assertion (i) is a direct consequence of (ii). Assertions (iii) and (iv) are proven in the same way. □

Together with Remark 13.3, the above Proposition gives us in particular the following.

Corollary 13.6. *Let $X = X(A, P)$ be non-toric. Then* $\mathrm{Aut}(X)$ *acts almost transitively on X if and only if X admits horizontal P-roots.*

Besides the obvious subgroups $G_k \subseteq \mathrm{Aut}(X)^0$ acting almost transitively on X, we sometimes also encounter the following more hidden family of two-dimensional subgroups of $\mathrm{Aut}(X)^0$.

Construction 13.7. Let $X = X(A, P)$ be non-toric with $x^- \in X$ and P normalized. Assume $\rho \geq 1$, $\zeta = 1$ and consider

$$\mathrm{Aut}(X)^0 \;=\; \left(\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta\right) \rtimes_\psi \mathbb{K}^*.$$

Then every choice of a non-zero element $w_\rho \in \mathbb{K}$ gives rise to a one-dimensional subgroup of $\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$ via

$$U(w_\rho) \;:=\; \{(s^\rho w_1, s^{\rho-1} w_2, \ldots, s w_\rho, s);\; s \in \mathbb{K}\},$$

$$w_k \;:=\; \frac{1}{(\rho - k + 1)} \binom{\rho - 1}{k - 1} w_\rho.$$

If $l_{0n_0} = \rho$ and $l_{1n_1} = 1$ hold, then $U(w_\rho) \subseteq \mathrm{Aut}(X)^0$ is normalized by $\mathbb{K}^*$ and thus gives rise to a two-dimensional subgroup

$$G(w_\rho) \;:=\; U(w_\rho) \rtimes_\psi \mathbb{K}^* \;\subseteq\; \mathrm{Aut}(X)^0.$$

Remark 13.8. In the setting of Construction 13.7, assume $\rho = l_{0n_0} = l_{1n_1} = 1$. Then the unit component of the automorphism group of X is given by

$$\mathrm{Aut}(X)^0 \;=\; \mathbb{K}^2 \rtimes_\psi \mathbb{K}^*, \qquad \psi(t) \;=\; \mathrm{diag}(t^{-1}, t^{-1}).$$

Moreover, the subgroups U_1, U_2 and $U(w_1)$, where $w_1 \in \mathbb{K}^*$ are precisely the lines through the origin of $\mathbb{K}^2$.

The fact that $U(w_\rho) \subseteq \mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$ is indeed a subgroup as well as the condition for being normalized by $\mathbb{K}^*$ are direct consequences of the subsequent two more general observations.

Proposition 13.9. *Let $X = X(A, P)$ be non-toric with $x^- \in X$ and P normalized. Assume $\rho \geq 1$ and $\zeta = 1$. Consider a subset of $\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$ of the form*

$$U(w) \ := \ \{(w(s), s);\ s \in \mathbb{K}\}, \qquad w(s) \ := \ (s^{\gamma_1} w_1, \ldots, s^{\gamma_\rho} w_\rho),$$

with a given non-zero $w = (w_1, \ldots, w_\rho) \in \mathbb{K}^\rho$ and given integers $\gamma_1 > \ldots > \gamma_\rho > 0$. Then $U(w) \subseteq \mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$ is a subgroup if and only if $U(w) = U(w_\rho)$ holds.

Proof. First, recall from Theorem 0.1 the matrix $A(s)$ defining the twisting homomorphism $\varphi\colon \mathbb{K}^\zeta \to \mathrm{Aut}(\mathbb{K}^\rho)$. The subset $U(w)$ of $\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$ is a subgroup if and only if

$$(w(r), r) \circ (w(s), s) \ = \ (w(r) + A(r) \cdot w(s), r + s) \ = \ (w(r+s), r+s)$$

holds for any two $r, s \in \mathbb{K}$. Now, for $k = 1, \ldots, \rho$, the k-th coordinate of the second of the above equalities gives us the following identities of polynomials p_k and q_k in r, s, altogether characterizing the subgroup property of $U(w)$:

$$\begin{aligned} p_k(r,s) \ &:= \ w_k r^{\gamma_k} + \sum_{i=k}^{\rho} \binom{i-1}{k-1} w_i r^{i-k} s^{\gamma_i} \\ &= \ w_k r^{\gamma_k} + \sum_{i=0}^{\rho-k} \binom{k+i-1}{k-1} w_{k+i} r^i s^{\gamma_{k+i}} \\ &= \ w_k (r+s)^{\gamma_k} \\ &= \ \sum_{i=0}^{\gamma_k} \binom{\gamma_k}{i} w_k r^i s^{\gamma_k - i} \\ &=: \ q_k(r,s). \end{aligned}$$

We claim that if $U(w)$ is a subgroup of $\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$, then $w_1, \ldots, w_\rho \in \mathbb{K}^*$ holds. Otherwise, let k be minimal with $w_k = 0$. Then $p_k = q_k$ implies $w_i = 0$ for $i = k, \ldots, \varrho$. Due to $w \neq 0$, we have $k > 1$. The equation $p_{k-1} = q_{k-1}$ yields

$$r^{\gamma_{k-1}} + s^{\gamma_{k-1}} \ = \ \sum_{i=0}^{\gamma_{k-1}} \binom{\gamma_{k-1}}{i} r^i s^{\gamma_{k-1} - i}.$$

This is only possible for $\gamma_{k-1} = 1$. As we have $\gamma_1 > \ldots > \gamma_\rho > 0$, we conclude $k - 1 = \rho$. A contradiction to the choice of k. Thus, if $U(w)$ is a subgroup of $\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$, then we must have $w_1, \ldots, w_\rho \in \mathbb{K}^*$.

This reduces our task to showing that a given $U(w)$ with $w_1, \ldots, w_\rho \in \mathbb{K}^*$ is a subgroup of $\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$, if and only if $U(w)$ equals $U(w_\rho)$ from Construction 13.7. Comparing the number of terms of p_k and q_k, we obtain

$\gamma_k = \rho - k + 1$. Now, comparing the coefficients of p_k and q_k leads to the following identities, characterizing the subgroup property of $U(w)$ by

$$(\star) \qquad \binom{k+i-1}{k-1} w_{k+i} \;=\; \binom{\gamma_k}{i} w_k \;=\; \binom{\rho-(k-1)}{i} w_k,$$

where $k = 1, \ldots, \rho$ and $i = 0, \ldots, \gamma_k$. In particular, taking k and $i = \rho - k$, the identities $(\star)$ bring us to the conditions

$$w_\rho \;=\; w_{k+(\rho-k)} \;=\; \binom{\rho-k+1}{\rho-k}\binom{\rho-1}{k-1}^{-1} w_k,$$

which in turn are equivalent to the defining conditions of $U(w_\rho)$ from Construction 13.7. Conversely, we retrieve the characterizing identities $(\star)$ from the above conditions by an explicit computation:

$$\begin{aligned} \binom{k+i-1}{k-1} w_{k+i} \;&=\; \binom{k+i-1}{k-1}\binom{\rho-1}{k+i-1}\binom{\rho-(k+i)+1}{\rho-(k+i)}^{-1} w_\rho \\ &=\; \binom{\rho-(k-1)}{i} w_k. \end{aligned}$$

□

Proposition 13.10. *Let $X = X(A, P)$ be non-toric with $x^- \in X$ and P normalized. Assume $\rho \geq 1$, $\zeta = 1$. For any one-dimensional closed subgroup $U \subseteq \mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$ neither contained in $\mathbb{K}^\rho$ nor in $\mathbb{K}^\zeta$ the following statements are equivalent.*

(i) *The group $U \subseteq \mathrm{Aut}(X)^0$ is normalized by $\mathbb{K}^*$.*
(ii) *We have $l_{0n_0} = \rho$, $l_{1n_1} = 1$ and $U = U(w_\rho)$.*

Proof. First assume that U is normalized by $\mathbb{K}^*$. By assumption we have $\zeta = 1$ and the projection $U \to U_{\rho+\zeta} \cong \mathbb{K}$ is surjective. As U is unipotent, $U \to U_{\rho+\zeta}$ is an isomorphism. In particular, there is a unique element $w = (w_1, \ldots, w_\rho, 1) \in U$ with $w_k \in \mathbb{K}$ and $w_k \neq 0$ at least once. As U is normalized by $\mathbb{K}^*$, we obtain

$$(0, t) \circ (w, 1) \circ (0, t^{-1}) \;=\; (\psi(t)(w), 1) \;=\; (t^{\gamma_1} w_1, \ldots, t^{\gamma_\rho} w_\rho, t^{l_1}, 1)$$

for all $t \in \mathbb{K}^*$, where we set $\gamma_k := l_{0n_0} - (k-1) l_{1n_1}$ and $l_1 := l_{1n_1}$ for the moment. The right hand side gives us a parametric representation of the variety $U \subseteq \mathbb{K}^\rho \times \mathbb{K}^\zeta$. Thus, setting $c_k := w_k^{l_1}$, we obtain defining equations for $U \subseteq \mathbb{K}^\rho \times \mathbb{K}^\zeta$ by

$$T_k^{l_1} - c_k T_{\rho+\zeta}^{\gamma_k} \;=\; 0, \qquad k = 1, \ldots, \rho.$$

Observe that γ_k and l_1 are coprime due to smoothness of $x^- \in X$; see Proposition 6.10. Since at least one of the w_k is non-zero and $0 \in \mathbb{K}^\rho \times \mathbb{K}^\zeta$ is a smooth point of U, we conclude $l_1 = 1$. Thus, setting $l_0 = l_{0n_0}$, we have

$$U = \{(w(s), s);\ s \in \mathbb{K}\} \subseteq \mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta, \quad w(s) := (s^{l_0} w_1, s^{l_0-1} w_2, \ldots, s^{l_0-\rho+1} w_\rho).$$

Since U is a subgroup of $\mathbb{K}^\rho \times \mathbb{K}^\zeta$, Proposition 13.9 yields that we have $l_0 = \rho$ and the w_k arise from a $w_\rho \in \mathbb{K}^*$ as in Construction 13.7. Conversely, if (ii) holds, then one directly checks that $U = U(w_\rho)$ is normalized by $\mathbb{K}^*$. □

PROPOSITION 13.11. *Let $X = X(A, P)$ be non-toric with $x^- \in X$ and P normalized. Assume $\rho \geq 1$, $\zeta = 1$ and $l_{0n_0} = \rho$, $l_{1n_1} = 1$. Then $G(w_\rho) \subseteq \mathrm{Aut}(X)^0$ acts almost transitively and the isotropy group of a general $x \in X$ is cyclic of order $l_{0n_0} = \rho$.*

PROOF. Set for short $G = G(w_\rho)$. It suffices to show that for the general point $x \in X$, the isotropy group G_x is cyclic of order l_{0n_0}. For this note first that any element $\vartheta \in U(w_\rho)$ decomposes as

$$\vartheta \ = \ \vartheta_\rho \circ \vartheta_\zeta, \qquad \vartheta_\rho \in \mathbb{K}^\rho, \qquad \vartheta_\zeta \in \mathbb{K}^\zeta.$$

Using Lemma 13.4, we see that there are an $\vartheta \in U(w_\rho)$ and a $\mathbb{K}^*$-general $x \in X$ such that $\vartheta_\zeta(x) \in D_{0n_0}$. Applying Lemma 13.4 again shows $\vartheta_\rho(\vartheta_\zeta(x)) \in D_{0n_0}$. As in the proof of Proposition 13.5, we conclude that G_x is cyclic of order l_{0n_0}. $\square$

PROPOSITION 13.12. *Let $X = X(A, P)$ be non-toric with $x^- \in X$ and P normalized. Let $G \subseteq \mathrm{Aut}(X)^0$ be a two-dimensional subgroup containing $\mathbb{K}^*$ and acting almost transitively on X.*

(i) *If $G \subseteq \mathbb{K}^\rho \rtimes_\psi \mathbb{K}^*$ or $G \subseteq \mathbb{K}^\zeta \rtimes_\psi \mathbb{K}^*$ holds, then G is equal to one of the subgroups $G_1, \ldots, G_{\rho+\zeta}$.*
(ii) *If neither $G \subseteq \mathbb{K}^\rho \rtimes_\psi \mathbb{K}^*$ nor $G \subseteq \mathbb{K}^\zeta \rtimes_\psi \mathbb{K}^*$, then $l_{0n_0} = \rho$, $l_{1n_1} = 1$ and $G = G(w_\rho)$ with $w_\rho \in \mathbb{K}^*$.*

Up to conjugation, items (i) and (ii) list all closed two-dimensional subgroups of $\mathrm{Aut}(X)^0$ that act almost transitively on X and have a maximal torus of dimension one.

PROOF. We show (i). For $G \subseteq \mathbb{K}^\zeta \rtimes_\varphi \mathbb{K}^*$, the group G equals $G_{\rho+\zeta}$ by dimension reasons. Assume $G \subseteq \mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^*$. By assumption, $p\colon G \to \mathbb{K}^*$ is surjective. Thus, $U := \ker(p)$ is a one-dimensional subgroup of $\mathbb{K}^\rho$ and hence a line. Moreover, we find an element $(w, 1) \in G$, where $w \in U$. Then, for all $t \in \mathbb{K}^*$, we have

$$(0, t) \circ (w, 1) \circ (0, t^{-1}) \ = \ (\psi(t)(w), 1) \ = \ (t^{\gamma_1} w_1, \ldots, t^{\gamma_\rho} w_\rho, 1),$$

where we set $\gamma_k := \tilde{l}_{0\tilde{n}_0} - (k-1) l_{1n_1}$ as before and evaluate the twisting homomorphism $\psi\colon \mathbb{K}^* \to \mathrm{Aut}(\mathbb{K}^\rho)$ according to its definition. In particular,

$$(t^{\gamma_1} w_1, \ldots, t^{\gamma_\rho} w_\rho) \ \in \ U \ \subseteq \ \mathbb{K}^\rho$$

holds for all $t \in \mathbb{K}^*$. As the γ_k are pairwise distinct, we see that U is one of the U_i. Now, using again the definition of the twisting homomorphism $\psi\colon \mathbb{K}^* \to \mathrm{Aut}(\mathbb{K}^\rho)$, we arrive at the assertion.

We turn to (ii). Surjectivity of $p\colon G \to \mathbb{K}^*$ implies that $U := \ker(p)$ is a one-dimensional subgroup of $\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$. Because of $\mathbb{K}^* \subseteq G$, the subgroup $U \subseteq \mathrm{Aut}(X)^0$ is normalized by $\mathbb{K}^*$. Proposition 13.10 shows $l_{0n_0} = \rho$ and $l_{1n_1} = 1$ as well as $U = U(w_\rho)$. In particular, we arrive at $G = G(w_\rho)$. $\square$

PROPOSITION 13.13. *Consider $X = X(A, P)$ with $x^- \in X$ and P normalized. Then each of the subgroups $\mathbb{K}^\rho, \mathbb{K}^\zeta \subseteq \mathrm{Aut}(X)^0$ acts with orbits of dimension at most one. Moreover, any subgroup $G \subseteq \mathbb{K}^\rho \rtimes_\psi \mathbb{K}^\zeta$ containing U_1 and $U_{\rho+\zeta}$ acts almost transitively.*

Proof. Consider $X' \leftarrow \tilde{X} \rightarrow X$ as provided by Theorem 11.1 and Proposition 12.3. Then $\mathbb{K}^\rho$ is generated by the root groups stemming from horizontal P-roots at $(x^-, 0, 1)$, see Propositions 10.2 and 10.3. Thus, the corresponding subgroup of $\mathrm{Aut}(X')$ is generated by the root groups coming from Demazure roots at a common ray. Looking at the resulting root groups of X' in Cox coordinates, we directly see that $\mathbb{K}^\rho$ acts with at most one-dimensional orbits. For $\mathbb{K}^\zeta$ and G, we succeed by the same idea. □

Proposition 13.14. *Let $X = X(A, P)$ be non-toric with $x^- \in X$ and P normalized. Assume $\rho \geq 1$ and $\zeta = 1$ and consider $G := U_1 \rtimes_\varphi U_{\rho+\zeta} \subseteq \mathrm{Aut}(X)^0$.*

(i) *The group G is isomorphic to the vector group $\mathbb{K}^2$.*
(ii) *The G-action turns X into an equivariant G-compactification.*
(iii) *The subgroup $G \subseteq \mathrm{Aut}(X)^0$ is normalized by $\mathbb{K}^* \subseteq \mathrm{Aut}(X)^0$.*

Proof. Assertion (i) can be directly deduced from the structure of $\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$. We show (ii). According to Proposition 13.13, the group G acts with an open orbit. In particular, its general isotropy group is finite, hence trivial due to (i). Assertion (iii) is again directly checked. □

Proof of Theorem 0.2. Clearly, (i) implies (ii). Assume that (ii) holds. Then we infer from Proposition 13.1 that there must be horizontal P-roots. Thus, we may assume that there is a fixed point $x^- \in X$ and that P is normalized. Then we have $\rho + \zeta > 0$ and this translates to (iii), see Proposition 12.2. Now, if (iii) holds, then there is a horizontal P-root and an associated root group as in Remark 13.3. By Proposition 13.5, this yields an almost transitive action of $G = \mathbb{K} \rtimes \mathbb{K}^* \subseteq \mathrm{Aut}(X)^0$ on X. So, we made our way back to (i).

We turn to the supplement concerning the case that a two-dimensional subgroup $G \subseteq \mathrm{Aut}(X)$ acts almost transitively on X. First note that G is either solvable with one-dimensional maximal torus or G is unipotent. Thus, we either can assume by suitably conjugating that $\mathbb{K}^* \subseteq G$ holds or we must have $G \cong \mathbb{K} \rtimes \mathbb{K}$ and hence $G \cong \mathbb{K}^2$. Then (iv) is covered by Propositions 13.5, 13.11 and 13.12. For (v) observe first that both series of inequalities being valid means $\rho \geq 1$ and $\zeta = 1$ due to Proposition 12.2. Then the assertion is a direct consequence of (iv) and Proposition 13.14. □

Finally, we descibe all the subgroups $G \subseteq \mathrm{Aut}(X)$ that are isomorphic to the vector group $\mathbb{K}^2$ and act almost transitively on X.

Proposition 13.15. *Let $X = X(A, P)$ be non-toric with $x^- \in X$ and P normalized. Assume $\rho > 1$ and $\zeta = 1$. For $0 \neq w_\rho \in \mathbb{K}$, set*

$$V(w_\rho) \ := \ U_1 U(w_\rho) \ \subseteq \ \mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta,$$

where U_1 and $U(w_\rho)$ are the subgroups of $\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$ from Constructions 13.2 and 13.15. Then the following holds.

(i) *$V(w_\rho)$ is a subgroup of $\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$, isomorphic to $\mathbb{K}^2$.*
(ii) *$V(w_\rho)$ is normalized by $\mathbb{K}^*$ in $\mathrm{Aut}(X)^0$ if and only if $l_{0n_0} = \rho$ and $l_{1n_1} = 1$.*
(iii) *$V(w_\rho)$ acts almost transitively on X.*

Proof. We show (i). Clearly, $U_1 \cap U(w_\rho)$ contains only the zero element. Moreover, we directly see that the each element of U_1 commutes with each element of $U(w_\rho)$. Thus, $V(w_\rho)$ is a subgroup of $\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta$ isomorphic to the vector group $\mathbb{K}^2$. Assertion (ii) holds because U_1 is normalized by $\mathbb{K}^*$ and $U(w_\rho)$ is normalized by $\mathbb{K}^*$; see Proposition 13.10 for the latter. For (iii), we use Theorem 5.4 and Lemma 13.4 to show that D_{0n_0} and D_{1n_1} lie in the orbit of $V(w_\rho)$ through $x^- \in X$. Thus the orbit of V_ρ through $x^- \in X$ is open in X. □

Remark 13.16. Let $X = X(A, P)$ be non-toric with $x^- \in X$ and P normalized. Assume $\rho \geq 1$ and $\zeta = 1$.

(i) If $l_{0n_0} = \rho$ and $l_{1n_1} = 1$ hold, then we have a one-parameter family of one-dimensional unipotent subgroups

$$\{U(w_\rho);\ w_r \in \mathbb{K}^*\} \ = \ \{tU(1)t^{-1};\ t \in \mathbb{K}^*\} \ \subseteq \ \mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta,$$

and we have a one-parameter family of subgroups isomorphic to the vector group $\mathbb{K}^2$:

$$\{tV(1)t^{-1};\ t \in \mathbb{K}^*\} \ \subseteq \ \mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta.$$

(ii) If $l_{0n_0} \neq \rho$ or $l_{1n_1} \neq 1$ holds, then we have a two-parameter family of one-dimensional unipotent subgroups

$$\{tU(w_\rho)t^{-1};\ w_\rho \in \mathbb{K}^*, t \in \mathbb{K}^*\} \ \subseteq \ \mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta,$$

and we have a two-parameter family of subgroups isomorphic to the vector group $\mathbb{K}^2$:

$$\{tV(w_\rho)t^{-1};\ w_\rho \in \mathbb{K}^*, t \in \mathbb{K}^*\} \ \subseteq \ \mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta.$$

Proposition 13.17. *Let $X = X(A, P)$ be non-toric with $x^- \in X$ and P normalized. Then X admits additive actions if and only if $\rho \geq 1$ and $\zeta = 1$. In this case, the additive actions on X are given by the groups $G = U_1 \rtimes_\varphi U_{\rho+\zeta}$ and, up to conjugation by elements from $\mathbb{K}^*$, the groups $G = V(w_\varrho)$, where $w_\rho \in \mathbb{K}^*$.*

Proof. The first statement is clear by Propositions 13.14 and 13.13. For the second one, we consider once more $X' \leftarrow \tilde{X} \rightarrow X$ as provided by Theorem 11.1 and Proposition 12.3. This realizes $\mathrm{Aut}(X)^0$ as a subgroup of the automorphism group of the smooth toric surface X' in such a way that $\mathbb{K}^*$ becomes a subtorus of the (two-dimensional) acting torus $\mathbb{T}'$ of X'. In particular, $U_1 \rtimes_\varphi U_{\rho+\zeta}$ and the family $\mathbb{V}$ defined by $V(1)$ in the sense of Remark 13.16 show up in of $\mathrm{Aut}(X')$, where $\mathbb{V}$ is a locally closed subvariety isomorphic to a torus of dimension one or two. According to [**17**, Thm. 3], there are only two additive actions on X' up to conjugation by $\mathbb{T}'$. One them is normalized by $\mathbb{T}'$. Proposition 4.4 (iii) tells us that this is $U_1 \rtimes_\varphi U_{\rho+\zeta}$. The other additive action G' has a non-trivial orbit $\mathbb{T}' * G'$ under the $\mathbb{T}'$-action on $\mathrm{Aut}(X')$ via conjugation. Thus, $\mathbb{T}' * G'$ is isomorphic as a variety either to a one-dimensional torus or to a two-dimensional torus. This reflects exactly the cases (i) and (ii) of Remark 13.16. Thus, injectivity of the morphism $\mathrm{Aut}(X)^0 \rightarrow \mathrm{Aut}(X')^0$ gives the assertion. □

14. Examples

In this section we want to construct rational projective $\mathbb{K}^*$-surfaces $X(A,P)$ with horizontal and vertical P-roots to gain a deeper understanding of Theorem 0.1.

In the whole section we consider horizontal P-roots at elliptic fixed points x^- and vertical P-roots at a parabolic fixed point curve D^+. Moreover the defining matrices P are adapted to the source or sink, when considering horizontal P-roots or vertical P-roots, respectively. Additionally in the first case the defining matrix P is normalized.

The first two examples show that there are surfaces with horizontal P-roots at $(x^-,0,1)$, but not at $(x^-,1,0)$ and vice versa. In particular, there is no relation between the exponents ρ and ζ of $\mathrm{Aut}(X)^0 = \left(\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta\right) \rtimes_\psi \mathbb{K}^*$.

Remark 14.1. Let $X(A,P)$ be a rational projective $\mathbb{K}^*$-surface with an elliptic fixed point $x^- \in X(A,P)$. Recall the mutually inverse bijection between integers in the interval $\Gamma(i_0,i_1)$ and horizontal P-roots at (x^-,i_0,i_1) as seen in Proposition 9.11.

For $0 \le i,k \le r$, define rational numbers

$$\eta_k \ := \ -\frac{1}{l_{kn_k} m^-}, \qquad\qquad \xi_i \ := \ \begin{cases} 0, & n_i = 1, \\ \frac{1}{l_{in_i}(m_{in_i-1}-m_{in_i})}, & n_i \ge 2. \end{cases}$$

For any two $0 \le \iota,\kappa \le r$, we define the intersections

$$\Delta(\iota,\kappa) \ = \ \bigcap_{i \ne \iota} [\xi_i, \eta_\kappa] \ \subseteq \ \mathbb{Q}_{\ge 0}.$$

We find horizontal P-roots at (x^-,i_0,i_1) if and only if there are integers $\gamma \in \Delta(i_0,i_1)$ such that $\gamma d_{i_1 n_{i_1}} \equiv -1 \bmod l_{i_1 n_{i_1}}$.

Example 14.2. We consider the following rational projective $\mathbb{K}^*$-surface $X(A,P)$ with defining data:

$$P \ := \ \begin{bmatrix} -3 & 5 & 2 & 0 & 0 \\ -3 & 0 & 0 & 1 & 1 \\ -2 & 3 & 1 & 1 & 0 \end{bmatrix}, \qquad A \ := \ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

The only horizontal P-root is given by the linear form $u = (-1,0,2)$. It is a horizontal P-root at $(x^-,1,0)$. In particular there are no horizontal P-roots at $(x^-,0,1)$ since the following holds:

$$\xi_1 \ = \ 5, \quad \xi_2 \ = \ 1, \quad \eta_1 \ = \ 3, \quad \text{i.e. } \Delta(0,1) \ = \ \bigcap_{i=1,2} [\xi_i, 3] \ = \ \emptyset.$$

Example 14.3. We consider the following rational projective $\mathbb{K}^*$-surface $X(A,P)$ with defining data:

$$P \ := \ \begin{bmatrix} -3 & 2 & 0 & 0 \\ -3 & 0 & 3 & 1 \\ -2 & 1 & 1 & 0 \end{bmatrix}, \qquad A \ := \ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

The only horizontal P-root at x^- is given by the linear form $u = (-2,0,3)$. It is a horizontal P-root at $(x^-,0,1)$. Moreover, there is no horizontal P-root at $(x^-, 1, 0)$ since the following holds:

$$\xi_0 \;=\; 0, \quad \xi_2 \;=\; 3, \quad \eta_0 \;=\; 2, \qquad \text{i.e. } \Delta(0,1) \;=\; \bigcap_{i=0,2} [\xi_i, 2] \;=\; \emptyset.$$

Note that x^- is smooth, in particular we find that smoothness of x^- does not imply $\zeta = 1$.

Recall that for a $\mathbb{K}^*$-surface with P normalized the exponent ρ in the automorphism group can be expressed in terms of self intersection numbers of invariant curves, see Theorem 0.1:

$$\rho \;=\; \max\left(0,\; \left\lfloor l_{1n_1}^{-1} \min_{i \neq 0}(l_{in_i} D_{in_i}^2 + (l_{in_i} - l_{1n_1}) D_{in_i} D_{1n_1}) - c(x^-) \right\rfloor + 1\right)$$

The next example illustrates that it is necessary to use the integral part in the formula.

Example 14.4. Let $X(A,P)$ be the smooth rational projective $\mathbb{K}^*$-surface defined by the following matrices:

$$P \;:=\; \begin{bmatrix} -1 & -2 & -3 & 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & -2 & -3 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}, \qquad A \;:=\; \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Note that there is precisely one horizontal P-root at $(x^-, 0, 1)$, given by the linear form $u = (-2, 0, 3)$, i.e. we obtain $\rho = 1$.

Since $X(A,P)$ is smooth, we have $c(x^-) = 0$. Moreover we have:

$$\begin{aligned} l_{1n_1} D_{1n_1}^2 + (l_{1n_1} - l_{1n_1}) D_{1n_1} D_{1n_1} \;&=\; l_{1n_1} D_{1n_1}^2, \;=\; 2 \\ l_{2n_2} D_{2n_2}^2 + (l_{2n_2} - l_{1n_1}) D_{2n_2} D_{1n_1} \;&=\; 1. \end{aligned}$$

Therefore by the formula above, ρ is the maximum of zero and the following expression:

$$\left\lfloor l_{1n_1}^{-1} \min(1,2) \right\rfloor + 1 \;=\; \left\lfloor \frac{1}{2} \right\rfloor + 1 \;=\; 1.$$

The next three examples show that for every $(\rho, \zeta) \in \mathbb{Z}_{\geq 0} \times \{0,1\}$ there is a rational projective $\mathbb{K}^*$-surface $X(A,P)$.

Example 14.5. Let $\rho \in \mathbb{Z}_{\geq 1}$ and consider the rational projective $\mathbb{K}^*$-surface $X(A, P(\rho))$ with the following defining data:

$$P(\rho) \;:=\; \begin{bmatrix} -4\rho & 2 & 0 & 0 \\ -4\rho & 0 & 1 & 1 \\ -2\rho - 1 & 1 & 1 & 0 \end{bmatrix}, \quad A \;:=\; \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Observe that there are ρ-many distinct horizontal P-roots at $(x^-, 0, 1)$ given by the following linear forms:

$$u(\gamma) \;=\; (-1 - \gamma, 0, 1 + 2\gamma), \qquad 0 \leq \gamma \leq \rho - 1.$$

Note that since the elliptic fixed point x^- is not smooth we know by Proposition 9.12 (i) that there is no horizontal P-root at $(x^-, 1, 0)$. In particular, we have $\mathrm{Aut}(X(A,P))^0 = \mathbb{K}^\rho \rtimes_\psi \mathbb{K}^*$.

Example 14.6. Let $\rho \in \mathbb{Z}_{\geq 1}$ and consider the rational projective $\mathbb{K}^*$-surface $X(A, P(\rho))$ with the following defining data:

$$P(\rho) \;:=\; \begin{bmatrix} -(2\rho-1) & 2 & 0 & 0 \\ -(2\rho-1) & 0 & 1 & 1 \\ -\rho & 1 & 1 & 0 \end{bmatrix}, \qquad A \;:=\; \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Observe that there are ρ-many distinct horizontal P-roots at $(x^-, 0, 1)$ given by the following linear forms:

$$u(\gamma) \;=\; (-1-\gamma, 0, 1+2\gamma), \qquad 0 \leq \gamma \leq \rho - 1.$$

Furthermore the linear form $u = (-1, 0, 2)$ defines a horizontal P-root at $(x^-, 1, 0)$. In particular, we have $\mathrm{Aut}(X(A,P))^0 = (\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}) \rtimes_\psi \mathbb{K}^*$.

Example 14.7. Let $\rho \in \mathbb{Z}_{\geq 1}$ and consider the rational projective $\mathbb{K}^*$-surface $X(A, P(\rho))$ with the following defining data:

$$P(\rho) \;:=\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ -\rho & -\rho-1 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}, \qquad A \;:=\; \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Oow observe that there are ρ-many distinct vertical P-roots at D^+ essential to the indices $0, 1$ given by the following linear forms:

$$u(\alpha) \;=\; (\alpha, 0, -1), \qquad 0 \leq \gamma \leq \rho - 1.$$

In particular, we have $\mathrm{Aut}(X(A,P))^0 = \mathbb{K}^\rho \rtimes_\psi \mathbb{K}^*$.

The last examples gives an inside in the results of Section 10. In the first part of this Section we studied rational projective $\mathbb{K}^*$-surfaces with quasismooth elliptic fixed points x^- such that $l_{in_i} = 1$ for more than $r-1$ indices.

Proposition 10.5 gave relations among horizontal P-roots, which we illustrate with the following example

Example 14.8. Let $X = X(A,P)$ be the rational projective $\mathbb{K}^*$-surface with the following defining data:

$$P \;:=\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \end{bmatrix}, \qquad A \;:=\; \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

There are six horizontal P-roots at the elliptic fixed point x^- given as:

linear form u	pair of indices i_0, i_1
$u_0 = (0,0,1)$	$i_0 = 1$, $i_1 = 0$
u_0	$i_0 = 2$, $i_1 = 0$
$u_1 = (-1,0,1)$	$i_0 = 0$, $i_1 = 1$
u_1	$i_0 = 2$, $i_1 = 1$
$u_2 = (0,-1,1)$	$i_0 = 0$, $i_1 = 2$
u_2	$i_0 = 1$, $i_1 = 2$

We note that for any pair of indices $i \neq \iota$ we have $(u_i)_{1,\iota} = u_\iota$. Thus, setting $\tau(i_0, i_1) := (u_{i_1}, i_0, i_1, (2,2,2))$ for the corresponding Demazure P-root and $\{i_0, i_1, i_2\} = \{0, 1, 2\}$ we find the following representation according to Theorem 5.4:

$$\lambda_{\tau(i_0,i_1)}(s) \;=\; \lambda_{u_{i_1}}(s) \circ \lambda_{u_{i_2}}(-s)|_X.$$

The following equalities show that the pair of indices $0, 1$ is a pair of generating indices:

$$\lambda_{\tau(0,2)}(s) \;=\; \lambda_{\tau(0,1)}(-s), \qquad \lambda_{\tau(1,2)}(s) \;=\; \lambda_{\tau(1,0)}(-s),$$

$$\lambda_{\tau(2,0)}(s) \;=\; \lambda_{\tau(1,0)}(s) \circ \lambda_{\tau(0,1)}(-s), \qquad \lambda_{\tau(2,1)}(s) \;=\; \lambda_{\tau(0,1)}(s) \circ \lambda_{\tau(1,0)}(-s).$$

Note that the first line above is exactly the first statement of Proposition 10.5 and the second line is exactly the second statement of the same Proposition.

15. The Gorenstein log del Pezzo case

The Gorenstein log del Pezzo $\mathbb{K}^*$-surfaces have been classified in [**32**], see also [**2**, Chapter 5.4.4].

We want to use Theorem 0.1 to obtain all unit components of the automorphism groups of Gorenstein log del Pezzo $\mathbb{K}^*$-surfaces and compare the results to recent classifications of surfaces with infinite automorphism groups and equivariant compactifications.

Algorithm 15.1. *Input*: matrices A and P defining a rational projective $\mathbb{K}^*$-surface $X(A, P)$.

- Test whether there is a parabolic fixed point curve of positive self intersection. If there is one, D^+ say, do the following:
  - For every $i = 0, \ldots, r$ find a Hilbert basis for $\mathrm{cone}(v_{i1}, e_{r+1})$.
  - Calculate $c(D^+)$ and ρ as in Theorem 0.1 (i).
  - If $\rho > 0$, return $\mathbb{K}^\rho$ and ψ as in Theorem 0.1 (i).
- Test whether there is a quasismooth elliptic fixed point x. If there is one, x^- say, do the following:
  - Normalize the matrix P.
  - Compute the Hilbert basis of $\mathrm{cone}((-l_{0n_0}, d_{0n_0}), (l_{1n_1}, d_{1n_1}))$ as in Construction 11.3.
  - If x^- is simple, compute ρ as given in Theorem 0.1 (ii).
  - Set $\zeta = 1$ if the conditions of Theorem 0.1 (ii) hold, else set $\zeta = 0$.
  - If $\rho > 0$ or $\zeta > 0$, return $\mathbb{K}^\rho \rtimes \mathbb{K}^\zeta$ and φ, ψ as in Theorem 0.1 (ii).
- If none of the above procedures yield $\rho \neq 0$ or $\zeta \neq 0$, return $\mathbb{K}^*$.

Remark 15.2. We use the following short notation for the automorphism ψ described in Theorem 0.1:

$$\psi_{q_1,\ldots,q_k} \colon \mathbb{K}^* \to \mathrm{Aut}(\mathbb{K}^\rho \rtimes \mathbb{K}^\zeta), \qquad t \mapsto \mathrm{diag}(t^{q_1}, \ldots, t^{q_k}).$$

Furthermore for $\zeta = 1$ observe that the automorphism $\varphi \colon \mathbb{K} \to \mathrm{Aut}(\mathbb{K}^\rho)$ is uniquely determined by ρ. Hence we will omit it and write $\mathbb{K}^\rho \rtimes \mathbb{K}$ for $\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}$.

Proposition 15.3. *The following table lists the Cox ring $\mathcal{R}(X)$ and the unit component of the automorphism group of all non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surfaces with $\mathrm{Aut}(X)^0 \neq \mathbb{K}^*$:*

no.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$[w_1, \ldots, w_r]$	$\mathrm{Aut}(X)^\circ$
1	$\frac{\mathbb{K}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 5 & 2 & 3\end{bmatrix}$	$(\mathbb{K}^2 \rtimes \mathbb{K}) \rtimes_{\psi_{3,1,2}} \mathbb{T}$
2	$\frac{\mathbb{K}[T_1,\ldots,T_4]}{\langle T_1^2T_2+T_3^3+T_4^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 4 & 2 & 3\end{bmatrix}$	$\mathbb{K}^2 \rtimes_{\psi_{3,2}} \mathbb{T}$
3	$\frac{\mathbb{K}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^2\rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1 & 3 & 1 & 2\\ \bar{1} & \bar{1} & \bar{0} & \bar{1}\end{bmatrix}$	$\mathbb{K} \rtimes_{\psi_1} \mathbb{T}$
4	$\frac{\mathbb{K}[T_1,\ldots,T_4]}{\langle T_1^3T_2+T_3^3+T_4^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 3 & 2 & 3\end{bmatrix}$	$\mathbb{K} \rtimes_{\psi_3} \mathbb{T}$
5	$\frac{\mathbb{K}[T_1,\ldots,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}^2$	$\begin{bmatrix}1 & 3 & 1 & 2 & 2\\ 1 & 1 & 0 & 2 & 1\end{bmatrix}$	$(\mathbb{K}^2 \rtimes \mathbb{K}) \rtimes_{\psi_{2,1,1}} \mathbb{T}$
6	$\frac{\mathbb{K}[T_1,\ldots,T_5]}{\langle T_1T_2+T_3^2T_4+T_5^2\rangle}$	$\mathbb{Z}^2$	$\begin{bmatrix}1 & 3 & 1 & 2 & 2\\ 0 & 2 & 1 & 0 & 1\end{bmatrix}$	$\mathbb{K}^2 \rtimes_{\psi_{2,1}} \mathbb{T}$
7	$\frac{\mathbb{K}[T_1,\ldots,T_5]}{\langle T_1T_2+T_3T_4+T_5^3\rangle}$	$\mathbb{Z}^2$	$\begin{bmatrix}1 & 2 & 1 & 2 & 1\\ 1 & -1 & -1 & 1 & 0\end{bmatrix}$	$\mathbb{K} \rtimes_{\psi_1} \mathbb{T}$
8	$\frac{\mathbb{K}[T_1,\ldots,T_5]}{\langle T_1T_2+T_3^3T_4+T_5^2\rangle}$	$\mathbb{Z}^2$	$\begin{bmatrix}1 & 3 & 1 & 1 & 2\\ 1 & 1 & 0 & 2 & 1\end{bmatrix}$	$\mathbb{K} \rtimes_{\psi_1} \mathbb{T}$
9	$\frac{\mathbb{K}[T_1,\ldots,T_5]}{\langle T_1^2T_2+T_3^2T_4+T_5^2\rangle}$	$\mathbb{Z}^2$	$\begin{bmatrix}1 & 2 & 1 & 2 & 2\\ 1 & 0 & 0 & 2 & 1\end{bmatrix}$	$\mathbb{K} \rtimes_{\psi_2} \mathbb{T}$
10	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\mathbb{Z}^3$	$\begin{bmatrix}1 & -1 & -1 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1 & 0 & 1\end{bmatrix}$	$\mathbb{K}^2 \rtimes_{\psi_{1,1}} \mathbb{T}$
11	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$	$\mathbb{Z}^3$	$\begin{bmatrix}1 & 2 & 1 & 2 & 1 & 1\\ 1 & 0 & 0 & 1 & 0 & 1\\ 1 & -1 & -1 & 1 & 0 & 0\end{bmatrix}$	$\mathbb{K} \rtimes_{\psi_1} \mathbb{T}$

REMARK 15.4. Recall the group structure of $(\mathbb{K}^\rho \rtimes_\varphi \mathbb{K}^\zeta) \rtimes_\psi \mathbb{T}$ as given in Theorem 0.1. The following holds:

(i) For $\rho = 2$ and $\zeta = 1$ take two elements $(r, s), (r', s') \in \mathbb{K}^2 \rtimes_\varphi \mathbb{K}$. The group operation is given as follows:

$$(r, s) \circ (r', s') \;=\; (r_1 + r_1' + sr_2', r_2 + r_2', s + s').$$

(ii) For $\rho = 1$, $\zeta = 0$ consider the semidirect product $\mathbb{K} \rtimes_{\psi_1} \mathbb{K}^*$, where ψ_1 is defined as seen in Remark 15.2. For two elements $(r, t), (r', t') \in \mathbb{K} \rtimes_{\psi_1} \mathbb{K}^*$ we have

$$(r, t) \circ (r', t') \;=\; (r + tr', tt').$$

REMARK 15.5. For the general linear group $\mathrm{GL}_n(\mathbb{K})$ recall the following definitions:

(i) The set of all upper triangular matrices $B_n \subseteq \mathrm{GL}_n(\mathbb{K})$ is a subgroup of $\mathrm{GL}_n(\mathbb{K})$.
(ii) The set of upper triangular matrices with $1's$ on the diagonal $U_n \subseteq \mathrm{GL}_n(\mathbb{K})$ is a subgroup of $\mathrm{GL}_n(\mathbb{K})$.
(iii) Let C denote the center of $\mathrm{GL}_n(\mathbb{K})$. Then the *projective linear group* $\mathbb{P}\mathrm{GL}_n(\mathbb{K})$ is given as $\mathrm{GL}_n(\mathbb{K})/C$.

LEMMA 15.6. *Consider the projective linear group* $\mathbb{P}\mathrm{GL}_n(\mathbb{K})$ *and the following subgroups:*

$$\mathbb{B}_n \;:=\; B_n/C \;\subseteq\; \mathbb{P}\mathrm{GL}_n(\mathbb{K}), \qquad \mathbb{U}_n \;:=\; U_n/(U_n \cap C) \;\subseteq\; \mathbb{P}\mathrm{GL}_n(\mathbb{K})$$

Then the following is true:

(i) *The group* $\mathbb{K} \rtimes_{\psi_1} \mathbb{K}^*$ *is isomorphic to* $\mathbb{B}_2$.
(ii) *The group* $\mathbb{K}^2 \rtimes_{\varphi_2} \mathbb{K}^*$ *is isomorphic to* $\mathbb{U}_3$.

PROOF. For the first statement we consider the following bijective map:

$$f\colon \mathbb{K} \rtimes_{\psi_1} \mathbb{K}^* \to \mathbb{B}_2, \quad (s,t) \mapsto \begin{bmatrix} t & s \\ 0 & 1 \end{bmatrix}$$

We show that the map f is a group homomorphism:

$$\begin{aligned} f((s,t)\cdot(s',t')) &= f(s+ts', tt') \\ &= \begin{bmatrix} tt' & s+ts' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t & s \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} t' & s' \\ 0 & 1 \end{bmatrix} \\ &= f(s,t)\cdot f(s',t') \end{aligned}$$

This shows the first statement. For the second statement we consider the bijective map g given as follows:

$$g\colon \mathbb{K}^2 \rtimes_{\psi_2} \mathbb{K}^* \to \mathbb{U}_3, \quad (x_1,x_2,y) \mapsto \begin{bmatrix} 1 & y & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Again, we show that this map is a group homomorphism:

$$\begin{aligned} g((x_1,x_2,y)\cdot(x_1',x_2',y')) &= g(x_1+x_1'+yx_2', x_2+x_2', y+y') \\ &= \begin{bmatrix} 1 & y+y' & x_1+x_1'+yx_2' \\ 0 & 1 & x_2+x_2' \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & y & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & y' & x_1' \\ 0 & 1 & x_2' \\ 0 & 0 & 1 \end{bmatrix} \\ &= g(x_1,x_2,y)\cdot g(x_1',x_2',y'). \end{aligned}$$

This shows the second statement and therefore ends the proof. $\square$

COROLLARY 15.7. *The following table lists the degree $\mathcal{K}_X^2$, the Picard number $\rho(X)$ and the singularity type $S(X)$ of all non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface with* $\mathrm{Aut}(X)^0 \neq \mathbb{K}^*$*:*

no. in Prop 15.3	*no. in* [**10**]	$\mathcal{K}_X^2$	$\rho(X)$	$S(X)$	$\mathrm{Aut}(X)^\circ$
1	*36*	*5*	*1*	A_4	$\mathbb{U}_3 \rtimes \mathbb{T}$
2	*24*	*4*	*1*	D_5	$\mathbb{K}^2 \rtimes \mathbb{T}$
3	*15*	*3*	*1*	A_1A_5	$\mathbb{B}_2$
4	*14*	*3*	*1*	E_6	$\mathbb{K} \rtimes \mathbb{T}$
5	*43*	*6*	*2*	A_2	$\mathbb{U}_3 \rtimes \mathbb{T}$
6	*37*	*5*	*2*	A_3	$\mathbb{K}^2 \rtimes \mathbb{T}$
7	*18*	*4*	*2*	A_1A_3	$\mathbb{B}_2$
8	*27*	*4*	*2*	A_4	$\mathbb{B}_2$
9	*26*	*4*	*2*	D_4	$\mathbb{K} \rtimes \mathbb{T}$
10	*45*	*6*	*3*	A_1	$\mathbb{K}^2 \rtimes \mathbb{T}$
11	*39*	*5*	*3*	A_2	$\mathbb{B}_2$

REMARK 15.8. Observe that the surfaces in the lists of Corollary 15.7 are exactly the non-toric $\mathbb{K}^*$-surfaces with $\mathrm{Aut}(X)^0 \neq \mathbb{K}^*$ that occur in the big table of [**10**].

Recall that statements (iv) and (v) of Theorem 0.2 give conditions on the defining matrix P of a rational projective $\mathbb{K}^*$-surface $X(A,P)$ to be an equivariant compactification:

(i) $X(A,P)$ is an equivariant compactification of $\mathbb{K} \rtimes \mathbb{K}^*$ if and only if there is a horizontal P-root at (x^-, i_0, i_1) with $l_{i_1 n_{i_1}} = 1$.

(ii) $X(A,P)$ is an equivariant compactification of $\mathbb{K}^2$ if and only if there are horizontal P-roots at (x^-, i_0, i_1) and (x^-, i_1, i_0).

We check these coonditions

Corollary 15.9. *The following table lists the degree $\mathcal{K}_X^2$, the singularity type $S(X)$ and the set $\mathcal{I}(X)$ of index pairs (i_0, i_1) with horizontal P-roots at (x^-, i_0, i_1) of all non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface that are equivariant group compactification for some semidirect product $\mathbb{K} \rtimes \mathbb{T}$.*

The index pairs are printed in bold type if and only if $l_{i_1 n_{i_1}} = 1$. Furthermore the varieties are equivariant compactifications of $\mathbb{K}^2$ if and only if there is an asterisk in the last column.

no. in Prop 15.3	$\mathcal{K}_X^2$	$S(X)$	$(l_{0n_0}, l_{1n_1}, l_{2n_2})$	$\mathcal{I}(X)$	
5	*6*	A_2	$(1,1,2)$	$(0,2), (\mathbf{2},\mathbf{0}), (1,2), (\mathbf{2},\mathbf{1})$	$\star$
10	*6*	A_1	$(1,1,1)$	***all***	$\star$
11	*5*	A_2	$(1,1,1)$	***all***	
6	*5*	A_3	$(1,1,2)$	$(\mathbf{2},\mathbf{0}), (1,2), (\mathbf{2},\mathbf{1})$	$\star$
7	*4*	A_1A_3	$(1,1,3)$	$(\mathbf{2},\mathbf{0}), (\mathbf{2},\mathbf{1})$	
9	*4*	D_4	$(1,1,2)$	$(\mathbf{2},\mathbf{0}), (\mathbf{2},\mathbf{1})$	

Additionally, precisely the following non-toric Gorenstein log del Pezzo surfaces are equivariant compactifications of a homogeneous space for some semidirect product $\mathbb{K} \rtimes \mathbb{T}$.

no. in Prop 15.3	$\mathcal{K}_X^2$	$S(X)$	$(l_{0n_0}, l_{1n_1}, l_{2n_2})$	$\mathcal{I}(X)$	
1	*5*	A_4	$(1,3,2)$	$(1,2), (2,1)$	$\star$
2	*4*	D_5	$(1,3,2)$	$(1,2), (2,1)$	$\star$
8	*4*	A_4	$(1,1,2)$	$(1,2), (2,1)$	
3	*3*	A_1A_5	$(1,4,2)$	$(1,2)$	
4	*3*	E_6	$(1,3,2)$	$(1,2)$	

Remark 15.10. Observe that the surfaces which occur in the lists of Corollary 15.9 are precisely the non-toric ones in [**16**, Theorem 1.1]. To compare the first table with the result given there note that all Gorenstein log del Pezzo surfaces of degree ≥ 7 are toric, furthermore so are the surfaces of degree 6 and singularity type A_1A_2 and $2A_1$ and of degree 5 and singularity type A_1A_2.

Moreover, the surfaces with asterisks are precisely the non-toric ones occuring in [**15**, Theorem]. In particular, all Gorenstein log del Pezzo surfaces that are $\mathbb{K}^2$-compactifications admit an effective $\mathbb{K}^*$-operation.

CHAPTER 2

The almost homogeneous log del Pezzo case

1. Outline of the chapter

In this chapter we develop an algorithm to classify almost homogeneous log del Pezzo $\mathbb{K}^*$-surfaces. The results have been achieved in joint work with Daniel Hättig.

As described in the introduction the algorithm consists of three steps:

(i) Find all almost k-hollow polygons.
(ii) Find all combinatorially minimal almost homogeneous, almost k-hollow LDP complexes.
(iii) Build all almost homogeneous, almost k-hollow LDP complexes.

In the whole chapter a k-fold point is a point in $k\mathbb{Z}^n$. We give an outline of the following chapter: The first two sections are dedicated to the classification of almost k-hollow polygons, i.e. polygons whose only interior k-fold point is the origin. The main idea is to classify certain minimal polygons and inductively "grow" these minimal polygons by successive addition of vertices. These ideas have been developed in [**33**] in a general manner and used to classify all toric Fano threefolds. Daniel Hättig adapted these ideas to an algorithm classifying all almost k-hollow polygons, presented in Section 2, and run it successfully for $k = 2$. In Section 3 we concentrate on algorithmic aspects to make the described algorithm feasible for larger k. The algorithm provided a classification of all almost 3-hollow polygons presented in Theorem 12.1; there are exactly 910786.

Sections 4-6 establish a one-to-one correspondence between certain polyhedral complexes and log del Pezzo $\mathbb{K}^*$-surfaces. This correspondence relies on the anticanonical complex introduced in [**8**, **31**]. In Section 4 we give a short summary of the results achieved on anticanonical complexes for varieties of complexity one. In the following section we develop arithmetic conditions necessary to show the one-to-one correspondence of so called LDP complexes with log del Pezzo $\mathbb{K}^*$-surfaces. We collect some of the results already known in [**32**]; For convenience, we show the full proofs. Last we present some useful properties of LDP complexes in Section 6. The section ends with the defintion of a standard form for LDP complexes, providing a quick way to compare these complexes.

We go on with the second and third part of the algorithm: In Section 7 we introduce a way to remove vertices of an LDP complex such that the remaining vertices still define an LDP complex. Reversing this process yields a possibility to "grow" LDP complexes from minimal ones, similar to the process for polygons. For the corresponding $\mathbb{K}^*$-surfaces applying this process means contracting a divisor: We show that the contraction of a divisor of a 1/3-log canonical del Pezzo $\mathbb{K}^*$-surfaces yields a 1/3-log canonical del Pezzo $\mathbb{K}^*$-surfaces.

Sections 8 investigates step (ii) of the algorithm described: Removing vertices from a given LDP complex ends in a non-toric combinatorially minimal LDP complex or a toric LDP complex. In the former case, Proposition 8.2 states constraints, which are an improvement on results in [**32**]. In the latter case Proposition 8.5 formulates conditions for toric LDP complexes that are contractions of non-toric LDP complexes. The following Section 9 specializes to almost homogeneous LDP complexes and classifies the combinatorially minimal ones.

Last, in Section 10 we show that there are only finitely many LDP complexes corresponding to almost homogeneous $1/k$-log canonical $\mathbb{K}^*$-surfaces. This has already been shown in [**6**] for general $1/k$-log canonical surfaces. However, our proof is constructive relying on the algorithm described. Hence it can be implemented in a computer algebra system to achieve a classification of all almost homogeneous $1/k$-log canonical $\mathbb{K}^*$-surfaces. The general algorithm is described in Section 11 and we state our results in Section 12.

2. Hättig's results on almost k-hollow lattice polygons

We start this section by describing a process to successively deconstruct almost k-hollow lattice polygons, see Construction 2.6. This process ends in combinatorially minimal almost k-hollow polygons which have been classified in [**24**], see Proposition 2.9. The main result of this first part, Proposition 2.5, provides us with upper bounds for the vertices of almost k-hollow polygons.

This algorithm has been developed by Daniel Hättig and successfully used to classify all almost 2-hollow polygons up to unimodular transformation.

Definition 2.1. Let $n, k \in \mathbb{Z}_{\geq 1}$ and consider a convex rational polytope $\mathsf{P} \subseteq \mathbb{Q}^n$. The set of vertices of P is denoted by $\mathcal{V}(\mathsf{P})$, the relative interior by P° and the boundary by $\partial\mathsf{P}$. We call P

(i) a *lattice polytope*, if $\mathcal{V}(\mathsf{P}) \subseteq \mathbb{Z}^n$.
(ii) a *lattice polygon*, if P is a lattice polytope and $n = 2$.
(iii) *k-hollow*, if $\mathsf{P}^\circ \cap k\mathbb{Z}^2 = \emptyset$,
(iv) *almost k-hollow*, if $\mathsf{P}^\circ \cap k\mathbb{Z}^2 = \{(0,0)\}$.

Definition 2.2. The group $\mathrm{GL}_n(\mathbb{Z})$ of *unimodular transformations in* $\mathbb{Q}^n$ acts on the set of lattice polytopes in $\mathbb{Q}^n$. Lattice polytopes P_1 and P_2 are called *unimodular equivalent*, if $\mathsf{P}_2 \in \mathrm{GL}_n(\mathbb{Z}) \cdot \mathsf{P}_1$.

Remark 2.3. Let $A \in \mathrm{GL}_n(\mathbb{Z})$ and P be a lattice polytope. Then the following hold.

- $A(\mathbb{Z}^n) = \mathbb{Z}^n$.
- $A(k\mathbb{Z}^n) = k\mathbb{Z}^n$.
- $A(\mathcal{V}(\mathsf{P})) = \mathcal{V}(A(\mathsf{P}))$.
- $A(\partial\mathsf{P}) = \partial A(\mathsf{P})$.
- $A(\mathsf{P}^\circ) = A(\mathsf{P})^\circ$.
- $\mathrm{vol}(\mathsf{P}) = \mathrm{vol}(A(\mathsf{P}))$.

Therefore, the number of vertices, the number of (interior) lattice points and the number of (interior) k-fold lattice points are invariant under the action of $\mathrm{GL}_n(\mathbb{Z})$.

REMARK 2.4. For $r \in \mathbb{Q}$, the *ball of radius r centered at the origin* $B_r(0) \subseteq \mathbb{Q}^2$ is the following subset of $\mathbb{Q}^2$:

$$B_r(0) \ := \ \{x \in \mathbb{Q}^2;\ |x| \leq r\}.$$

PROPOSITION 2.5. *Let P be an almost k-hollow lattice polygon. Then there is a unimodular transformation $A \in \mathrm{GL}_2(\mathbb{Z})$ such that $\mathcal{V}(A(\mathsf{P})) \subseteq B_r \cap \mathbb{Z}^2$ with $r := k^2\sqrt{(2k+1)^2+1}$.*

In particular, the number of almost k-hollow polygons up to unimodular transformation is finite.

CONSTRUCTION 2.6. Let P be a lattice polygon. For a vertex $p \in \mathcal{V}(\mathsf{P})$ we define the convex set

$$\mathsf{P}^p := \mathrm{conv}\left((\mathsf{P} \cap \mathbb{Z}^2) \setminus \{p\}\right).$$

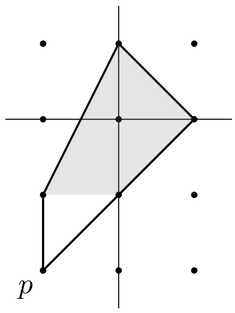


FIGURE 1. A polygon P and P^p in grey as in Construction 2.6.

REMARK 2.7. Let P be a lattice polygon and $p \in \mathsf{P}$. Then the following holds:

(i) The convex set P^p is a lattice polygon.
(ii) We have $\mathsf{P}^p \subsetneq \mathsf{P}$.
(iii) If P is almost k-hollow, then so is P^p.

DEFINITION 2.8. A lattice polygon P is *combinatorially minimal* if for every point $p \in \mathbb{Z}^2$ with $p \in \mathcal{V}(\mathsf{P})$ we have $0 \notin \mathsf{P}^p$.

PROPOSITION 2.9 (Compare [**24**]). *The combinatorially minimal almost k-hollow lattice polygons are up to unimodular equivalence precisely the following:*

$$\begin{aligned} \mathsf{P}_a &:= \mathrm{conv}(e_1, e_2, -ae_1 - e_2), \quad a \in \{1, \ldots, 2k\}, \\ \mathsf{P}_{2k+1} &:= \mathrm{conv}(\pm e_1, \pm e_2). \end{aligned}$$

REMARK 2.10. For the polygons $\mathsf{P}_a \subseteq \mathbb{Q}^2$ of Proposition 2.9 the following holds, where $r_a \in \mathbb{Q}$ is the maximal radius such that $B_{r_a}(0) \subseteq \mathsf{P}$:

a	$\lvert\mathcal{V}(\mathsf{P}_a)\rvert$	$\mathrm{vol}(\mathsf{P}_a)$	r_a
$\leq 2k$	3	$1+\frac{a}{2}$	$\frac{1}{\sqrt{(a+1)^2+1}}$
$2k+1$	4	2	$\frac{1}{\sqrt{2}}$

Construction 2.11. Let $\mathsf{P} \subseteq \mathbb{Q}^2$ be a lattice polygon. The polygon $\mathsf{P}^{p_1,\dots,p_r}$ *deconstructed along* $p_1,\dots,p_r$ is defined inductively as follows:

(i) Take a point $p_1 \in \mathcal{V}(\mathsf{P})$ and consider P^{p_1} as in Construction 2.6.
(ii) Take a point $p_i \in \mathcal{V}(\mathsf{P}^{p_1,\dots,p_{i-1}})$ and set $\mathsf{P}^{p_1,\dots,p_i} := (\mathsf{P}^{p_1,\dots,p_{i-1}})^{p_i}$.

Deconstructible points $p_1,\dots,p_r$ are points that admit this construction i.e. they suffice $p_i \in \mathcal{V}(\mathsf{P}^{p_1,\dots,p_{i-1}})$ for all $i = 1,\dots,r$.

Lemma 2.12. *For every almost k-hollow lattice polygon P there is a unimodular transformation $A \in \mathrm{GL}_2(\mathbb{Z})$ and an integer $1 \leq a \leq 2k+1$ such that $\mathsf{P}_a \subseteq A(\mathsf{P})$, where P_a is one of the polygons as in Proposition 2.9.*

Proof. For an almost k-hollow lattice polygon P one takes decontructile points as in Construction 2.11 until one arrives at a combinatorially minimal lattice polygon P_0. The latter is almost k-hollow by Remark 2.7 (iii). Thus it is unimodular equivalent to one of the polygons given in Proposition 2.9, i.e. one finds a unimodular transformation $A \in \mathrm{GL}_2(\mathbb{Z})$ such that $\mathsf{P}_a = A(\mathsf{P}_0)$ for some $1 \leq a \leq 2k+1$. Since $\mathsf{P}_0 \subseteq \mathsf{P}$, we have shown the statement. □

Remark 2.13 (Minkowski's Theorem). Let $S \subseteq \mathbb{Q}^n$ be a convex, centrally symmetric set with a volume greater than 2^n. Then there is a lattice point in S besides the origin.

Corollary 2.14. *Let $\mathsf{P} \subseteq \mathbb{Q}^2$ be an almost 1-hollow polygon and let $B_r(0) \subseteq \mathsf{P}$. Then for every $p \in \mathbb{Z}^2$ such that $\mathsf{P}(p)$ is almost 1-hollow we find $|p| \leq 2r^{-1}$.*

Proof. Consider the convex set $\mathrm{conv}(B_r(0), p)^\circ \subseteq \mathsf{P}(p)^\circ$ and note that it contains a lattice point if and only if there is a lattice point in $S := \mathrm{conv}(B_r(0), -p, p)^\circ$. The latter set is centrally symmetric. Hence by Remark 2.13 we have $\mathrm{vol}(S) = 2r|p| \leq 4$ since otherwise there is a lattice point in $\mathrm{conv}(B_r(0), p)^\circ \subseteq \mathsf{P}^\circ$. This contradicts the almost 1-hollowness of P. Reformulating the inequality yields the statement. □

Lemma 2.15. *Let P be an almost k-hollow lattice polygon containing P_a. For every vertex $v \in \mathcal{V}(\mathsf{P})$ we find:*

$$
\begin{aligned}
|v| &\leq k^2\sqrt{(a+1)^2+1}, && \text{if } a \leq 2k,\\
|v| &\leq k^2\sqrt{2}, && \text{if } a = 2k.
\end{aligned}
$$

Proof. Since $\mathsf{P}_a \subseteq \mathsf{P}$ note that $\mathsf{P}_a(v) \subseteq \mathsf{P}$. We consider the following polygon:

$$\mathsf{P}'_a(v) := \{p \in \mathbb{Z}^2;\ kp \in \mathsf{P}_a(v)\}.$$

Note that $\mathsf{P}'_a(v)$ is almost hollow if and only if $\mathsf{P}_a(v)$ is almost k-hollow. Moreover, for the radii r_a in Remark 2.10 we have $B_{r_a/k}(0) \subseteq \mathsf{P}'_a(v)$ and $k^{-1}v \in \mathsf{P}'_a(v)$. By Corollary 2.14 we find $|k^{-1}v| \leq 2(r_a/k)^{-1}$. Reformulating the last inequality yields the statement. □

PROOF OF PROPOSITION 2.5. By Lemma 2.12 we find a unimodular transformation $A \in \mathrm{GL}_2(\mathbb{Z})$ such that $\mathsf{P}_a \subseteq A(\mathsf{P})$. For the latter polygon Lemma 2.15 yields the upper bound. Since the upper bound limits the number of possible vertices, we infer that the number of almost k-hollow polygons, up to unimodular equivalence, is finite. □

The objective of the following parts of this section is to develop Construction 2.23. Here we establish an algorithm that classifies (after finitely many steps) all almost k-hollow polygons up to unimodular equivalence. This is achieved by reversing the process described in Construction 2.6.

DEFINITION 2.16. Let P be a lattice polygon in $\mathbb{Q}^2$ and let $p \in \mathbb{Z}^2 \setminus (\mathsf{P} \cap \mathbb{Z})$ be a point. The *p-expansion* of P is the following set:

$$\mathsf{P}(p) \ := \ \mathrm{conv}(\mathcal{V}(\mathsf{P}) \cup \{p\}).$$

REMARK 2.17. Let P be a lattice polytope in $\mathbb{Q}^2$ and let $p \in \mathbb{Z}^2 \setminus \mathsf{P}$ be a point. Then the following statements are true:

(i) We find $\mathcal{V}(\mathsf{P}(p)) \subseteq \mathcal{V}(\mathsf{P}) \cup \{p\}$, thus $\mathsf{P}(p)$ is a lattice polytope.
(ii) We have $\mathsf{P} \subsetneq \mathsf{P}(p)$, in particular if $0 \in \mathsf{P}$, then $0 \in \mathsf{P}(p)$.
(iii) If $\mathsf{P}(p)$ is almost k-hollow, then so is P.

DEFINITION 2.18. Let $\mathsf{P} \subseteq \mathbb{Q}^2$ be a lattice polygon and let $p_1, \ldots, p_r \in \mathbb{Z}^2$ be points. The *expansion of* P *along* $(p_1, \ldots, p_r)$ is defined inductively as follows:

$$\mathsf{P}(p_1, \ldots, p_r) \ := \ \mathsf{P}(p_1, \ldots, p_{r-1})(p_r).$$

LEMMA 2.19. *Let $\mathsf{P} \subseteq \mathbb{Q}^2$ be a polygon with points $p_1, \ldots, p_r \in \mathsf{P}$. Then we find*

$$\mathsf{P}^{p_1,\ldots,p_r}(p_1, \ldots, p_r) \ = \ \mathsf{P}.$$

PROOF. Observe that $\mathsf{P}^{p_1,\ldots,p_r} \subseteq \mathsf{P}$, thus the inclusion $\mathsf{P}^{p_1,\ldots,p_r}(p_1, \ldots, p_r) \subseteq \mathsf{P}$ holds since $p_1, \ldots, p_r \in \mathsf{P}$.

For the other inclusion note that $\mathcal{V}(\mathsf{P}^{p_1,\ldots,p_r}) \supseteq \mathcal{V}(\mathsf{P}) \setminus \{p_1, \ldots, p_r\}$, therefore we have:

$$\begin{aligned} \mathsf{P}^{p_1,\ldots,p_r}(p_1, \ldots, p_r) &= \mathrm{conv}\left(\mathcal{V}(\mathsf{P}^{p_1,\ldots,p_r}) \cup \{p_1, \ldots, p_r\}\right) \\ &\supseteq \mathrm{conv}(\mathcal{V}(\mathsf{P}) \setminus \{p_1, \ldots, p_r\} \cup \{p_1, \ldots, p_r\}) \\ &= \mathsf{P}. \end{aligned}$$

□

DEFINITION 2.20. Let $\mathsf{P} \subseteq \mathbb{Q}^2$ be an almost k-hollow polygon. The *k-search space* is the following set

$$S_k(\mathsf{P}) \ := \ \{p \in \mathbb{Z}^2 \setminus (\mathsf{P} \cap \mathbb{Z});\ \mathsf{P}(p) \text{ is almost } k\text{-hollow}\}.$$

LEMMA 2.21. *Consider an almost k-hollow lattice polygon $\mathsf{P} \subseteq \mathbb{Q}^2$. The following statements hold:*

(i) *The set $S_k(\mathsf{P}_a)$ is finite for all $1 \leq a \leq 2k+1$.*
(ii) *For a point $p \in \mathcal{V}(\mathsf{P})$ we have $p \in S_k(\mathsf{P}^p)$.*
(iii) *For an almost k-hollow lattice polygon P' with $\mathsf{P} \subsetneq \mathsf{P}'$ we have $S_k(\mathsf{P}') \subsetneq S_k(\mathsf{P})$.*

Proof. The first statement follows with Lemma 2.15 and the second is an immediate consequence of Lemma 2.19.

For the third statement note that we have $\mathsf{P}(p) \subseteq \mathsf{P}'(p)$, which yields the containment $S_k(\mathsf{P}') \subseteq S_k(\mathsf{P})$. The sets are not equal, since we find a lattice point $p \in \mathsf{P}' \setminus \mathsf{P}$. Thus we have $p \in S_k(\mathsf{P}) \setminus S_k(\mathsf{P}')$. □

Lemma 2.22. *Let P be an almost k-hollow lattice polygon with k-search space $S_k(\mathsf{P})$. Then for every almost k-hollow lattice polygon P' that contains P we find $p_1, \ldots, p_r \in S_k(\mathsf{P})$ such that $\mathsf{P}' = \mathsf{P}(p_1, \ldots, p_r)$.*

Proof. Let $p_1, \ldots, p_r$ be the vertices of P' that are not contained in P, then $\mathsf{P}(p_1, \ldots, p_r) = \mathsf{P}'$. □

Construction 2.23. Let $(\mathsf{P}_i, S_k(\mathsf{P}_i))$ be a pair, where P_i is a combinatorially minimal lattice polygon as given in Proposition 2.9 and $S_k(\mathsf{P}_i)$ its k-search space. Set

$$\mathcal{L}_0 \ := \ \{(\mathsf{P}_i, S_k(\mathsf{P}_i);\ i = 1, \ldots, 2k+1\} \text{ and } \mathcal{L} \ := \ \{\mathsf{P}_i;\ i = 1, \ldots, 2k+1\}.$$

While $|\mathcal{L}_0| \neq 0$ do the following:

(i) Take a pair $(\mathsf{P}, S) \in \mathcal{L}_0$.
(ii) Set $S' := \{p \in S;\ \mathsf{P}(p) \text{ is almost } k\text{-hollow}\}$.
(iii) For all $p \in S'$ test whether there is a polygon $\mathsf{P}' \in \mathcal{L}$ such that P' and $\mathsf{P}(p)$ are unimodular equivalent.
(iv) If not, add the pair $(\mathsf{P}(p), S' \setminus (\mathsf{P} \cap \mathbb{Z}^2))$ to $\mathcal{L}_0$ and $\mathsf{P}(p)$ to $\mathcal{L}$.

This algorithm ends after finitely many steps and yields the set $\mathcal{L}$ of all almost k-hollow lattice polygons, up to unimodular equivalence.

Proof. We want to show that this construction yields all almost k-hollow lattice polygons, up to unimodular equivalence. Observe that by Lemma 2.12 is suffices to find all almost k-hollow polygons containing P_a for every $1 \leq a \leq 2k+1$ and Lemma 2.22 states that these are given by $\mathsf{P}_a(p_1, \ldots, p_r)$ where $p_1, \ldots, p_r \in S_k(\mathsf{P}_a)$.

Thus we show that for a given index a the construction yields all polygons $\mathsf{P}_a(p_1, \ldots, p_r)$. Note that in the second step we only remove those points from S that do yield non almost k-hollow lattice polygons, in particular we have $S_k(\mathsf{P}(p)) = S' \setminus (\mathsf{P} \cap \mathbb{Z}^2)$ by Lemma 2.21 (iii).

Furthermore in step (iii) note that if there is a lattice polygon $\mathsf{P}' \in \mathcal{L}$ unimodular equivalent to P, it has been found in a previous step and its k-search space $S_k(\mathsf{P}')$ has been already computed, in particular all polygons $\mathsf{P}'(p_1', \ldots, p_r')$ lie in the set $\mathcal{L}$ when the construction ends. Thus removing P from the process, we do not lose polygons $\mathsf{P}(p_1, \ldots, p_r)$ up to unimodular equivalence.

Last note that since $S_k(\mathsf{P}_a)$ is finite by Lemma 2.21 (i) and $S' \subsetneq S$ the construction terminates after finitely many steps. □

3. Algorithmic aspects for polygons

For growing k, the steps needed when running the algorithm established in Construction 2.23 increase profoundly and the task to classify polygons becomes infeasible in reasonable time.

The following considerations significantly lower the number of computations performed. We concentrated on three aspects of the algorithm:

(i) Improving the test on unimodular equivalence of two polygons.
(ii) Improving the test on almost k-hollowness of polygons.
(iii) Removing certain point from the k-search space to prevent unnecessary computations.

This finally yields a faster method to find all almost k-hollow polygons, see Construction 3.14.

We start with the first aspect and introduce the *standard vertex matrix* for polygons, a useful tool to check whether two polygons are unimodular equivalent.

DEFINITION 3.1. Let $v_1, \ldots, v_r$ be vertices of a lattice polygon P. We set $v_0 := v_r$ and $v_{r+1} := v_1$.

(i) The vertices are *in adjacent order* if for every $1 \le i \le r$ the lines through v_{i-1}, v_i and v_i, v_{i+1} lie in the boundary of P.
(ii) For vertices $v_1, \ldots, v_r$ in adjacent order choose a $\mathbb{Z}$-linear basis $w_1, w_2 \in \mathbb{Z}^2$ such that

$$v_1 \;=\; a_1 w_1, \qquad v_2 \;=\; a_2 w_1 + b_2 w_2, \qquad 0 \le a < b,$$
$$v_i = a_i w_1 + b_i w_2, \quad a_i, a_i \in \mathbb{Z}.$$

The *Hermite vertex matrix* $H(v_1, \ldots, v_r)$ is the $(2 \times r)$-matrix with columns $[a_i, b_i]$, $1 \le i \le r$, i.e. we have:

$$H(v_1, \ldots, v_r) \;:=\; \begin{bmatrix} a_1 & a_2 & \cdots & a_r \\ 0 & b_2 & \cdots & b_r \end{bmatrix}.$$

(iii) The *standard vertex matrix* $\mathcal{SV}(\mathsf{P})$ is the Hermite vertex matrix $H(v_1, \ldots, v_r)$ that is lexicographically minimal among all Hermite vertex matrices for all adjacent orders of $v_1, \ldots, v_r$.

REMARK 3.2. Let $v_1, \ldots, v_r$ be vertices of a lattice polygon P in adjacent order. Then the following is true:

(i) For every tuple of vertices $v_1, \ldots, v_r$ there is a unique Hermite vertex matrix $H(v_1, \ldots, v_r)$.
(ii) Let $D_n \subseteq S_n$ the dyhedral group of order $2n$. The standard vertex matrix is the matrix $H(v_1, \ldots, v_r)$ such that

$$H(v_1, \ldots, v_r) \;\le_{\mathrm{lex}}\; H(v_{\sigma(1)}, \ldots, v_{\sigma(1)}) \qquad \text{for every } \sigma \in D_n.$$

(iii) Every polygon P has a unique standard vertex matrix since the lexicographic order is a total order on $(\mathbb{Z}^2)^r$.

LEMMA 3.3. *Let $\mathsf{P}, \mathsf{P}' \subseteq \mathbb{Q}^2$ be two convex polygons. Then the following statements are equivalent.*

(i) *There is a unimodular transformation $A \in \mathrm{GL}_2(\mathbb{Z})$ such that $A(\mathsf{P}) = \mathsf{P}'$.*
(ii) *We have $r = r'$ and their standard vertex matrices coincide.*

PROOF. The first implication is clear. For the second implication assume that the vertices $v_1, \ldots, v_r$ of P and $v_1', \ldots, v_r'$ of P' are numerated as they are in its standard vertex matrix. Choose a $\mathbb{Z}$-linear basis $w_1, w_2 \in \mathbb{Z}^2$ as in Definition 3.1 (ii). Thus, there are two unimodular matrices $B, B' \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$Bv_i \;=\; a_i w_1 + b_i w_2 \;=\; B' v_i'.$$

In particular, for $A := B^{-1}B'$ we find $A(\mathsf{P}) = \mathsf{P}'$. $\square$

Remark 3.4. Let P be a lattice polygon and let S be a set of lattice polygons. Checking whether there is a lattice polygon $\mathsf{P}' \in S$ that is unimodular equivalent to P means performing (possibly) $|S|$-many tests on unimodular equivalence, i.e. finding matrices $A \in \mathrm{GL}_2(\mathbb{Z})$ such that $A(\mathsf{P}) = \mathsf{P}'$.

If for every $\mathsf{P}' \in S$ its standard vertex matrix is known, the test on unimodular equivalence boils down to finding the standard vertex matrix of P. This is especially useful, when a large amount of polygons have to be tested on unimodular equivalence.

We turn to the second aspect and find a quick way to test whether triangles, i.e. lattice polygons with three vertices, are almost k-hollow. For a fixed triangle the computation needed to test k-hollowness decrease with increasing k, see Remark 3.8.

Since for every polygon P there is a triangulation, this implies a method to test almost k-hollowness of P. Observe that for the algorithm in Construction 2.23 it suffices to check (almost) k-hollowness of triangles (see Lemma 3.10).

Definition 3.5. Consider two numbers $a \in \mathbb{Q}$ and $k \in \mathbb{Z}$. We set:

$$\lfloor a \rfloor_k \ := \ \max\left(l \in k\mathbb{Z}, l \leq q\right), \qquad \lceil a \rceil_k \ := \ \min\left(u \in k\mathbb{Z}, u \geq q\right).$$

Lemma 3.6. *Let $a, b \in \mathbb{Q}$ be rational numbers and let $k \in \mathbb{Z}_{\geq 1}$ be an integer. Then the following holds:*

(i) *The number of k-fold points in $[a, b]$ is given by:*

$$\rho_k \ := \ \max\left(0, \frac{\lfloor b \rfloor_k - \lceil a \rceil_k}{k} + 1\right).$$

(ii) *There are no k-fold points in (a, b) if and only if one of the followng statements hold.*
   (a) $\rho_k = 0$.
   (b) $\rho_k = 1$ *and* $\lceil a \rceil_k = a$ *or* $\lfloor b \rfloor_k = b$.
   (c) $\rho_k = 2$ *and* $\lceil a \rceil_k = a$ *and* $\lfloor b \rfloor_k = b$.

Proof. The first statement is clear and the second statement follows immediately using the first statement and the fact that for $\rho_k \neq 0$ and $(a, b) \cap k\mathbb{Z} = \emptyset$ the only possible k-fold points in $[a, b] \setminus (a, b)$ are given by a, b. $\square$

Corollary 3.7. *Let $\mathsf{T} \subseteq \mathbb{Q}^2$ be a triangle and let $n \in k\mathbb{Z}$ be an integer. Let $I(n, \mathsf{T}) \subseteq \mathbb{Q}$ be the open interval in $\mathbb{Q}$ such that*

$$(\{n\} \times \mathbb{Z}) \cap \mathsf{T}^\circ \ = \ \{n\} \times I(n, \mathsf{T}).$$

Then T is k-hollow if and only if for every $n \in k\mathbb{Z}$ the intervall $I(n, \mathsf{T})$ does not contain any k-fold points.

Construction 3.8. Let $\mathsf{T} \subseteq \mathbb{Q}^2$ be a triangle with vertices

$$v_1 := [x_1, y_1], \qquad v_2 := [x_2, y_2], \qquad v_3 := [x_3, y_3].$$

Then test the following:

(i) Set $x_l := \max(x_1, x_2, x_3)$ and $x_u := \min(x_1, x_2, x_3)$.

(ii) For every $\lceil x_l \rceil_k \leq n \leq \lfloor x_u \rfloor_k$ such that $n \in k\mathbb{Z}$ test whether $I(n, \mathsf{T})$ contains k-fold points following Lemma 3.6.

If one of the tests in step (ii) fails, T is not k-hollow, otherwise it is k-hollow.

REMARK 3.9. Let $\mathsf{T} \subseteq \mathbb{Q}^2$ be a triangle with vertices

$$v_1 := [x_1, y_1], \qquad v_2 := [x_2, y_2], \qquad v_3 := [x_3, y_3].$$

We want to check whether T is k-hollow. A canoncial way to do so is to find all integer points in T and check whether they are contained in $(k\mathbb{Z})^2$. We preceed with the following steps:

(i) We define the following numbers:

$$x_l := \min(x_1, x_2, x_3), \qquad x_u := \min(x_1, x_2, x_3),$$

$$y_l := \min(y_1, y_2, y_3), \qquad y_u := \min(y_1, y_2, y_3).$$

(ii) For every pair $[x, y] \in \mathbb{Z}^2$ such that $x_l \leq x \leq x_u$ and $y_l \leq y \leq y_u$ test whether $[x, y] \in \mathsf{T}^\circ$.

(iii) For every $[x, y] \in \mathsf{T}^\circ$ test whether $[x, y] \in (k\mathbb{Z})^2$.

If one of the tests in step (iii) is true, T is not k-hollow, otherwise T is k-hollow.

Note that for a triangle T that is not k-hollow, checking k-hollowness with this steps means performing $(y_l - y_u + 1)(x_l - x_u + 1)$ tests. Using the algorithm as described in Construction 3.8 we only have to perform $\frac{\lfloor x_u \rfloor_k - \lceil x_l \rceil_k}{k} + 1$ calculations in $\mathbb{Q}$. In particular, for increasing k the number of tests performed decreases.

LEMMA 3.10. *Consider a k-hollow polygon P and a point $p \in \mathbb{Z}^2$ such that $\mathsf{P}(p)$ has more than three vertices.*

Let $v_1, v_2 \in \mathcal{V}(\mathsf{P}(p))$ be the vertices such that $\operatorname{conv}(p, v_1)$, $\operatorname{conv}(p, v_2)$ are the facets of $\mathsf{P}(p)$ containing p. Then the following is equivalent:

(i) *The polygon $\mathsf{P}(p)$ is k-hollow.*

(ii) *The triangle $\operatorname{conv}(p, v_1, v_2)$ is k-hollow and there are no k-fold points in the relative interior of $\operatorname{conv}(v_1, v_2)$.*

PROOF. The convex polygon $\mathsf{P}(p)$ can be split in the following way:

$$\mathsf{P}(p) = \operatorname{conv}(v_1, v_2, p) \cup \mathsf{P} \setminus \operatorname{conv}(v_1, v_2, p).$$

Note that the latter set is contained in P, therefore it does not contain any k-fold points, since P is k-hollow. Furthermore since $\mathsf{P}(p)$ has more than three vertices we have $\operatorname{conv}(v_1, v_2) \not\subseteq \partial\mathsf{P}(p)$ Thus it suffices to check k-hollowness of $\operatorname{conv}(v_1, v_2, p)$ and that there are no k-fold points in $\operatorname{conv}(v_1, v_2)$. $\square$

REMARK 3.11. All described statements and algorithms have versions to test almost k-hollowness. In all cases one needs to pay attention to the origin being the only k-fold point in the polygon.

Last, we turn to the k-search space. When the Algorithm in Construction 2.23 has found an almost k-hollow polygon P, it computes the k-search space $S_k(\mathsf{P})$ in step (iv) without considering other polygons that had been

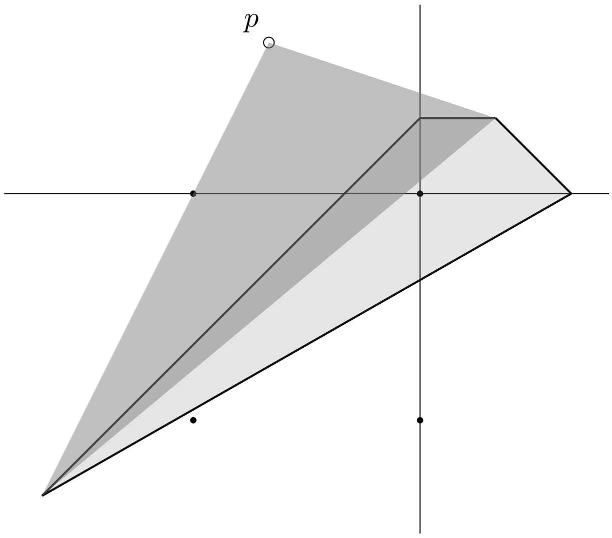


FIGURE 2. An almost k-hollow polygon P and $\mathsf{P}(p)$ for a point $p \in \mathbb{Z}^2$; points in $(k\mathbb{Z})^2$ are drawn in bold. To check whether $\mathsf{P}(p)$ is almost k-hollow it suffices to test the triangle in dark grey.

found in the previous step leading to unnecessary computations. Removing suitable point from the k-search space yields a drop in the number of computations, see Figure 3 and Example 3.13.

LEMMA 3.12. *Consider a k-hollow polygon P and a set $\tilde{S}$ of points in $\mathbb{Z}^2$. For a point $p' \in \tilde{S}$ we define the following:*

$$\mathsf{P}' := \mathsf{P}(p'), \qquad \tilde{S}(\mathsf{P}') := \left(\tilde{S} \setminus (\mathsf{P}' \cap \mathbb{Z}^2)\right) \setminus \{p \in \tilde{S}; \mathsf{P}' \subseteq \mathsf{P}(p)\}.$$

Then the following containment holds:

$$\{\mathsf{P}'(q); q \in \tilde{S} \setminus (\mathsf{P}' \cap \mathbb{Z}^2)\} \ \subseteq \ \{\mathsf{P}(p);\, p \in \tilde{S}\} \cup \{\mathsf{P}'(q);\, q \in \tilde{S}(\mathsf{P}')\}$$

PROOF. Let $q \in \tilde{S} \setminus (\mathsf{P}' \cap \mathbb{Z}^2)$. If we have $\mathsf{P}' \subseteq \mathsf{P}(q)$, then note that $\mathsf{P}'(q) = \mathsf{P}(q)$, i.e. $\mathsf{P}'(q)$ lies in the first set. Otherwise we have $q \in \tilde{S}(\mathsf{P}')$, thus $\mathsf{P}(q)$ lies in the second set. $\square$

EXAMPLE 3.13. Let $\mathsf{P} = \mathsf{P}_1 = \mathrm{conv}(e_1, e_2, -e_1 - e_2)$ be one of the combinatorially minimal lattice polygons as seen in Proposition 2.9. For the 3-search space of P we find:

$$|S_3(P)| = 2\,424.$$

We take $[-1, 0] \in S_3(\mathsf{P})$ and set $\mathsf{P}' = \mathsf{P}([-1, 0])$. We note the following:

$$|S_3(\mathsf{P}) \setminus (\mathsf{P}' \cap \mathbb{Z})| = 2\,423, \qquad |\tilde{S}(\mathsf{P}')| = 2\,271.$$

Furthermore we find the following statement:

$$\sum_{p \in S_3(P)} |S_3(\mathsf{P}) \setminus \{\mathsf{P} \cap \mathbb{Z}\}| \ = \ 5\,716\,935, \qquad \sum_{p \in S_3(P)} |\tilde{S}(\mathsf{P}(p))| \ = \ 5\,560\,530.$$

Note that we save 156 405 tests on k-hollowness in this first step.

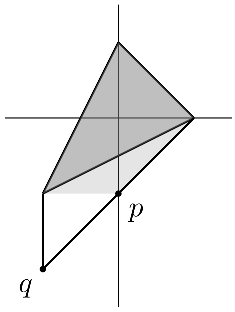


FIGURE 3. The polygon P_1 in grey and the polygons $\mathsf{P}' := \mathsf{P}(p)$ in light grey and $\mathsf{P}(p,q)$. Note that we have $\mathsf{P}'(q) = \mathsf{P}(q)$.

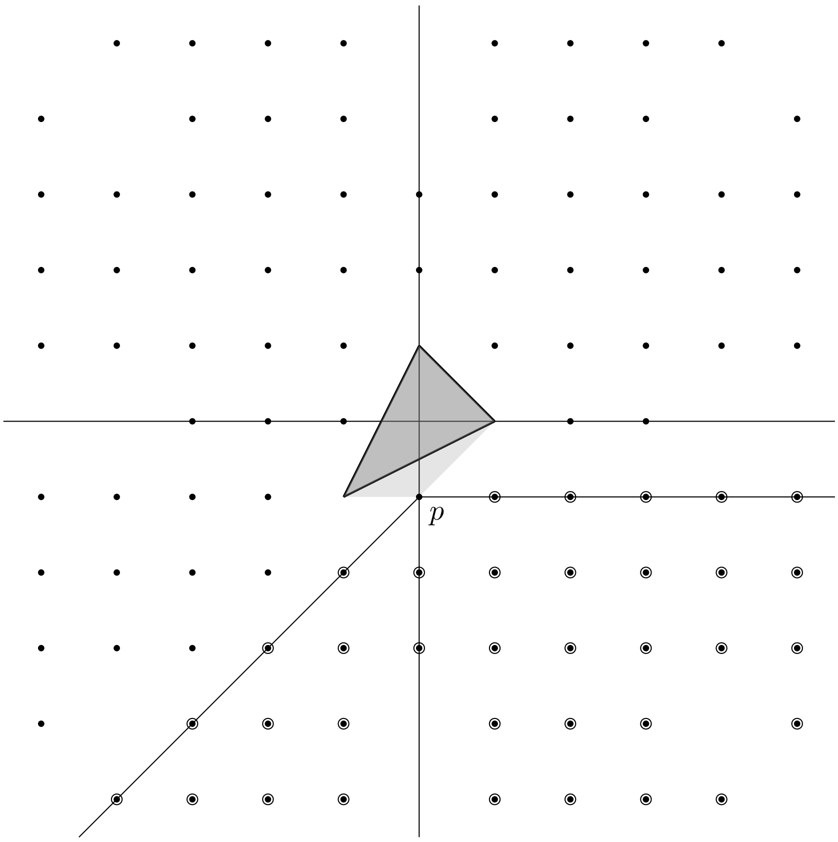


FIGURE 4. The combinatorially minimal polygon P_1 and its 3-search space $S_3(\mathsf{P}_1)$. For $\mathsf{P}' := \mathsf{P}_1(p)$ all points encircled correspond to points $q \in S_3(\mathsf{P}_1)$ such that $\mathsf{P}'(q) = \mathsf{P}_1(q)$.

CONSTRUCTION 3.14. Consider the combinatorially minimal polygons P_a as in Proposition 2.9 and their k-search spaces $S_k(\mathsf{P}_a)$.

- Set $\mathcal{L} := \{\mathcal{SV}(\mathsf{P}_a);\ a = 1, \ldots, 2k+1\}$.

- For $a = 1, \dots, 2k+1$ set $\mathcal{L}_0 := \{(\mathsf{P}_a, S_k(\mathsf{P}_a)\}$.
  While $|\mathcal{L}_0| \neq 0$ do the following:
  (i) Take a pair $(\mathsf{P}, S) \in \mathcal{L}_0$ and test whether $\mathsf{P}_j \subseteq \mathsf{P}$ for $j = 1, \dots, a-1$.
  (ii) If none of the polygons P_j is contained in P set
  $$S' := \{p \in S;\ \mathsf{P}(p) \text{ is } k\text{-hollow}\}.$$
  The test on almost k-hollowness is performed using the triangle described in Lemma 3.10 and the method described in Corollary 3.7.
  (iii) For all $p \in S'$ compute $\mathcal{SV}(\mathsf{P}(p))$.
  (iv) If $\mathcal{SV}(\mathsf{P}(p)) \notin \mathcal{L}$, add the following to the sets $\mathcal{L}_0$ and $\mathcal{L}$:
  $$\Big(\mathsf{P}(p), \tilde{S}(\mathsf{P}(p))\Big) \text{ to } \mathcal{L}_0, \qquad \mathcal{SV}(\mathsf{P}(p)) \text{ to } \mathcal{L},$$
  where $\tilde{S}(\mathsf{P}(p))$ is the set described in Lemma 3.12.

This algorithm ends after finitely many steps and yields the set $\mathcal{L}$ of all almost k-hollow lattice polygons, up to unimodular equivalence.

Proof. We want to show that this yields all almost k-hollow lattice polygons. Note that this construction is a modification of Construction 2.23 which computes all almost k-hollow lattice polygons. Thus it suffices to show that the modification only prevents redundant computations.

We do this by induction on a. We first note that step (ii) and (iii) check almost k-hollowness and unimodular equivalence as in Construction 2.23.

We turn to step (iv) For $a = 1$ the only difference to Construction 2.23 is the space $\tilde{S}$. Note that Lemma 3.12 shows that we do not miss polygons using the smaller set $\tilde{S}$ since for $q \in S' \setminus \tilde{S}(\mathsf{P}(p))$ all polygons $\mathsf{P}(p, q)$ have already been found. Therefore by Lemma 2.22 this first step computes all almost k-hollow polygons containing P_1.

Now for $a > 1$ note that for $j = 1, \dots, a-1$ all polygons containg P_j have already been found, therefore it suffices to consider the polygons not containing P_j, in particular the first step only removes redundant computations. The remaining is seen as for $a = 1$. □

4. The anticanonical complex

First we summarize the combinatorial theory of normal rational Fano varieties with torus action of complexity one developed in [**8**, **25**, **29**].

Construction 4.1. Fix $r \in \mathbb{Z}_{\geq 1}$, a sequence $n_0, \dots, n_r \in \mathbb{Z}_{\geq 1}$, set $n := n_0 + \ldots + n_r$, and fix integers $m \in \mathbb{Z}_{\geq 0}$ and $0 < s < n + m - r$. The input data are matrices

$$A = [a_0, \dots, a_r] \in \mathrm{Mat}(2, r{+}1; \mathbb{K}), \qquad P = \begin{bmatrix} L & 0 \\ d & d' \end{bmatrix} \in \mathrm{Mat}(r{+}s, n{+}m; \mathbb{Z}),$$

where A has pairwise linearly independent columns and P is built from an $(s \times n)$-block d, an $(s \times m)$-block d' and an $(r \times n)$-block L of the form

$$L \;=\; \begin{bmatrix} -l_0 & l_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \dots & l_r \end{bmatrix}, \qquad l_i \;=\; (l_{i1}, \dots, l_{in_i}) \;\in\; \mathbb{Z}_{\geq 1}^{n_i}$$

such that the columns of P are pairwise different vectors generating $\mathbb{Q}^{r+s}$ as a cone. Consider the polynomial algebra

$$\mathbb{K}[T_{ij}, S_k] \ := \ \mathbb{K}[T_{ij}, S_k;\ 0 \le i \le r,\ 1 \le j \le n_i, 1 \le k \le m].$$

Denote by $\mathfrak{I}$ the set of all triples $I = (i_1, i_2, i_3)$ with $0 \le i_1 < i_2 < i_3 \le r$ and define for any $I \in \mathfrak{I}$ a trinomial

$$g_I \ := \ g_{i_1,i_2,i_3} \ := \ \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}, \qquad T_i^{l_i} \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}.$$

Let P^* denote the transpose of P, consider the factor group $K := \mathbb{Z}^{n+m}/\mathrm{im}(\mathrm{P}^*)$ and the projection $Q\colon \mathbb{Z}^{n+m} \to K$. We define a K-grading on $\mathbb{K}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) \ := \ w_{ij} \ := \ Q(e_{ij}), \qquad \deg(S_k) \ := \ w_k \ := \ Q(e_k).$$

Then the trinomials g_I just introduced are K-homogeneous, all of the same degree. In particular, we obtain a K-graded factor algebra

$$R(A, P) \ := \ \mathbb{K}[T_{ij}, S_k] \ / \ \langle g_I;\ I \in \mathfrak{I} \rangle.$$

Remark 4.2. The K-graded ring $R(A, P)$ of Construction 4.1 is a complete intersection: with $g_i := g_{i,i+1,i+2}$ we have

$$\langle g_I;\ I \in \mathfrak{I} \rangle \ = \ \langle g_0, \ldots, g_{r-2} \rangle, \qquad \dim(R(A, P)) \ = \ n + m - (r - 1).$$

Definition 4.3. Consider a K-graded algebra $R(A, P)$ from Construction 4.1. The *anticanonical class* $R(A, P)$ is

$$\kappa(A, P) \ := \ \sum_{i,j} w_{ij} \ + \ \sum_k w_k \ - \ (r - 1) \sum_{j=0}^{n_0} l_{0j} w_{0j} \ \in \ K$$

and the *moving cone* of $R(A, P)$ in $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$ is

$$\mathrm{Mov}(A, P) \ := \ \bigcap_{i,j} \mathrm{cone}(w_{uv}, w_t;\ (u, v) \ne (i, j)) \ \cap \ \bigcap_k \mathrm{cone}(w_{uv}, w_t;\ t \ne k).$$

Construction 4.4. Let (A, P) be Fano data. The K-grading on $\mathbb{K}[T_{ij}, S_k]$ defines an action of the quasitorus $H := \mathrm{Spec}\, \mathbb{K}[K]$ on $\bar{Z} := \mathbb{K}^{n+m}$ and

$$\mathrm{Spec}\, R(A, P) \ \cong \ \bar{X} \ := \ V(g_I;\ I \in \mathfrak{I}) \ \subseteq \ \bar{Z}$$

is an H-invariant closed subvariety. We now pass to GIT quotients. The set of H-semistable points of $\bar{Z}$ associated with $\kappa(A, P)$ is

$$\hat{Z}_c \ := \ \{z \in \bar{Z};\ f(z) \ne 0 \text{ for some } f \in \mathbb{K}[T_{ij}, S_k]_{\nu\kappa(A,P)},\ \nu \in \mathbb{Z}_{>0}\} \ \subseteq \ \underline{Z}.$$

The intersection $\hat{X} := \bar{X} \cap \hat{Z}_c$ is an open H-invariant set in $\bar{X}$ and the quotient map $\hat{Z}_c \to Z_c := \hat{Z}_c /\!\!/ H$ yields a commutative diagram

$$\begin{tikzcd} \hat{X} \arrow[r] \arrow[d, "/\!\!/ H"'] & \hat{Z}_c \arrow[d, "/\!\!/ H"] \\ X(A, P) \arrow[r] & Z_c \end{tikzcd}$$

with a Fano variety $X(A,P)$ embedded into the projective toric variety Z_c. Dimension, divisor class group, anticanonical class and Cox ring of $X(A,P)$ are given by

$$\dim(X(A,P)) = s+1, \qquad \mathrm{Cl}(X(A,P)) \;\cong\; K,$$

$$-\mathcal{K}_X \;=\; \kappa(A,P), \qquad \mathcal{R}(X) \;\cong\; R(A,P).$$

For a face $\delta_0 \preceq \delta$ of the orthant $\delta \subseteq \mathbb{Q}^{n+m}$, let $\delta_0^* \preceq \delta$ denote the complementary face and call δ_0 *relevant* if we have

$$\kappa(A,P) \;\in\; Q(\delta_0^*)^\circ.$$

Then we obtain the describing fans $\hat{\Sigma}_c$ in $\mathbb{Z}^{n+m}$ and Σ_c in $\mathbb{Z}^{r+s}$ of $\hat{Z}_c$ and Z_c respectively as

$$\hat{\Sigma} \;:=\; \{\delta_1 \preceq \delta_0;\; \delta_0 \preceq \delta \text{ relevant}\}, \qquad \Sigma \;:=\; \{\sigma \preceq P(\delta_0);\; \delta_0 \preceq \delta \text{ relevant}\}.$$

The subtorus $T \subseteq \mathbb{T}^{r+s}$ of the acting torus of Z_c associated with the sublattice $\mathbb{Z}^s \subseteq \mathbb{Z}^{r+s}$ leaves $X(A,P)$ invariant and the induced T-action on $X(A,P)$ is of complexity one.

By the results of [**25**, **29**] every normal rational Fano variety with a torus action of complexity one arises from this construction.

Remark 4.5. Consider the case $r = 1$ in Constructions 4.1 and 4.4. Then the defining matrix P is of the form

$$P \;=\; \begin{bmatrix} -l_0 & l_1 & 0 \\ d_0 & d_1 & d' \end{bmatrix},$$

the algebra $R(A,P)$ equals $\mathbb{K}[T_{ij}, S_k]$, we have $X(A,P) = Z_c$ and T is a one-codimensional subtorus of $\mathbb{T}^{r+s}$. Moreover, every action of a one-codimensional subtorus on a toric Fano variety can be represented this way.

Remark 4.6. The following elementary column and row operations on the defining matrix P do not change the isomorphy type of the associated Fano variety $X(A,P)$; we call them *admissible operations*:

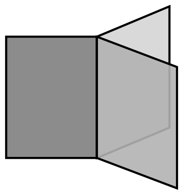
(i) swap two columns inside a block $v_{ij_1}, \ldots, v_{ij_{n_i}}$,
(ii) swap two whole column blocks $v_{ij_1}, \ldots, v_{ij_{n_i}}$ and $v_{i'j_1}, \ldots, v_{i'j_{n_{i'}}}$,
(iii) add multiples of the upper r rows to one of the last s rows,
(iv) any elementary row operation among the last s rows,
(v) swap two columns inside the d' block.

The operations of type (iii) and (iv) do not change the associated ring $R(A,P)$, whereas the types (i), (ii), (v) correspond to certain renumberings of the variables of $R(A,P)$ keeping the (graded) isomorphy type.

Construction 4.7. Let $X = X(A,P)$ be obtained from Construction 4.4. The *tropical variety* of X is the fan $\mathrm{trop}(X)$ in $\mathbb{Q}^{r+s}$ consisting of the cones

$$\lambda_i \;:=\; \mathrm{cone}(v_{i1}) + \mathrm{lin}(e_{r+1}, \ldots, e_{r+s}) \text{ for } i \;=\; 0, \ldots, r, \qquad \lambda \;:=\; \lambda_0 \cap \ldots \cap \lambda_r,$$

where $v_{ij} \in \mathbb{Z}^{r+s}$ denote the first n columns of P and $e_k \in \mathbb{Z}^{r+s}$ the k-th canonical basis vector; we call λ_i a *leaf* and λ the *lineality part* of $\mathrm{trop}(X)$.



trop(X) for $r = 2$

We say that a face $\delta_0 \preceq \delta$ of the orthant $\delta \subseteq \mathbb{Q}^{n+m}$ is an *X-face*, if it is relevant and the relative interior of $P(\delta_0)$ intersects $\mathrm{trop}(X)$. Define a fan Σ in $\mathbb{Z}^{r+s}$ by setting

$$\Sigma \ := \ \{\sigma \preceq P(\delta_0); \ \delta_0 \preceq \delta \ X\text{-face}\}.$$

Then the toric variety and Z associated with Σ is *minimal toric ambient variety* of X, that means, the smallest open toric subvariety of Z_c containing X as a closed subvariety.

Definition 4.8. Let $X = X(A, P)$ arise from Construction 4.1 and denote by Σ the fan of the minimal toric ambient variety Z of X. Define a (rational) polyhedron

$$B(-\mathcal{K}_X) \ := \ Q^{-1}(-\mathcal{K}_X) \cap \mathbb{Q}^{n+m}_{\geq 0} \ \subseteq \ \mathbb{Q}^{n+m}$$

and let $B := B(g_0) + \ldots + B(g_{r-2}) \subseteq \mathbb{Q}^{n+m}$ be the Minkowski sum of the Newton polytopes $B(g_i)$ of the relations $g_0, \ldots, g_{r-2}$ of $R(A, P)$. Let $e_Z \in \mathbb{Z}^{n+m}$ denote the sum over the canonical basis vectors e_{ij} and e_k of $\mathbb{Z}^{n+m}$.

(i) The *anticanonical polyhedron* of X is the dual polyhedron $A_X \subseteq \mathbb{Q}^{r+s}$ of the polyhedron

$$B_X \ := \ (P^*)^{-1}(B(-\mathcal{K}_X) + B - e_\Sigma) \ \subseteq \ \mathbb{Q}^{r+s}.$$

(ii) The *anticanonical complex* of X is the coarsest common refinement of polyhedral complexes

$$A^c_X \ := \ \mathrm{faces}(A_X) \sqcap \Sigma \sqcap \mathrm{trop}(X).$$

(iii) The *relative interior* of A^c_X is the interior of its support with respect to the intersection $\mathrm{Supp}(\Sigma) \cap \mathrm{trop}(X)$.

(iv) The *relative boundary* ∂A^c_X is the complement of the relative interior of A^c_X in A^c_X.

Theorem 4.9. *Let $X = X(A, P)$ arise from Construction 4.1. Then the following statements hold.*

(i) *A^c_X contains the origin in its relative interior and all primitive generators of the fan Σ are vertices of A^c_X.*

(ii) *X has at most log terminal singularities if and only if the anticanonical complex A^c_X is bounded.*

(iii) *X has at most canonical singularities if and only if 0 is the only lattice point in the relative interior of A^c_X.*

(iv) *X has at most terminal singularities if and only if 0 and the primitive generators v_ϱ for $\varrho \in \Sigma^{(1)}$ are the only lattice points of A^c_X.*

(v) *X has at most $1/k$-log canonical singularities if and only if 0 is the only k-fold lattice point in the relative interior of A^c_X.*

Construction 4.10. Let $X = X(A, P)$ arise from Construction 4.1 and let Σ the fan of the minimal toric ambient variety Z. Write $v_{ij} := P(e_{ij})$ and $v_k := P(e_k)$ for the columns of P. Consider a pointed cone of the form

$$\tau \;=\; \operatorname{cone}(v_{0j_0}, \dots, v_{rj_r}) \;\subseteq\; \mathbb{Q}^{r+s},$$

that means that τ contains exactly one v_{ij} for every $i = 0, \dots, r$. We call such τ a *P-elementary cone* and associate the following numbers with τ:

$$\ell_{\tau,i} \;:=\; \frac{l_{0j_0} \cdots l_{rj_r}}{l_{ij_i}} \text{ for } i = 0, \dots, r, \qquad \ell_\tau \;:=\; (1-r) l_{0j_0} \cdots l_{rj_r} + \sum_{i=0}^{r} \ell_{\tau,i}.$$

Moreover, we set

$$v(\tau) \;:=\; \ell_{\tau,0} v_{0j_0} + \ldots + \ell_{\tau,r} v_{rj_r} \;\in\; \mathbb{Z}^{r+s}, \qquad \varrho(\tau) \;:=\; \mathbb{Q}_{\ge 0} \cdot v(\tau) \;\in\; \mathbb{Q}^{r+s}.$$

We denote by $\mathrm{T}(A, P)$ the set of all P-elementary cones $\tau \in \Sigma$. For a given $\sigma \in \Sigma$, we denote by $\mathrm{T}(\sigma)$ the set of all P-elementary faces of σ.

Remark 4.11. Let $X = X(A, P)$ arise from Construction 4.1. Let Σ be the fan of the minimal toric ambient variety of X and $\lambda_0, \dots, \lambda_r \subseteq \operatorname{trop}(X)$ the leaves of the tropical variety of X. As in [**8**, Def. 4.1], we say that

(i) a cone $\sigma \in \Sigma$ is a *leaf cone* if $\sigma \subseteq \lambda_i$ holds for some $i = 0, \dots, r$,
(ii) a cone $\sigma \in \Sigma$ is called *big* if $\sigma \cap \lambda_i^\circ \neq \emptyset$ holds for all $i = 0, \dots, r$.

Observe that a given cone $\sigma \in \Sigma$ is big if and only if σ contains some P-elementary cone as a subset.

Proposition 4.12. *Let $X = X(A, P)$ arise from Construction 4.1, let Σ be the fan of the minimal toric ambient variety Z, denote by $\lambda_0, \dots, \lambda_r$ the leaves of $\operatorname{trop}(X)$ and by $\lambda = \lambda_0 \cap \ldots \cap \lambda_r$ its lineality part.*

(i) *The fan $\Sigma \sqcap \operatorname{trop}(X)$ consists of the cones $\sigma \cap \lambda$ and $\sigma \cap \lambda_i$, where $\sigma \in \Sigma$ and $i = 0, \dots, r$. Here, one always has $\sigma \cap \lambda \preceq \sigma \cap \lambda_i$.*
(ii) *The fan $\Sigma \sqcap \operatorname{trop}(X)$ is a subfan of the normal fan of the polyhedron B_X. In particular, for every cone $\sigma \cap \lambda_i$, there is a vertex $u_{\sigma,i} \in B_X$ with*

$$\partial A_X^{\mathrm{c}} \cap \sigma \cap \lambda_i \;=\; \{v \in \sigma \cap \lambda_i;\ \langle u_{\sigma,i}, v\rangle = -1\}.$$

(iii) *If a P-elementary cone τ is contained in some $\sigma \in \Sigma$, then τ is simplicial, $v(\tau) \in \tau^\circ$ holds, $\varrho(\tau)$ is a ray, $\varrho(\tau) = \tau \cap \lambda$ holds as well as $\mathbb{Q}\varrho(\tau) = \mathbb{Q}\tau \cap \lambda$.*
(iv) *Let $\sigma \in \Sigma$ be any cone. Then, for every $i = 0, \dots, r$, the set of extremal rays of $\sigma \cap \lambda_i \in \Sigma \sqcap \operatorname{trop}(X)$ is given by*

$$(\sigma \cap \lambda_i)^{(1)} \;=\; \{\varrho(\sigma_0);\ \sigma_0 \in \mathrm{T}(\sigma)\} \cup \{\varrho \in \sigma^{(1)};\ \varrho \subseteq \lambda_i\}.$$

(v) *The set of rays of $\Sigma \sqcap \operatorname{trop}(X)$ consists of the rays of Σ and the rays $\varrho(\sigma_0)$, where $\sigma_0 \in \mathrm{T}(A, P, \Phi)$.*
(vi) *If a P-elementary cone τ is contained in some $\sigma \in \Sigma$, then the minimum value among all $\langle u, v(\tau)\rangle$, where $u \in B_X$, equals $-\ell_\tau$.*
(vii) *Let the P-elementary cone τ be contained in $\sigma \in \Sigma$. Then $\varrho(\tau) \not\subseteq A_X^{\mathrm{c}}$ holds if and only if $\ell_\tau > 0$ holds; in this case, $\varrho(\tau)$ leaves A_X^{c} at $v(\tau)' = \ell_\tau^{-1} v(\tau)$.*

(viii) *The vertices of A_X^c are the primitive generators of Σ, i.e. the columns of P, and the points $v(\sigma_0)' = \ell_{\sigma_0}^{-1} v(\sigma_0)$, where $\sigma_0 \in \mathrm{T}(A, P, \Phi)$ and $\ell_{\sigma_0} > 0$.*

REMARK 4.13. Recall the structure of a rational projective $\mathbb{K}^*$-surface as described in Section 6 of the last chapter. Several aspects of the description in Construction 4.1 simplify:

First, the lower part $[d, d']$ of the matrix P is just a row and we have $m \leq 2$. We arrange P to be *slope ordered*, that means that for each $0 \leq i \leq r$, we order the block $v_{i1}, \ldots, v_{in_i}$ of columns in such a way that

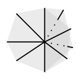
$$m_{i1} \; > \; \ldots \; > \; m_{ini}, \qquad \text{where } m_{ij} \; := \; \frac{d_{ij}}{l_{ij}}.$$

Observe that the defining fan Σ of the ambient toric variety Z is basically unique and needs no extra specification. More precisely, the rays of Σ are the cones over the columns of P and we always have the following maximal cones, which are leaf cones

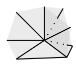
$$\tau_{ij} \; := \; \mathrm{cone}(v_{ij}, v_{ij+1}) \; \in \; \Sigma, \qquad i = 0, \ldots, r, \; j = 1, \ldots, n_i - 1.$$

Writing $v^+ := v_1 = (0, \ldots, 0, 1)$ and $v^- := v_2 = (0, \ldots, 0, -1)$ for the columns of P that arise for $m = 1, 2$, the collection of maximal cones of Σ is complemented depending on the value of m as follows

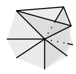
$$m = 2: \quad \text{(p-p)} \quad \begin{aligned} \tau_i^+ \; &:= \; \mathrm{cone}(v^+, v_{i1}) \\ \tau_i^- \; &:= \; \mathrm{cone}(v_{in_i}, v^-) \end{aligned}$$



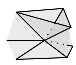
$$m = 1: \quad \text{(p-e)} \quad \begin{aligned} \tau_i^+ \; &:= \; \mathrm{cone}(v^+, v_{i1}) \\ \sigma^- \; &:= \; \mathrm{cone}(v_{0n_0}, \ldots, v_{rn_r}) \end{aligned}$$



$$\text{(e-p)} \quad \begin{aligned} \sigma^+ \; &:= \; \mathrm{cone}(v_{01}, \ldots, v_{r1}) \\ \tau_i^- \; &:= \; \mathrm{cone}(v_{in_i}, v^-) \end{aligned}$$



$$m = 0: \quad \text{(e-e)} \quad \begin{aligned} \sigma^+ \; &:= \; \mathrm{cone}(v_{01}, \ldots, v_{r1}) \\ \sigma^- \; &:= \; \mathrm{cone}(v_{0n_0}, \ldots, v_{rn_r}) \end{aligned}$$



The cones σ^+ and σ^-, if they exists, are P-elementary big cones. Furthermore all big cones are of this form. Last we fix the following notation:

$$m_{ij} \; := \; \frac{d_{ij}}{l_{ij}}, \qquad m^+ \; := \; \sum_{i=0}^{r} m_{i1}, \qquad m^- \; := \; \sum_{i=0}^{r} m_{in_i},$$

$$l^+ \; := \; l_{01} \cdots l_{r1}, \qquad l^- \; := \; l_{0n_0} \cdots l_{rn_r},$$

$$\bar{l}^+ \; := \; \sum_{i=0}^{r} \frac{1}{l_{i1}} - (r-1), \qquad \bar{l}^- \; := \; \sum_{i=0}^{r} \frac{1}{l_{in_i}} - (r-1).$$

LEMMA 4.14. *A del-Pezzo $\mathbb{K}^*$-surface $X(A, P)$ is log terminal if and only if the following statements are true:*

(i) *If there is an elliptic fixed point x^+, then $\bar{l}^+ > 0$.*
(ii) *If there is an elliptic fixed point x^-, then $\bar{l}^- > 0$.*

PROOF. Note that by Theorem 4.9 the $\mathbb{K}^*$-surface is log terminal if and only if the anticanonical complex is bounded.

By Proposition 4.12 the anticanonical complex is bounded if and only for all elementary big cones σ we have $\ell_\sigma > 0$. Since in dimension two the only elementary big cones correspond to elliptic fixed points, the statement follows with the equivalences below:

$$\ell_{\sigma^+} \ > \ 0 \quad \Leftrightarrow \quad \bar{l}^+ \ > \ 0, \qquad \ell_{\sigma^-} \ > \ 0 \quad \Leftrightarrow \quad \bar{l}^- \ > \ 0.$$

We exemplarily show the first equivalence. By definition of ℓ_{σ^+} in Construction 4.10 we have

$$\ell_{\sigma^+,i} \ = \ \frac{l^+}{l_{i1}} \quad \text{for } i = 0, \ldots, r, \qquad \ell_{\sigma^+} \ = \ (1-r)l^+ + \sum_{i=0}^{r} \ell_{\sigma^+,i}.$$

The last equality yields $\ell_{\sigma^+} = l^+\bar{l}^+$. Since $l^+ > 0$, the equivalence follows. $\square$

REMARK 4.15. Let $X(A,P)$ be a del Pezzo $\mathbb{K}^*$-surface with a bounded anticanonical complex and an elliptic fixed point, say σ^+. For the vectors defined in Construction 4.10 we obtain:

$$\begin{aligned} v(\sigma^+) \ &= \ \ell_{\sigma^+,0}v_{01} + \ldots + \ell_{\sigma^+,r}v_{r1} \\ &= \ l^+ \left(\frac{d_{01}}{l_{01}} + \cdots \frac{d_{01}}{l_{01}} \right) e_{r+1} \\ &= \ l^+ m^+ e_{r+1}. \end{aligned}$$

The analagous statement is true for an elliptic fixed point σ^-. As we have seen in the proof of Lemma 4.14 we have $\ell_{\sigma^+} = l^+\bar{l}^+$. Thus, we achieve

$$\ell_{\sigma^\pm}^{-1} v(\sigma^\pm) \ = \ m^\pm/\bar{l}^\pm e_{r+1}.$$

LEMMA 4.16. *Let $X(A,P)$ be a del Pezzo $\mathbb{K}^*$-surface with a bounded anticanonical complex. Set:*

$$v^+ \ := \ \begin{cases} e_{r+1}, & \textit{type (p-p) or (p-e)}, \\ m^+/\bar{l}^+ e_{r+1}, & \textit{type (p-p) or (p-e)}. \end{cases}$$

$$v^- \ := \ \begin{cases} -e_{r+1}, & \textit{type (e-p) or (p-p)}, \\ m^-/\bar{l}^- e_{r+1}, & \textit{type (p-e) or (e-e)}. \end{cases}$$

Then the anticanonical complex is the polyhedral complex given by the following polytopes lying in the leaves λ_i.

$$\mathsf{P}_i \ := \ A_X^c \cap \lambda_i \ = \ \mathrm{conv}(v^+, v_{i1}, \ldots, v_{in_i}, v^-).$$

Furthermore these polytopes intersect in the lineality part λ, where we have

$$A_X^c \cap \lambda \ = \ \mathrm{conv}[v^+, v^-].$$

PROOF. Since A_X^c is bounded we find the vertices of A_X^c by Proposition 4.12 (viii). They are given as the primitive ray generators v_{ij}, v_1, v_2 of the minimal ambient toric variety and for every elementary big cone τ we have another vertex given by $\ell_\tau^{-1}v(\tau)$.

If there are parabolic fixed point curves $D^\pm$ note that its corresponding primitive ray generators v_1, v_2 coincide with v^+, v^- as defined. The case of an elliptic fixed point is treated in Remark 4.15. $\square$

5. LDP-Complexes

Recall that a lattice polytope is called *Fano* if the origin lies in its interior. For every projective toric variety Z_Σ with fan $\Sigma \subseteq N_\mathbb{Q}$ there is a Fano polytope P_Z defined by:

$$P_Z \ := \ \mathrm{conv}(v_\rho;\, v_\rho \text{ primitive ray generator of } \Sigma) \ \subseteq \ N_\mathbb{Q}.$$

There is a well known one-to-one correspondence:

$$\begin{array}{rcl} \{\text{Fano polytopes}\} & \longleftrightarrow & \{\text{Fano toric varieties}\} \\ \mathsf{P} & \mapsto & Z_{\Sigma(\mathsf{P})} \\ \mathsf{P}_Z & \leftarrow\!\shortmid & Z \end{array}$$

Here, $\Sigma(\mathsf{P}) \subseteq N_\mathbb{Q}$ is the face fan of P.

In this section we establish a similar one-to-one correspondence for $\mathbb{K}^*$-surfaces in Theorem 5.10. In fact, they correspond to polyhedral complexes, so called LDP complexes.

Remark 5.1. For the rest of this chapter we fix the following notation:

Fix an integer $r \in \mathbb{Z}_{\geq 0}$, consider the rational vector space $\mathbb{Q}^{r+1}$ and set $e_0 := -e_1 \cdots - e_r$, where e_i is the i-th standard basis vector. We define the following subsets in $\mathbb{Q}^{r+1}$:

$$\lambda \ := \ \mathrm{cone}(\pm e_{r+1}), \qquad \lambda_i \ := \ \lambda + \mathrm{cone}(e_i).$$

Moreover, for every $0 \leq i \leq r$ consider primitive vectors $v_{i1}, \ldots, v_{in_i} \in \lambda_i \setminus \lambda$ and write

$$v_{ij} := l_{ij} e_i + d_{ij} e_{r+1}, \quad \text{ where } \frac{d_{ij}}{l_{ij}} > \frac{d_{ik}}{l_{ik}} \text{ whenever } j > k$$

with coprime integers $l_{ij} \in \mathbb{Z}_{\geq 1}$, $d_{ij} \in \mathbb{Z}$. Furthermore we fix the following notation:

$$m_{ij} \ := \ \frac{d_{ij}}{l_{ij}}, \qquad m^+ \ := \ \sum_{i=0}^r m_{i1}, \qquad m^- \ := \ \sum_{i=0}^r m_{in_i},$$

$$\bar{l}^+ \ := \ \sum_{i=0}^r \frac{1}{l_{i1}} - (r-1), \qquad \bar{l}^- \ := \ \sum_{i=0}^r \frac{1}{l_{in_i}} - (r-1).$$

Last, for any point in λ we set:

$$v^+ = d^+ e_{r+1}, \text{ if } d^+ > 0, \qquad v^- = d^- e_{r+1}, \text{ if } d^- < 0.$$

Definition 5.2. An *LDP precomplex* is a polyhedral complex $\mathcal{L}$ of $r+1$ convex polytopes $\mathsf{P}_i \subseteq \lambda_i$ of dimension 2 such that the following holds:

(i) The origin lies in the relative interior of $\mathcal{L}$.

(ii) The intersection of any two polygons $\mathsf{P}_i, \mathsf{P}_k$ coincides and we have $\mathsf{P}_i \cap \mathsf{P}_k = \mathrm{cone}(v^+, v^-) \subseteq \lambda$.

(iii) For every non-primitive vertex v of $\mathcal{L}$ we have $v \in \lambda$.

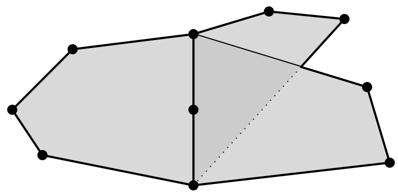
An *LDP complex* of type (p-e) is an LDP precomplex such that the following assumptions hold:

(iv) The inequalities $\bar{l}^- > 0$ and $m^+ < \bar{l}^+$ hold.

(v) The equalities $m^- = \bar{l}^- d^-$ and $d^+ = 1$ hold.

An *LDP complex* of type (e-e) is an LDP precomplex such that the following assumptions hold:

(iv) The inequalities $\bar{l}^- > 0$ and $\bar{l}^+ > 0$ hold.
(v) The equalities $m^- = \bar{l}^- d^-$ and $m^+ = \bar{l}^+ d^+$ hold.



An LDP complex in $\mathbb{Q}^3$.

Remark 5.3. We want to investigate the definition of LDP complexes by the following two remarks:

(i) There are LDP complexes of type (p-e) with primitive vertex v^-. Consider the LDP complex defined by the following vertices:

$$v_{01} = [-5,-5,6], \quad v_{02} = [-5,-5,-6], \quad v_{11} = [2,0,1],$$
$$v_{21} = [0,3,2], \quad v^+ = [0,0,1], \quad v^- = [0,0,-1].$$

Then note that condition (v) is fulfilled:

$$m^- \;=\; -\frac{6}{5}+\frac{1}{2}+\frac{2}{3} \;=\; -\frac{1}{30} \;=\; \frac{1}{6}+\frac{1}{2}+\frac{2}{3}-1 \;=\; \bar{l}^- d^-.$$

(ii) For LDP complexes $\mathcal{L}$ the following assumption is true:

$$\mathcal{V}(\mathrm{conv}(\mathcal{L})) = \{v_{ij}, v^+, v^-\}.$$

Observe that the converse is not true: Consider the complex given by the vertices

$$v_{01} = [-5,-5,6], \quad v_{02} = [-5,-5,-6], \quad v_{11} = [2,0,1],$$
$$v_{21} = [0,3,2], \quad v^+ = [0,0,1], \qquad v^- = [0,0,-1].$$

Then we find $\mathcal{V}(\mathrm{conv}(\mathcal{L})) = \{v_{ij}, v^+, v^-\}$, but this complex is not an LDP complex since condition (iv) is violated:

$$m^+ \;=\; \frac{71}{30} \;>\; \frac{1}{30} \;=\; \bar{l}^+.$$

Definition 5.4. A unimodular transformation $A \in \mathrm{GL}_{r+1}(\mathbb{Z})$ on $\mathbb{Q}^{r+1}$ is *LDP-preserving* if the following two conditions apply:

(i) We have $A(\lambda) \subseteq \lambda$.
(ii) For every $1 \le i \le r$ there is a $1 \le j \le r$ such that $A(\lambda_i) \subseteq \lambda_j$.

Remark 5.5. Let $A \in \mathrm{GL}_{r+1}(\mathbb{Z})$ be an LDP-preserving unimodular transformation. Then it can be written as a series of the following operations:

(i) For a pair $0 \le j,k \le r$ the unimodular transformation of the following form:

$$A(j,k)\colon\ \mathbb{Q}^{r+1} \to \mathbb{Q}^{r+1}, \qquad e_i \mapsto \begin{cases} e_k, & i = j,\\ e_j, & i = k,\\ e_i, & i \ne j,k.\end{cases}$$

(ii) For an index $0 \le k \le r$ and an integer $a \in \mathbb{Z}$ the unimodular transformation $A(a; r+1, k) \in \mathrm{GL}_2(\mathbb{Z})$ defined by:

$$A(a;k)\colon\ \mathbb{Q}^{r+1} \to \mathbb{Q}^{r+1}, \qquad e_i \mapsto \begin{cases} e_{r+1} + ae_k, & i = j, \\ e_i, & i \neq j. \end{cases}$$

(iii) The unimodular transformation $A^{\pm} \in \mathrm{GL}_2(\mathbb{Z})$ defined as follows:

$$A^{\pm}\colon\ \mathbb{Q}^{r+1} \to \mathbb{Q}^{r+1}, \qquad e_i \mapsto \begin{cases} e_i, & i \neq r+1, \\ -e_{r+1}, & i = r+1. \end{cases}$$

Note that these notions coincide with the admissible operations (ii), (iii) and (iv) of Remark 4.6.

Construction 5.6. Let $\mathcal{L}$ be an LDP complex. Then the following construction yields a well defined $\mathbb{K}^*$-surface:

(i) If the LDP complex is of type (p-e), let $P(\mathcal{L})$ be the matrix with columns e_{r+1} and v_{ij} for $0 \le i \le j$ and $1 \le j \le n_i$, and set:

$$P(\mathcal{L}) \ :=\ [v_{ij}, e_{r+1}], \qquad A(\mathcal{L}) \ :=\ \begin{bmatrix} 1 & 0 & -1 & -1 & \cdots & -1 \\ 0 & 1 & -1 & -2 & \cdots & 1-r \end{bmatrix}.$$

(ii) If the LDP complex is of type (e-e), let $P(\mathcal{L})$ be the matrix with columns v_{ij} for $0 \le i \le j$ and $1 \le j \le n_i$, and set:

$$P(\mathcal{L}) \ :=\ [v_{ij}], \qquad A(\mathcal{L}) \ :=\ \begin{bmatrix} 1 & 0 & -1 & -1 & \cdots & -1 \\ 0 & 1 & -1 & -2 & \cdots & 1-r \end{bmatrix}.$$

Proof. We need to show that the columns of $P(\mathcal{L})$ generate $\mathbb{Q}^{r+1}$ as a cone. Note that for every $0 \le i \le r$ there is a primitive ray generator $v_{ij} \in \lambda_i$. Thus it suffices to show that $\pm e_{r+1} \in \mathrm{cone}(v_{ij})$. Since the origin lies in the interior of $\mathcal{L}$ we find vectors $v^+, v^- \in \lambda$ such that $v^+ = d^+ e_{r+1}$ with $d^+ > 0$ and $v^- = d^- e_{r+1}$ with $d^- < 0$. Hence $\pm e_{r+1} \in \mathrm{cone}(v_{ij})$. □

Remark 5.7. Let $X = X(A, P)$ be a log del Pezzo $\mathbb{K}^*$-surface. Then the anticanonical complex A_X^c is an LDP precomplex:

(i) By the first statement of Theorem 4.9 the origin lies in the relative interior of A_X^c.
(ii) The anticanonical complex suffices the second condition by Lemma 4.16.
(iii) The only non-primitive vertices of A_X^c are given as $v^{\pm} \in \lambda$.

Definition 5.8. Let $\mathcal{L}$ be an LDP complex. Set $n_i \in \mathbb{Z}_{\ge 1}$ to be the number of vertices of P_i lying in $\lambda_i \setminus \lambda$.

(i) The LDP complex is *irredundant* if $l_{in_i} n_i \neq 1$ for all $0 \le i \le r$. Otherwise, it is called *redundant*.
(ii) The LDP complex is *non-toric* if it is irredundant and we have $r \ge 2$.
(iii) The LDP complex is *toric* if it is irredundant and we have $r = 1$.

Remark 5.9. Let $\mathcal{L}$ be a toric LDP complex with polygons $\mathsf{P}_0, \mathsf{P}_1$. Note that we obtain the following:

(i) If $\mathcal{L}$ is of type (e-e), we have $v^+ \in \mathrm{conv}(v_{01}, v_{11})$.
(ii) For both types we obtain $v^- \in \mathrm{conv}(v_{0n_0}, v_{1n_1})$.

In particular we have $\mathsf{P}_0 \cup \mathsf{P}_1$ that is a lattice polygon containing the origin with primitive vertices $v_{01}, \dots, v_{0n_0}$ and $v_{11}, \dots, v_{1n_1}$. In particular, every lattice polygon is an LDP complex.

THEOREM 5.10. *Up to LDP-preserving unimodular transformations and isomorphism of $\mathbb{K}^*$-surfaces there are mutually inverse bijections:*

$$\begin{array}{rcl} \{\textit{LDP complexes}\} & \longleftrightarrow & \{\textit{log del Pezzo } \mathbb{K}^*\textit{-surfaces}\} \\ \mathcal{L} & \mapsto & X(A(\mathcal{L}), P(\mathcal{L})) \\ A_X^c & \leftarrow\!\shortmid & X \end{array}$$

To show this statement we start by finding conditions on the defining data of a rational projective $\mathbb{K}^*$-surface $X(A,P)$ to be Fano, i.e. conditions on the entries of the defining matrix P.

REMARK 5.11 (Kleiman's criterium for ampleness). Let D be a divisor of a normal complete variety of dimension two. Then D is ample if and only if $D \cdot C > 0$ for all effective curves C.

PROPOSITION 5.12. *Let $X(A,P)$ be a log terminal rational projective $\mathbb{K}^*$-surface. We set*

$$l_{i0} \;:=\; l_{in_i+1} \;:=\; 0, \qquad d_{i0} \;:=\; \frac{m^+}{\bar{l}^+}, \qquad \textit{and} \qquad d_{in_i+1} \;:=\; \frac{m^-}{\bar{l}^-}.$$

Then the following equivalences hold:

(i) *If there is a parabolic fixed point curve $D^\pm$ we have*

$$-\mathcal{K}_X \cdot D^\pm > 0 \qquad \Leftrightarrow \qquad \pm m^\pm < \bar{l}^\pm.$$

(ii) *For a divisor D_{ij}, the intersection product $-\mathcal{K}_X \cdot D_{ij}$ is positive if and only if*

$$(l_{ij} - l_{ij+1})(d_{ij-1} - d_{ij}) - (d_{ij} - d_{ij+1})(l_{ij-1} - l_{ij}) \;>\; 0.$$

In particular, $X(A,P)$ is a del Pezzo surface if and only if all of the above inequalities hold.

REMARK 5.13. For $X = X(A,P)$, the intersection numbers of the orbit closures $D_{ij} \subseteq X$ and possible parabolic fixed point curves $D^+, D^- \subseteq X$ vanish in all but the stated cases. They are given by:

$$D_{ij} \cdot D_{ij+1} \;=\; \frac{1}{l_{ij} l_{ij+1}} \frac{1}{m_{ij} - m_{ij+1}}, \qquad D_{i1} \cdot D^+ \;=\; \frac{1}{l_{i1}}, \qquad D_{in_i} \cdot D^- \;=\; \frac{1}{l_{in_i}}.$$

$$D_{i1} \cdot D_{k1} \;=\; \begin{cases} \frac{1}{l_{i1} l_{k1}} \left(\frac{1}{m^+} - \frac{1}{m^-} \right), & \text{(e-e) with } n_i n_k = 1, \\ \frac{1}{l_{i1} l_{k1} m^+}, & \text{(e-e) with } n_i n_k \neq 1 \text{ or (e-p).} \end{cases}$$

$$D_{in_i} \cdot D_{kn_k} \;=\; \begin{cases} \frac{1}{l_{in_i} l_{kn_k}} \left(\frac{1}{m^+} - \frac{1}{m^-} \right), & \text{(e-e) with } n_i n_k = 1, \\ \frac{-1}{l_{in_i} l_{kn_k} m^-}, & \text{(e-e) with } n_i n_k \neq 1 \text{ or (p-e).} \end{cases}$$

Furthermore the self intersection numbers are given by

$$D_{i1}^2 = \begin{cases} \frac{1}{l_{i1}^2}\left(\frac{1}{m^+} - \frac{1}{m^-}\right), & \text{(e-e)}, \\ 0 & \text{(p-p)}, \\ \frac{1}{l_{i1}^2 m^+}, & \text{(e-p)}, \\ \frac{-1}{l_{i1}^2 m^-}, & \text{(p-e)}, \end{cases} \qquad \text{for } n_i = 1,$$

$$D_{ij}^2 = \begin{cases} \frac{1}{l_{ij}^2}\left(\frac{1}{m^+} - \frac{1}{m_{ij}-m_{ij+1}}\right), & j = 1 \text{ and (e-e) or (e-p)}, \\ \frac{-1}{l_{ij}^2(m_{ij}-m_{ij+1})}, & j = 1 \text{ and (p-p) or (p-e)}, \\ \frac{-(m_{ij-1}-m_{ij+1})}{l_{ij}^2(m_{ij-1}-m_{ij})(m_{ij}-m_{ij+1})}, & 1 < j < n_i, \\ \frac{-1}{l_{ij}^2(m_{ij-1}-m_{ij})}, & j = n_i \text{ and (p-p) or (e-p)}, \\ \frac{1}{l_{ij}^2}\left(-\frac{1}{m^-} + \frac{1}{m_{ij}-m_{ij-1}}\right), & j = n_i \text{ and (e-e) or (p-e)}, \end{cases} \qquad \text{for } n_i > 1,$$

$$(D^+)^2 = -m^+,$$
$$(D^-)^2 = m^-.$$

LEMMA 5.14 (Compare Proposition 4.24 in [**32**]). *Let $X := X(A,P)$ be a rational projective $\mathbb{K}^*$-surface and consider the anticanonical divisor $-\mathcal{K}_X$. Then the following is true:*

(i) *If there is a parabolic fixed point curve D^+ we have*
$$-\mathcal{K}_X \cdot D^+ > 0 \quad \Leftrightarrow \quad \bar{l}^+ - m^+ > 0.$$

(ii) *If there is a parabolic fixed point curve D^- we have*
$$-\mathcal{K}_X \cdot D^- > 0 \quad \Leftrightarrow \quad m^- + \bar{l}^- > 0.$$

(iii) *For some i with $n_i = 1$ we have the following cases:*
  (a) *If there are two elliptic fixed points x^- and x^+ we find:*
  $$-\mathcal{K}_X \cdot D_{i1} > 0 \quad \Leftrightarrow \quad m^+\bar{l}^- - m^-\bar{l}^+ > 0.$$
  (b) *If there is a parabolic fixed point curve D^+ and an elliptic fixed point x^- we have*
  $$-\mathcal{K}_X \cdot D_{i1} > 0 \quad \Leftrightarrow \quad m^- - \bar{l}^- > 0.$$
  (c) *If there is a parabolic fixed point curve D^- and an elliptic fixed point x^+ we have*
  $$-\mathcal{K}_X \cdot D_{i1} > 0 \quad \Leftrightarrow \quad \bar{l}^+ - m^+ > 0.$$
  (d) *If there are two parabolic fixed point curves D^- and D^+ we find that $-\mathcal{K}_X \cdot D_{i1} > 0$. The condition can be stated as follows:*
  $$l_{i1} + l_{i1} > 0.$$

(iv) *For some i with $n_i \geq 2$ we find:*
  (a) *If there is a parabolic fixed point curve D^+ we find*
  $$-\mathcal{K}_X \cdot D_{i1} > 0 \quad \Leftrightarrow \quad l_{i1} - l_{i2} + d_{i1}l_{i2} - d_{i2}l_{i1} > 0.$$

(b) *If there is an elliptic fixed point x^+ we find*

$$-\mathcal{K}_X \cdot D_{i1} > 0 \quad \Leftrightarrow \quad (l_{i1}d_{i2} - d_{i1}l_{i2})\bar{l}^+ + m^+ (l_{i2} - l_{i1}) > 0.$$

(c) *If $n_i \geq 3$ and $2 \leq j \leq n_i - 1$ we have*

$$-\mathcal{K}_X \cdot D_{ij} > 0 \quad \Leftrightarrow \quad (l_{ij} - l_{ij+1})(d_{ij-1} - d_{ij}) - (d_{ij} - d_{ij+1})(l_{ij-1} - l_{ij}) > 0.$$

(d) *If there is an elliptic fixed point x^- we find*

$$-\mathcal{K}_X \cdot D_{in_i} > 0 \quad \Leftrightarrow \quad (l_{in_i}d_{in_i-1} - d_{in_i}l_{in_i-1})\bar{l}^- + m^- (l_{in_i-1} - l_{in_i}) > 0.$$

(e) *If there is a parabolic fixed point curve D^- we find*

$$-\mathcal{K}_X \cdot D_{in_i} > 0 \quad \Leftrightarrow \quad l_{in_i-1} - l_{in_i} + d_{in_i-1}l_{in_i} - d_{in_i}l_{in_i-1} > 0.$$

Proof. We calculate the intersection product $-\mathcal{K}_X \cdot D_{ij}$ for every given case. We note that the anticanonical divisor can be written as follows

$$-\mathcal{K}_X = \sum_{i,j} D_{ij} + \sum_k D_k - (r-1)\sum_{j=1}^{n_{i_0}} l_{i_0j}D_{i_0j}, \qquad \text{where } 0 \leq i_0 \leq r.$$

We show the first statement and note that the second statement is proved analogously. For the index i_0 in the formula of $-\mathcal{K}_X$ given above we choose $i_0 = 0$. Then we have:

$$\begin{aligned} -\mathcal{K}_X \cdot D^+ &= \sum_{i=0}^{r} D^+ \cdot D_{i1} + (D^+)^2 - (r-1)l_{01}D^+ \cdot D_{01} \\ &= -m^+ + \sum_{i=0}^{r} \frac{1}{l_{i1}} - (r-1) \\ &= -m^+ + \bar{l}^+. \end{aligned}$$

For the statements (iii) and (iv) we choose $i_0 = i$ for the formula of $-\mathcal{K}_X$ given above, respectively. Note that we get the following expression:

$$-\mathcal{K}_X \cdot D_{ij} = D_{ij} \cdot \left(\sum_{\iota \neq i,j} D_{\iota j} + \sum_k D_k\right) + \sum_\kappa (1 - l_{i\kappa}(r+1))D_{ij} \cdot D_{i\kappa}$$

We show the assumption (iii) (a) by calculating the two summands as given:

$$\begin{aligned} D_{i1} \cdot \left(\sum_{\iota \neq i,j} D_{\iota j} + \sum_k D_k\right) &= \sum_{\iota \neq i} D_{i1} \cdot D_{\iota 1} + D_{i1} \cdot D_{\iota n_\iota} \\ &= \sum_{\iota \neq i} \frac{1}{l_{i1}l_{\iota 1}} \left(\frac{1}{m^+} - \frac{1}{m^-}\right) \\ \sum_\kappa (1 - l_{i\kappa}(r+1))D_{i1} \cdot D_{i\kappa} &= (1 + l_{i1}(r-1))D_{i1}^2 \\ &= \frac{1 + l_{i1}(r-1)}{l_{i1}^2} \left(\frac{1}{m^+} - \frac{1}{m^-}\right) \\ &= \frac{1}{l_{i1}^2} \left(\frac{1}{m^+} - \frac{1}{m^-}\right) + \frac{(r-1)}{l_{i1}} \left(\frac{1}{m^+} - \frac{1}{m^-}\right) \end{aligned}$$

Therefore, combining the two calculations we find:

$$\begin{aligned} -\mathcal{K}_X D_{ij} &= \sum_\iota \frac{1}{l_{i1}l_{\iota 1}}\left(\frac{1}{m^+}-\frac{1}{m^-}\right)+\frac{r-1}{l_{i1}}\left(\frac{1}{m^+}-\frac{1}{m^-}\right) \\ &= -\frac{1}{l_{i1}m^+m^-}\left(m^+\left(\sum_\iota \frac{1}{l_{\iota n_\iota}}-(r-1)\right)-m^-\left(\sum_\iota \frac{1}{l_{\iota 1}}-(r-1)\right)\right) \\ &= -\frac{1}{l_{i1}m^+m^-}\left(m^+\bar{l}^- - m^-\bar{l}^+\right) \end{aligned}$$

We go on by showing statement (iii) (b) and note that the assumption (iii) (c) is proved analagously. Again, we first consider the two summands as above:

$$\begin{aligned} D_{i1}\cdot\left(\sum_{\iota\neq i,j} D_{\iota j}+\sum_k D_k\right) &= \sum_{i\neq\iota} D_{i1}\cdot D_{in_i}+D_{i1}\cdot D^+ \\ &= \sum_{\iota\neq i}\frac{-1}{l_{i1}l_{\iota n_\iota}m^-}+\frac{1}{l_{i1}} \\ \sum_\kappa (1-l_{i\kappa}(r+1))D_{i1}\cdot D_{i\kappa} &= (1+l_{i1}(r-1))D_{i1}^2 \\ &= -\frac{1+l_{i1}(r-1)}{l_{i1}^2 m^-} \\ &= -\frac{1}{l_{i1}^2m^-}+\frac{(r-1)}{l_{i1}}\frac{1}{m^-} \end{aligned}$$

Adding the two equalities yields the intersection product $-\mathcal{K}_X\cdot D_{i1}$, namely:

$$\begin{aligned} -\mathcal{K}_X\cdot D_{i1} &= -\sum_\iota \frac{1}{l_{i1}l_{\iota n_\iota}m^-}+\frac{1}{l_{i1}}+(r-1)\frac{1}{l_{i1}m^-} \\ &= \frac{1}{l_{i1}m^+}\left(-\sum_\iota\frac{1}{l_{\iota n_\iota}}+(r-1)+m^-\right) \\ &= \frac{1}{l_{i1}m^+}\left(-\bar{l}^-+m^-\right) \end{aligned}$$

For the statement (iii) (d) we turn to the same summands as above:

$$\begin{aligned} D_{i1}\cdot\left(\sum_{\iota\neq i,j} D_{\iota j}+\sum_k D_k\right) &= D_{i1}\cdot(D^+ + D^-) \\ &= \frac{1}{l_{i1}}+\frac{1}{l_{i1}} \\ \sum_\kappa (1-l_{i\kappa}(r+1))D_{i1}\cdot D_{i\kappa} &= (1-l_{i1}(r+1))D_{i1}^2 \;=\; 0 \end{aligned}$$

Since the second summand vanishes we end up with the following equality for the intersection product, which can be expressed as follows:

$$-\mathcal{K}_X\cdot D_{i1} = \frac{1}{l_{i1}^2}\left(l_{i1}+l_{i1}\right)$$

We turn to the statements (iv) using the same methods as above. We will show statements (a), (b) and (c) and note that statements (d) and (e) are shown analoguosly to statements (b) and (a), respectively.

For the statement (a) we consider the second summand. Note that the only intersection products that do not vanish are given for $\kappa = 1, 2$. Therefore the second summand is given as follows:

$$\begin{aligned}
& (1 - l_{i1}(r-1))D_{i1}^2 + (1 - l_{i2}(r-1))D_{i1} \cdot D_{i2} \\
= \; & -\frac{1 - l_{i1}(r-1)}{l_{i1}^2(m_{i1} - m_{i2})} + \frac{1 - l_{i2}(r-1)}{l_{i1}l_{i2}(m_{i1} - m_{i2})} \\
= \; & \frac{-l_{i2} + l_{i1}l_{i2}(r-1) + l_{i1} - l_{i1}l_{i2}(r-1)}{l_{i1}^2 l_{i2}(m_{i1} - m_{i2})} \\
= \; & \frac{l_{i1} - l_{i2}}{l_{i1}^2 l_{i2}(m_{i1} - m_{i2})}
\end{aligned}$$

Now, since there is a parabolic fixed point D^+, the only non-vanishing component of the first summand is $D_{i1} \cdot D^+$, hence we find:

$$\begin{aligned}
-\mathcal{K}_X \cdot D_{i1} \; &= \; D_{i1} \cdot D^+ + (1 - l_{i1}(r-1))D_{i1}^2 + (1 - l_{i2}(r-1))D_{i1} \cdot D_{i2} \\
&= \; \frac{1}{l_{i1}} + \frac{l_{i1} - l_{i2}}{l_{i1}^2 l_{i2}(m_{i1} - m_{i2})} \\
&= \; \frac{l_{i1}l_{i2}(m_{i1} - m_{i2}) + (l_{i1} - l_{i2})}{l_{i1}^2 l_{i2}(m_{i1} - m_{i2})} \\
&= \; \frac{1}{l_{i1}^2 l_{i2}(m_{i1} - m_{i2})}\left(l_{i1} - l_{i2} + d_{i1}l_{i2} - d_{i2}l_{i1}\right)
\end{aligned}$$

For the statement (iv) (b) we consider the first summand of the intersection product $-\mathcal{K}_X \cdot D_{i1}$:

$$D_{i1} \cdot \left(\sum_{\iota \neq i, j} D_{\iota j} + \sum_k D_k \right) \; = \; \sum_{\iota \neq i} D_{i1} \cdot D_{\iota 1} \; = \; \sum_{\iota \neq i} \frac{1}{l_{i1} l_{\iota 1} m^+}$$

For the second summand we note, as before, that the only intersection products that do not vanish are given for $\kappa = 1, 2$. Therefore the second summand is given as follows:

$$\begin{aligned}
& (1 - l_{i1}(r-1))D_{i1}^2 + (1 - l_{i2}(r-1))D_{i1} \cdot D_{i2} \\
= \; & \frac{1 - l_{i1}(r-1)}{l_{i1}^2}\left(\frac{1}{m^+} - \frac{1}{m_{i1} - m_{i2}}\right) + \frac{1 - l_{i2}(r-1)}{l_{i1}l_{i2}} \frac{1}{m_{i1} - m_{i2}} \\
= \; & \frac{1}{l_{i1}m^+}\left(\frac{1}{l_{i1}} - (r+1)\right) + \frac{-l_{i2} + l_{i1}l_{i2}(r-1) + l_{i1} - l_{i1}l_{i2}(r-1)}{l_{i1}^2 l_{i2}(m_{i1} - m_{i2})} \\
= \; & \frac{1}{l_{i1}m^+}\left(\frac{1}{l_{i1}} - (r+1)\right) + \frac{l_{i1} - l_{i2}}{l_{i1}^2 l_{i2}(m_{i1} - m_{i2})}
\end{aligned}$$

Since the intersection number $-\mathcal{K}_X \cdot D_{i1}$ is the sum of the intersection numbers already calculated we find:

$$\begin{aligned}
-\mathcal{K}_X \cdot D_{i1} \; &= \; \sum_{\iota \neq i} \frac{1}{l_{i1} l_{\iota 1} m^+} + \frac{1}{l_{i1}m^+}\left(\frac{1}{l_{i1}} - (r+1)\right) + \frac{l_{i1} - l_{i2}}{l_{i1}^2 l_{i2}(m_{i1} - m_{i2})} \\
&= \; \frac{1}{l_{i1}m^+}\left(\sum_i \frac{1}{l_{i1}} - (r-1) + \frac{m^+(l_{i1} - l_{i2})}{l_{i1}l_{i2}(m_{i1} - m_{i2})}\right) \\
&= \; -\frac{1}{l_{i1}^2 l_{i2} m^+ (m_{i1} - m_{i2})}\Big((l_{i1}d_{i2} - d_{i1}l_{i2})\bar{l}^+ + m^+\left(l_{i2} - l_{i1}\right)\Big)
\end{aligned}$$

Last we turn to statement (iv) (c). Note that since $1 < j < n_i$ we find that the first summand vanishes since $D_{ij} \cdot D_{\iota\kappa} = 0$ for $i \neq \iota$. In order to calculate the second summand we consider the following equality:

$$\begin{aligned}
& -l_{ij-1}D_{ij-1} \cdot D_{ij} - l_{ij+1}D_{ij} \cdot D_{ij+1} \\
= \; & -\frac{l_{ij-1}}{l_{ij-1}l_{ij}(m_{ij-1} - m_{ij})} - \frac{l_{ij+1}}{l_{ij}l_{ij+1}(m_{ij} - m_{ij+1})} \\
= \; & \frac{-l_{ij-1}l_{ij+1}(m_{ij} - m_{ij+1}) - l_{ij+1}l_{ij-1}(m_{ij-1} - m_{ij})}{l_{ij-1}l_{ij}l_{ij+1}(m_{ij-1} - m_{ij})(m_{ij} - m_{ij+1})} \\
= \; & \frac{-(m_{ij-1} - m_{ij+1}))}{l_{ij}(m_{ij-1} - m_{ij})(m_{ij} - m_{ij+1})} \\
= \; & l_{ij}D_{ij}^2
\end{aligned}$$

This shows that the second summand can be written as $\sum_\kappa D_{ij}D_{i\kappa}$ and hence we find:

$$\begin{aligned}
& -\mathcal{K}_X \cdot D_{ij} \\
= \; & \sum_\kappa D_{ij}D_{i\kappa} \\
= \; & D_{ij-1} \cdot D_{ij} + D_{ij}^2 + D_{ij+1} \cdot D_{ij} \\
= \; & \frac{1}{l_{ij-1}l_{ij}(m_{ij-1} - m_{ij})} - \frac{(m_{ij-1} - m_{ij+1})}{l_{ij}^2(m_{ij-1} - m_{ij})(m_{ij} - m_{ij+1})} + \frac{1}{l_{ij}l_{ij+1}(m_{ij} - m_{ij+1})} \\
= \; & \frac{l_{ij}l_{ij+1}(m_{ij} - m_{ij+1}) - l_{ij-1}l_{ij+1}(m_{ij-1} - m_{ij+1}) + l_{ij-1}l_{ij}(m_{ij-1} - m_{ij})}{l_{ij-1}l_{ij}^2l_{ij+1}(m_{ij-1} - m_{ij})(m_{ij} - m_{ij+1})} \\
= \; & \frac{l_{ij-1}m_{ij-1}(l_{ij} - l_{ij+1}) + l_{ij}m_{ij}(l_{j+1} - l_{j-1}) + l_{ij+1}m_{ij+1}(l_{ij-1} - l_{ij})}{l_{ij-1}l_{ij}^2l_{ij+1}(m_{ij-1} - m_{ij})(m_{ij} - m_{ij+1})} \\
= \; & \frac{d_{ij-1}(l_{ij} - l_{ij+1}) + d_{ij}(l_{ij+1} - l_{ij-1}) + d_{ij+1}(l_{ij-1} - l_{ij})}{l_{ij-1}l_{ij}^2l_{ij+1}(m_{ij-1} - m_{ij})(m_{ij} - m_{ij+1})} \\
= \; & \frac{(l_{ij} - l_{ij+1})(d_{ij-1} - d_{ij}) - (d_{ij} - d_{ij+1})(l_{ij-1} - l_{ij})}{l_{ij-1}l_{ij}^2l_{ij+1}(m_{ij-1} - m_{ij})(m_{ij} - m_{ij+1})}
\end{aligned}$$

Note that in every case the expression for the intersection product $-\mathcal{K}_X \cdot D_{ij}$ or $-\mathcal{K}_X \cdot D^+$ is given as a product of a positive rational number and the terms stated. Hence we have shown the assumptions. □

Proof of Proposition 5.12. The statement follows immediately by casewise comparing the inequality with the ones given in Lemma 5.14. We exemplarily consider the case (iv) (b) of Lemma 5.14. Since $X(A, P)$ is log-terminal we can divide the inequality given by $\bar{l}^+$ to find:

$$\begin{aligned}
(l_{i1}d_{i2} - d_{i1}l_{i2}) + d^+(l_{i2} - l_{i1}) &= (l_{i1} - l_{i2})(d^+ - d_{i1}) - (d_{i1} - d_{i2})(0 - l_{i1}) \\
&= (l_{i1} - l_{i2})(d_{i0} - d_{i1}) - (d_{i1} - d_{i2})(l_{i0} - l_{i1})
\end{aligned}$$

This shows the statement. All other cases can be seen analaguously. □

The next objective of this section is to show the implications of Proposition 5.15 which yields that every del Pezzo surface possesses at most one parabolic fixed point curve.

Proposition 5.15 (Compare Lemma 5.7 in [**32**]). *Let $X(A,P)$ be a non-toric rational porjective $\mathbb{K}^*$-surface with irredunant defining matrix P. The the following implications hold:*

(i) *If $m^+ < \bar{l}^+$ then $m^- \leq -\bar{l}^-$.*
(ii) *If $m^- > -\bar{l}^-$ then $m^+ \geq \bar{l}^+$.*

Corollary 5.16. *Let $X(A,P)$ be a log terminal del Pezzo surface. Then there is at most one parabolic fixed point curve.*

Proof. Assume there is a del Pezzo surface $X(A,P)$ with two parabolic fixed point curves. Then Proposition 5.12 (i) yields that $m^+ < \bar{l}^+$ and $m^- > -\bar{l}^-$, a contradiction to Proposition 5.15. $\square$

Lemma 5.17. *Consider a defining matrix P of a rational projective $\mathbb{K}^*$-surface $X(A,P)$. Then the following inequalities hold:*

(i) *For the slopes m_{ij} we find:*

$$m_{ij} + \frac{l_{ij}-1}{l_{ij}} \;\geq\; \lceil m_{ij} \rceil, \qquad m_{ij} - \frac{l_{ij}-1}{l_{ij}} \;\geq\; \lfloor m_{ij} \rfloor.$$

(ii) *Furthermore the following inequality holds:*

$$\lceil m_{i1} \rceil - \lfloor m_{in_i} \rfloor \;\geq\; 1.$$

(iii) *We find that the following is true:*

$$\left(m^+ - \bar{l}^+\right) - \left(m^- + \bar{l}^-\right) + 4 \;\geq\; r+1.$$

Proof. To prove the first statement we write $d_{ij} = q_{ij} l_{ij} + r_{ij}$, where $q_{ij}, r_{ij} \in \mathbb{Z}$ and $0 \leq r_{ij} < l_{ij}$. We consider the case that $r_{ij} \neq 0$. Note that in this case we have $\lfloor m_{ij} \rfloor = q_{ij}$ and $\lceil m_{ij} \rceil = q_{ij}+1$. Thus we find:

$$\begin{aligned} m_{ij} + \frac{l_{ij}-1}{l_{ij}} &= \frac{(q_{ij}+1)l_{ij} + r_{ij} - 1}{l_{ij}} = (q_{ij}+1) + \frac{r_{ij}-1}{l_{ij}} \;\geq\; \lceil m_{ij} \rceil, \\ m_{ij} - \frac{l_{ij}-1}{l_{ij}} &= \frac{q_{ij} l_{ij} + r_{ij} - l_{ij} + 1}{l_{ij}} = q_{ij} + \frac{1-(l_{ij}-r_{ij})}{l_{ij}} \leq \lfloor m_{ij} \rfloor. \end{aligned}$$

Now for $r_{ij} = 0$, we find $l_{ij} = 1$, since $\gcd(l_{ij}, d_{ij})$, therefore the inequalities above also hold as shown here:

$$m_{ij} \pm \frac{l_{ij}-1}{l_{ij}} \;=\; m_{ij} \;=\; d_{ij} = \lceil m_{ij} \rceil = \lfloor m_{ij} \rfloor.$$

We turn to statement (ii). It is clear that the difference of the integers $\lceil m_{i1} \rceil, \lfloor m_{in_i} \rfloor$ is positive since $m_{i1} \geq m_{in_i}$, furthermore the difference vanishes only if $l_{i1} n_i = 1$ since the following holds:

$$\lceil m_{i1} \rceil - \lfloor m_{in_i} \rfloor = 0 \quad \Leftrightarrow \quad m_{i1} = m_{in_i} \text{ and } m_{i1} \in \mathbb{Z} \quad \Leftrightarrow \quad l_{i1} = 1 \text{ and } n_i = 1.$$

This contradicts irredundancy, i.e. the difference is greater than 1.

For the last statement we find the following series of inequalities where the first estimate stems from statement (ii) and the second one from statement (i):

$$\begin{aligned}
r+1 &\leq \sum_{i=0}^{r} \lceil m_{i1} \rceil - \lfloor m_{in_i} \rfloor \\
&\leq \sum_{i=0}^{r} m_{i1} + \frac{l_{i1}-1}{l_{i1}} - \left(\sum_{i=0}^{r} m_{in_i} - \frac{l_{in_i}-1}{l_{in_i}} \right) \\
&= \left(\sum_{i=0}^{r} m_{i1} - \sum_{i=0}^{r} \frac{1}{l_{i1}} + (r+1) \right) - \left(\sum_{i=0}^{r} m_{in_i} + \sum_{i=0}^{r} \frac{1}{l_{in_i}} - (r+1) \right) \\
&= \left(m^+ - \bar{l}^+ \right) - \left(m^- + \bar{l}^- \right) + 4.
\end{aligned}$$

$\square$

Proof of Proposition 5.15. We show that the following two inequalities cannot hold simultaneously:

$$m^+ - \bar{l}^+ < 0 \quad \text{and} \quad m^- + \bar{l}^- > 0$$

Therefore assume that the inequalities do hold. By Lemma 5.17 (iii) we find that $r+1 < 4$, i.e. $r = 2$, since the $\mathbb{K}^*$-surface is non-toric.

For $r = 2$ we consider the inequality as given in the proof of Lemma 5.17:

$$3 \;=\; r+1 \;\leq\; \sum_{i=0}^{r} \lceil m_{i1} \rceil - \lfloor m_{in_i} \rfloor \;<\; 4.$$

Since the sum is an integer, equality holds, i.e. $\sum_{i=0}^{r} \lceil m_{i1} \rceil - \lfloor m_{in_i} \rfloor = 3$, moreover the following inequality holds:

$$\begin{aligned}
2 \;>\; 2 + \left(m^+ - \bar{l}^+ \right) &= \sum_{i=0}^{r} \left(m_{i1} + \frac{l_{i1}-1}{l_{i1}} \right) \\
&\geq \sum \lceil m_{i1} \rceil \\
&\geq \sum \lceil m_{i1} \rceil - \lfloor m_{in_i} \rfloor \;=\; 3
\end{aligned}$$

Since the sum in the second line is an integer, we find $\sum \lceil m_{i1} \rceil = 3$ and therefore $\lfloor m_{in_i} \rfloor = 0$, i.e. $m_{in_i} \geq 0$ for all $i = 0, \ldots, r$. This contradicts $m^- < 0$. Therefore the two stated inequalities cannot hold simultaneously.

$\square$

Lemma 5.18. *Let $\mathsf{P}_i \subseteq \lambda_i$ be a convex polygon having a face $\operatorname{conv}(v^+, v^-) \subseteq \lambda$. Let $v_{ij} \in \lambda_i$ be vertices of P_i with*

$$v_{ij} \;=\; l_{ij} e_i + d_{ij} e_{r+1}, \qquad \frac{d_{i1}}{l_{i1}} \;>\; \cdots \;>\; \frac{d_{in_i}}{l_{in_i}}.$$

Set $v_{i0} := v^+$ and $v_{in_i+1} := v^-$. Then for every $j = 1, \ldots, n_i$ the following inequality holds:

$$(l_{ij} - l_{ij+1})(d_{ij-1} - d_{ij}) - (d_{ij} - d_{ij+1})(l_{ij-1} - l_{ij}) \;>\; 0.$$

Proof. Set H_{ij} to be the affine hyperplane that contains v_{ij} and v_{ij+1} and the closed halfspace H_{ij}^+ which contains the origin. This halfspace is defined by the linear form u_{ij} and the integer b_{ij} given as follows:

$$u_{ij} \;=\; (d_{ij+1} - d_{ij}, l_{ij} - l_{ij+1}) \quad \text{and} \quad b_{ij} \;=\; d_{ij+1}l_{ij} - d_{ij}l_{ij+1}.$$

Now note that since $b_{ij} < 0$ we find $v_{ij-1} \in H_{ij}^+ \setminus H_{ij}$ if and only if the following inequality holds:

$$(d_{ij+1} - d_{ij})l_{ij-1} + (l_{ij} - l_{ij+1})d_{ij-1} \;>\; d_{ij+1}l_{ij} - d_{ij}l_{ij+1}$$

After subtraction b_{ij} on both sides of the inequality we end up with the following inequality:

$$\begin{aligned} & (d_{ij+1} - d_{ij})l_{ij-1} + (l_{ij} - l_{ij+1})d_{ij-1} - d_{ij+1}l_{ij} + d_{ij}l_{ij+1} \;>\; 0 \\ \Leftrightarrow \quad & (l_{ij} - l_{ij+1})(d_{ij-1} - d_{ij}) - (d_{ij} - d_{ij+1})(l_{ij-1} - l_{ij}) \;>\; 0. \end{aligned}$$

□

Proof of Theorem 5.10. By Remark 5.9 every toric LDP complex is a Fano polygon. Since correspondence between Fano polygons and toric surfaces is well known, it only remains to show the correspondence between non-toric LDP complexes and non-toric $\mathbb{K}^*$-surfaces.

We first show that for an LDP complex the surface $X := X(A(\mathcal{L}), P(\mathcal{L}))$ is log del Pezzo. Note that by Construction 5.6 we obtain that X is a well defined $\mathbb{K}^*$-surface. Furthermore X is log terminal by Lemma 4.14 since we have $\bar{l}^- > 0$ for an LDP complex of type (p-e) and $\bar{l}^+\bar{l}^- > 0$ for an LDP complex of type (e-e) by condition (iv) of the Definition 5.2.

Last observe that convexity of the polygons P_i implies that the inequalities of Lemma 5.18 hold. These inequalities coincide with the ones given in Proposition 5.12 (ii). Moreover, for an LDP complex of type (p-e) condition (iv) states $m^+ < \bar{l}^+$, which is exactly the inequality in Proposition 5.12 (i), i.e. X is a del Pezzo surface.

Now for a log del Pezzo $\mathbb{K}^*$-surface X the anticanonical complex A_X^c is an LDP precomplex by Remark 5.7. Furthermore there is at most one parabolic fixed point curve by Corollary 5.16, which we can take as D^+.

If there is a parabolic fixed point curve, the conditions (iv) of Definition 5.2 is equivalent to X being log terminal (see Lemma 4.14) and the first condition of Proposition 5.12. and conditions (v) of Definition 5.2 follow since $v^+ = e_{r+1}$ is a vertex of A_X^c and $v^- = m^-/\bar{l}^- e_{r+1}$ by Lemma 4.16.

If there is no parabolic fixed point curve, the conditions (iv) of Definition 5.2 is equivalent to X being log terminal (see Lemma 4.14) and conditions (v) of Definition 5.2 follow since $v^+ = m^+/\bar{l}^+ e_{r+1}$ and $v^- = m^-/\bar{l}^- e_{r+1}$ by Lemma 4.16.

Last, for an LDP complex $\mathcal{L}$ set $X := X(A(\mathcal{L}), P(\mathcal{L}))$. It is clear that $\mathcal{L} = A_X^c$. Furthermore for a rational projective $\mathbb{K}^*$-surface $X(A, P)$ it is clear by definition that $P = P(A_X^c)$. Therefore, the maps are bijections. □

6. Properties of LDP complexes

In this section we collect some basic properties of LDP complexes. We start by considering redundant LDP complexes, i.e. LDP complexes such that $l_{i1}n_{i1} > 1$ for all $i = 0, \dots, r$.

Proposition 6.1. *For every non-toric rational projective $\mathbb{K}^*$-surface X there is an irredundant LDP complex $\mathcal{L}$ with $X \cong X(A(\mathcal{L}), P(\mathcal{L}))$.*

Construction 6.2. Let $\mathcal{L}$ be a redundant LDP complex with $r \geq 2$, i.e. there is an index $0 \leq \iota \leq r$ such that $l_{\iota 1} n_\iota = 1$. The *redundancy elimination at ι* is defined by the following steps:

(i) Apply a unimodular transformation $A(-d_{\iota 1}; \iota)$ to $\mathcal{L}$ and set $\mathcal{L}' = A\mathcal{L}$.

(ii) Set $\mathrm{pr}_\iota \colon \mathbb{Z}^{r+2} \to \mathbb{Z}^{r+1}$ defined as follows:

$$\mathrm{pr}_\iota \colon \mathbb{Z}^{r+1} \to \mathbb{Z}^r, \qquad e_i \mapsto \begin{cases} e_i, & i < \iota \\ 0, & i = \iota \\ e_{i-1}, & i > \iota. \end{cases}$$

Then the polygons $\mathrm{pr}_\iota(\mathsf{P}_i)$, $i \neq \iota$ define an LDP complex $\mathcal{L}''$ in $\mathbb{Z}^r$.

Successively applying eliminations of redundancies yields an irredundnant LDP complex or a toric LDP complex.

Construction 6.3. Let (A, P) be defining data as in Construction 4.1; in particular, P is built from the blocks L, 0, d and d'. By a *redundant extension* of (A, P) we mean defining data of the form

$$\tilde{A} \ := \ [a_0, \ldots, a_r, a_{r+1}], \qquad \tilde{P} \ := \ \begin{bmatrix} P & 0 & 0 \\ 0 & 1 & 0 \\ d & 0 & d' \end{bmatrix}$$

such that the column a_{r+1} of $\tilde{A}$ is not proportional to any of the columns $a_0, \ldots, a_r$ of A. The pair $(\tilde{A}, \tilde{P})$ satisfies the conditions of 4.1, we have a canonical isomorphism $\tilde{K} \cong K$ of grading groups and one of the associated graded $\mathbb{K}$-algebras:

$$R(\tilde{A}, \tilde{P}) \ \to \ R(A, P), \qquad T_{ij} \ \mapsto \ \begin{cases} T_{ij}, & i = 0, \ldots, r, \\ 0, & i = r+1, \end{cases} \qquad S_k \ \mapsto \ S_k.$$

In particular the rational projective $\mathbb{K}^*$-surfaces $X(A, P)$ and $X(\tilde{A}, \tilde{P})$ are isomorphic.

Proof of Proposition 6.1. Let (A, P) be a pair of defining data for X, i.e. $X \cong X(A, P)$ and consider the anticanonical complex $\mathcal{L} := A_X^c$. If there is an index $1 \leq i \leq r$ such that $l_{i1} n_i = 1$ apply an redundancy elimination on $\mathcal{L}$ as in Construction 6.2 to achieve an LDP complex $\mathcal{L}'$. Note that by Construction 6.3 we have $X(A(\mathcal{L}'), P(\mathcal{L}')) \cong X(A, P)$. Successively applying redundancy eliminations yields the statement. $\square$

Definition 6.4. An LDP complex $\mathcal{L}$ is *almost k-hollow* if the only k-fold point in the relative interior of $\mathcal{L}$ is the origin, i.e.

$$\mathcal{L}^\circ \cap (k\mathbb{Z})^{r+1} \ = \ \{0\}.$$

Remark 6.5. Consider the LDP complex $\mathcal{L}$ corresponding to a log del Pezzo surface X. Then note that by Theorem 4.9 the surface X has at most $1/k$-log canonical singularities if and only if $\mathcal{L}$ is almost k-hollow.

Lemma 6.6. *An LDP complex is almost k-hollow if and only if the following holds:*

(i) P_i *is* k*-hollow for every* $0 \le i \le r$.
(ii) $d^+ \le k$ *and* $d^- \ge -k$.

Proof. The points in the relative interior of $\mathcal{L}^\circ$ are exactly the points in the relative interior of P_i and in $\mathrm{conv}(v^-, v^+)$. Hence $\mathcal{L}$ is almost k-hollow if and only if if there are no k-fold points in P_i for all $0 \le i \le r$, i.e. P_i is k-hollow, and the only k-fold point in $\mathrm{conv}(v^-, v^+)$ is the origin. The latter is equivalent to (ii). $\square$

Remark 6.7. For an LDP complex consider the number $\bar{l}^+$ defined in Remark 5.1 and the condition $\bar{l}^+ > 0$.

Assume $l_{01} \ge \cdots \ge l_{r1}$. Then $l_{31} = \cdots = l_{r1} = 1$ holds and (l_{01}, l_{11}, l_{21}) is a platonic triple, i.e. one of the following tripels:

$$(l_{01}, l_{11}, 1), \quad (l_{01}, 2, 2), \quad (3,3,2), \quad (4,3,2), \quad (5,3,2).$$

Furthermore the following is true:

$$\bar{l}^+ \;=\; \sum_{i=0}^{r} \frac{1}{l_{i1}} - (r-1) \;=\; \begin{cases} \frac{1}{l_{01}} + \frac{1}{l_{11}}, & \text{if } (l_{01}, l_{11}, l_{21}) \;=\; (l_{01}, l_{11}, 1), \\ \frac{1}{l_{01}}, & \text{if } (l_{01}, l_{11}, l_{21}) \;=\; (l_{01}, 2, 2), \\ \frac{1}{6}, & \text{if } (l_{01}, l_{11}, l_{21}) \;=\; (3,3,2), \\ \frac{1}{12}, & \text{if } (l_{01}, l_{11}, l_{21}) \;=\; (4,3,2), \\ \frac{1}{30}, & \text{if } (l_{01}, l_{11}, l_{21}) \;=\; (5,3,2). \end{cases}$$

Note that the maximal value for $\bar{l}^+$ is given by 2. The same statements hold for $\bar{l}^-$.

Lemma 6.8. *Consider an LDP complex* $\mathcal{L}$ *with vertices* $v_{ij} = (l_{ij}, d_{ij})$ *such that* $l_{2n_2} = \cdots = l_{rn_r} = 1$ *and* $d_{2n_2} = \cdots = d_{rn_r} = 0$. *Consider the following* $\mathbb{Z}$*-linear map:*

$$\pi\colon \mathbb{Z}^{r+1} \to \mathbb{Z}^2, \qquad e_i \mapsto \begin{cases} e_1, & i = 1, \\ 0, & i \ne 1, r+1, \\ e_{r+1} & i = r+1. \end{cases}$$

Then $\pi(\mathcal{L}) \subseteq \mathbb{Z}^2$ *is a convex polygon with vertices* $\pi(v_{0j})$, $\pi(v_{1j})$ *and* $\pi(v^+)$. *In particular, the following statements are equivalent:*

(i) P_0 *and* P_1 *are* k*-hollow and* $(d^-, d^+) \cap k\mathbb{Z} = \{0\}$.
(ii) *The polygon* $\pi(\mathcal{L})$ *is almost* k*-hollow.*

Proof. To prove the first statement it suffices to show that for the vector $\pi(v^-)$ we have $\pi(v^-) \in \mathrm{conv}(\pi(v_{0n_0}), \pi(v_{1n_1}))$. Therefore we first consider the following sum of rational numbers:

$$\frac{1}{\bar{l}^- l_{0n_0}} + \frac{1}{\bar{l}^- l_{1n_1}} \;=\; \frac{1}{\bar{l}^-}\left(\frac{1}{l_{0n_0}} + \frac{1}{l_{1n_1}}\right) \;=\; \frac{\bar{l}^-}{\bar{l}^-} \;=\; 1.$$

Now note the following linear combination of vectors which proves the statement:

$$\begin{aligned}\frac{1}{\bar{l}^- l_{0n_0}}\pi(v_{0n_0}) + \frac{1}{\bar{l}^- l_{1n_1}}\pi(v_{1n_1}) &= \frac{1}{\bar{l}^- l_{0n_0} l_{1n_1}}\left(l_{1n_1}\pi(v_{0n_0}) + l_{0n_0}\pi(v_{1n_1})\right) \\ &= \frac{1}{\bar{l}^- l_{0n_0} l_{1n_1}}\left(l_{0n_0} l_{1n_1}(m_{0n_0} + m_{1n_1})e_2\right) \\ &= \frac{m^-}{\bar{l}^-}e_2 \ = \ d^- e_2 \ = \ \pi(v^-).\end{aligned}$$

This means that $\pi(v^-)$ can be written as linear combination of vectors in $\operatorname{conv}(\pi(v_{0n_0}), \pi(v_{1n_1}))$, furthermore the coefficients add up to 1, i.e. $\pi(v^-)$ is no vertex of $\pi(\mathcal{L})$.

The latter equivalence follows with Lemma 6.6. $\square$

The following part defines a standard form for rational projective $\mathbb{K}^*$-surfaces, see Definition 6.21. This definition is independent of ampleness of the anticanonical divisor. The corresponding statement also holds for LDP complexes. It yields a quick possibility to check whether two LDP complexes $\mathcal{L}$ and $\mathcal{L}'$ are unimodular equivalent, i.e. there is an LDP-preserving unimodular transformation A such that $A(\mathcal{L}) = \mathcal{L}'$.

The first part establishes a notion of symmetry for matrices P when applying an admissible operation of type (iv).

Definition 6.9. Let $X(A, P)$ be a rational projective $\mathbb{K}^*$-surface. We define the following:

(i) The *slopes* of P are the rational numbers $m_{ij} = \frac{d_{ij}}{l_{ij}}$.
(ii) The i-th *block* of P is the vector $m_i = (m_{i1}, \ldots, m_{in_i}) \in \mathbb{Q}^{n_i}$.
(iii) For every block we define the *normalized slope vectors*:

$$\begin{aligned}\sigma_i &:= (m_{i1} + a_i, \ldots, m_{in_i} + a_i), \\ \tau_i &:= (-m_{in_i} + b_i, \ldots, -m_{i1} + b_i),\end{aligned}$$

where $a_i = \lceil -m_{i1} \rceil \in \mathbb{Z}$ and $b_i = \lceil m_{in_i} \rceil \in \mathbb{Z}$.
(iv) A *symmetric* block is a block m_i such that $\sigma_i = \tau_i$. Otherwise the block is called *asymmetric*.

Example 6.10. Let $X(A, P)$ be the $\mathbb{K}^*$-surface with defining matrix

$$P := \begin{bmatrix} -3 & 2 & 0 & 0 \\ -3 & 0 & 1 & 1 \\ -2 & 1 & 1 & -1 \end{bmatrix}.$$

The matrix P consists of the following three blocks:

$$m_0 = (-2/3), \qquad m_1 = (1/2), \qquad m_2 = (1, -1).$$

For every $0 \leq i \leq 2$ we compute the normalized slope vectors σ_i and τ_i:

$$\sigma_0 = (1/3), \quad \tau_0 = (2/3), \quad \sigma_1 = (1/2), \quad \tau_1 = (1/2),$$

$$\sigma_2 = (0, -2), \quad \tau_2 = (0, -2).$$

Thus there are two symmetric blocks and one asymmetric block, namely block 0.

DEFINITION 6.11. Let m_i be an asymmetric block of a matrix P. Then we define the following:

(i) Let $1 \leq j \leq n_i$ be the smallest index such that the j-th entry of σ_i and τ_i differ.

The *pair of asymmetric slopes* is the following pair of rational numbers:

$$\Big(m_{ij} + c_i, -m_{in_{i-j+1}} + d_i\Big) \in \mathbb{Q}^2,$$

$$\text{where} \quad c_i := \lceil -m_{ij} \rceil, \quad d_i := \lceil m_{in_{i-j+1}} \rceil.$$

(ii) The block m_i *faces up* if $m_{ij} + c_i > -m_{in_{i-j+1}} + d_i$. Otherwise the block *faces down.*

EXAMPLE 6.12. We turn again to Example 6.10 and consider the defining matrix P given as follows:

$$P := \begin{bmatrix} -3 & 2 & 0 & 0 \\ -3 & 0 & 1 & 1 \\ -2 & 1 & 1 & -1 \end{bmatrix}.$$

Note that the only asymmetric block is block 0. Its pair of asymmetric slopes is given by $(1/3, 2/3)$, i.e. the block 0 faces down. The remaining blocks are symmetric.

DEFINITION 6.13. Let $X(A, P)$ be a rational projective $\mathbb{K}^*$-surface.

(i) The *symmetry vector* $\mathfrak{S}(m_i)$ of an asymmetric block m_i is σ_i if the block faces up and τ_i if the block faces down.
(ii) The matrix P is called *symmetric* if it is of type (e-e) or (p-p), we have $m^+ = -m^-$ and for every symmetry vector there are the same number of blocks facing up and facing down.

REMARK 6.14. For any matrix P consider the matrix P^- obtained by applying the admissible operation (iv) on P and rearranging the columns such that P^- is slope ordered.

For the rest of this part we use the notation m_{ij}^P and $m_{ij}^{P^-}$ when talking about the slopes of P and P^-, respectively. Moreover the sum of slopes is written as m_P^+ and $m_{P^-}^-$, respectively. It is clear that the following holds:

(i) For every slope we have $m_{ij}^P = -m_{in_i-j+1}^{P^-}$.
(ii) For the slope sum we obtain $m_P^+ = -m_{P^-}^-$ and $m_P^- = -m_{P^-}^+$.

LEMMA 6.15. *Let $X(A, P)$ be a rational projective $\mathbb{K}^*$-surface and consider P^-. Then the following conditions are equivalent.*

(i) *P can be obtained by applying a series of admissible operations of type (i) to (iii) and (v) on P^-.*
(ii) *P is symmetric.*

PROOF. Suppose that P can be brought to the form of P^- by admissible operations of type (i) to (iii) and (v). Hence we have

$$m_P^+ = m_{P^-}^+ = -m_P^-.$$

Consider the slope vector $(m_{i1}, \dots, m_{in_i})$ of a block of P such that no change of blocks is necessary to bring P^- to the form of P. That means there is an integer $a \in \mathbb{Z}$ such that the following holds

$$(m_{i1}, \dots, m_{in_i}) \;=\; (m_{in_i} + a, \dots, m_{i1} + a)\,.$$

Thus this block is symmetric by definition. Suppose on the other hand that $(m_{i1}, \dots, m_{in_i})$ is a slope vector of a block such that a change of blocks is necessary to bring P^- to the form of P. That means there is an integer $\iota \neq i$ and $a \in \mathbb{Z}$ such that

$$(m_{i1}, \dots, m_{in_i}) \;=\; (m_{\iota n_\iota} + a, \dots, m_{\iota 1} + a)$$

for some $\iota \neq i$ and $a \in \mathbb{Z}$. Note that the blocks have the same symmetry vector but differ by the directions they face to. We conclude that for every symmetry vector occurring there is the same number of blocks facing up and down, which proves that P is symmetric.

Now suppose that P is symmetric. Consider P^- and exchange blocks of P^- facing up with blocks of the same symmetry vector facing down, which yields a matrix $\tilde{P}^-$. By definition of symmetry vectors every block of $\tilde{P}^-$ now coincides with the blocks of P up to an admissible operation of type (iii). Hence we can assume that

$$m_{ij}^{\tilde{P}^-} \;=\; m_{ij}^{P} \quad \text{for all } i = 1, \dots, r.$$

Note that the 0-th blocks of P and $\tilde{P}^-$ are of the same symmetry type, i.e. they only differ by an integer vector in $\mathbb{Z}^{n_i}$. Since additionally $m^+ = -m^-$ we find $m_{01}^{\tilde{P}^-} = m_{01}^{P}$, thus the 0-th blocks coincide as well. $\square$

Remark 6.16. Let $X := X(A, P)$ be a rational projective $\mathbb{K}^*$-surface with a symmetric defining matrix P. Then the following two assertions hold:

(i) There is an automorphism $\varphi\colon X \to X$ switching source and sink, i.e. we have
$$\varphi(x^+) \;=\; x^- \qquad \text{or} \qquad \varphi(D^+) \;=\; \varphi(D^-).$$

(ii) The unit component of the automorphism group is trivial.

The latter follows since otherwise there are two quasismooth simple elliptic fixed points or two parabolic fixed point curves admitting vertical roots, a contradiction to Theorem 8.4 and Proposition 9.18, respectively.

Definition 6.17. Let P be a defining matrix of a rational projective $\mathbb{K}^*$-surface with asymmetric blocks $m_{i_1}, \dots, m_{i_k}$. Then we define the following:

(i) Let $\mathfrak{S}(m_i)$ be a symmetry vector. Set $k(\mathfrak{S}(m_i))^+$ to be the number of blocks of P with symmetry vector $\mathfrak{S}(m_i)$ facing up. Analagu-osly, we define $k(\mathfrak{S}(m_i))^-$ for blocks facing down.

(ii) The *matrix symmetry vector* $\mathfrak{S}(P)$ is the lexicographically maximal among all symmetry vectors $\mathfrak{S}(m_i)$ such that $k(\mathfrak{S}(m_i))^+ \neq k(\mathfrak{S}(m_i))^-$.

Lemma 6.18. *Let $X(A, P)$ be a rational projective $\mathbb{K}^*$-surface and consider P^- and the matrix symmetry vector $\mathfrak{S}(P)$ and $\mathfrak{S}(P^-)$. The following assertions hold:*

(i) *If P is of type (p-e), then P^- is of type (e-p).*
(ii) *If P is of type (e-e) or (p-p) and $m_P^+ > -m_P^-$, then P^- is of type (e-e) or (p-p) and $m_{P^-}^+ < -m_{P^-}^-$.*
(iii) *Suppose that P is of type (e-e) or (p-p) and $m_P^+ = -m_P^-$ holds. Assume further that for the symmetry vector $\mathfrak{S}(P)$ we have $k(\mathfrak{S}(P))^+ > k(\mathfrak{S}(P))^-$.*
Then P^- is of type (e-e) or (p-p) and $m_P^+ = -m_P^-$. Moreover for the symmetry vector $\mathfrak{S}(P^-)$ we have $k(\mathfrak{S}(P^-))^+ < k(\mathfrak{S}(P))^-$.

Proof. The first and the second statements are clear by the defintion of P^-. For the last statement note that every block in P that faces up yields a block in P^- with the same symmetry vector that faces down. Therefore we have

$$k(\mathfrak{S}(P^-))^+ \;=\; k(\mathfrak{S}(P))^- \;<\; k(\mathfrak{S}(P))^+ \;=\; k(\mathfrak{S}(P^-))^-.$$

□

Definition 6.19. Let $X(A,P)$ be a $\mathbb{K}^*$-surface. The matrix P *faces up* if one of the following conditions holds.

(i) The matrix P is of type (p-e).
(ii) The matrix P is of type (e-e) and $m^+ > -m^-$.
(iii) The matrix is of type (e-e) and $m^+ = -m^-$. Additionally, for the symmetry vector of P we have:

$$k(\mathfrak{S}(P))^+ \;>\; k(\mathfrak{S}(P))^-.$$

The matrix P *faces down* if P^- faces up.

Remark 6.20. Let P be a defining matrix of a rational projective $\mathbb{K}^*$-surface $X(A,P)$. Then Lemma 6.18 yields the following three cases:

(i) The matrix P is symmetric.
(ii) The matrix P faces up.
(iii) The matrix P^- faces up.

Definition 6.21. Let $X(A,P)$ be a rational projective $\mathbb{K}^*$-surface. The defining matrix P is in *standard form* if the following assumptions hold:

(i) The matrix is irredundant, i.e. $n_i l_{i1} > 1$ for all $i = 0,\ldots,r$.
(ii) The matrix P does not face down.
(iii) For the length of the arms of $X(A,P)$ we have:

$$n_{\max} \;:=\; n_0 \geq \cdots \geq n_r.$$

(iv) We have $0 \;\leq\; m_{i1} \;<\; 1$ for all $i = 1,\ldots,r$.
(v) Let ι be the largest index such that $n_\iota = n_{\max}$ and set $a := \lceil -m_{01} \rceil \in \mathbb{Z}$. Then for every $0 \leq i \leq \iota$ we have

$$(m_{01},\ldots,m_{0n_0}) \;\geq_{\mathrm{lex}}\; (m_{i1}-a,\ldots,m_{in_i}-a)\,.$$

(vi) For indices $i_1 \leq \cdots \leq i_k$ with $n_{i_1} = \cdots = n_{i_k}$ and $i_j \neq 0$, the tuples

$$\left(m_{i_1 1},\ldots,m_{i_1 n_{i_1}}\right),\ldots,\left(m_{i_k 1},\ldots,m_{i_k n_{i_k}}\right)$$

are in descending lexicographical order.

Proposition 6.22. *The standard form of a defining matrix P is unique. In particular, two defining matrices P and P' are equivalent if and only if their standard forms coincide.*

Proof. It is clear that every matrix P can be brought into standard form using admissible operations.

Suppose that P is in standard form and there is a distinct matrix P' in standard form such that P and P' are equivalent. We show that $P = P'$.

It is clear that admissible operations do not change the number of blocks with the same length. Thus observe that for $i = 1, \dots, r$ the blocks of P and P' coincide by conditions (ii), (iii) and (vi) of Definition 6.21.

For $i = 0$ note that by condition (v) the block is uniquely determined among all blocks of length $n_{\max}$. In particular, the 0-th block of P and P' coincide.

Last note that by Remark 6.20 and the second condition of Definition 6.21 P and P' both face up or they are symmetric. In the second case note that the standard form of P^- coincides with the one of P by Lemma 6.15 using the same arguments as above. □

7. Contraction of LDP complexes

In section 2, we investigated a deconstruction method of polygons, see Construction 2.6, that was a vital ingredient in the classification process for almost k-hollow polygons.

Now, our objective is to achieve a similar method for LDP complexes, see Construction 7.3: We define a process of removing vertices of an LDP complex without losing basic properties. Theorem 7.4 ensures that this process is well-behaved.

Moreover in Proposition 7.9, we observe that this process corresponds to contractions of divisors of $\mathbb{K}^*$-surfaces. In particular, in Theorem 7.7 we obtain that contractions of $\mathbb{K}^*$-surfaces preserve ampleness of the anticanonical divisor, log terminality and $1/k$-log canonicity.

Definition 7.1. Let $\mathcal{L}$ be an LDP complex consisting of the polygons P_i with n_i vertices in $\lambda_i \setminus \lambda$. If $n_\iota \geq 2$, we define the following:

$$m^+(\iota) \;=\; m^+ + m_{\iota 2} - m_{\iota 1}, \qquad m^-(\iota) \;=\; m^- + m_{\iota n_\iota - 1} - m_{\iota n_\iota},$$
$$\bar{l}^+(\iota) \;=\; \bar{l}^- + \frac{1}{l_{\iota 2}} - \frac{1}{l_{\iota 1}}, \qquad \bar{l}^-(\iota) \;=\; \bar{l}^- + \frac{1}{l_{\iota n_\iota - 1}} - \frac{1}{l_{\iota n_\iota}}.$$

Last if the rational numbers $\bar{l}^\pm(\iota)$ do not vanish we define

$$d^\pm(\iota) \;=\; \frac{m^\pm(\iota)}{l^\pm(\iota)}.$$

Definition 7.2. Let $\mathcal{L}$ be an LDP complex. A *combinatorially contractible* vertex v of $\mathcal{L}$ is a vertex sufficing one of the following conditions:

(i) The complex is of type (p-e) and for $v = v^+$ we have $m^+ > 0$.

(ii) For $v = v_{\iota 1}$ we have $n_\iota > 1$ and the complex is of type (p-e) or $m^+(\iota) > 0$.

(iii) For $v = v_{\iota j}$ with $j = 2, \dots, n_\iota - 1$.

(iv) For $v = v_{\iota n_\iota}$ we have $n_\iota > 1$ and $m^-(\iota) > 0$.

Construction 7.3. Let $\mathcal{L}$ be an irredundant LDP complex with a combinatorially contractible vertex v. We define the following polygons:

(i) If $\mathcal{L}$ is of type (p-e) and $v = v^+$, for all $i = 0, \ldots, r$ set

$$\mathsf{P}_i^v \ := \ \operatorname{conv}\left(\mathcal{V}(\mathsf{P}_i) \setminus \{v\} \cup \{m^+/\bar{l}^+ e_{r+1}\}\right).$$

(ii) If $\mathcal{L}$ is of type (e-e) and $v = v_{\iota 1}$ we set

$$\begin{aligned} \mathsf{P}_\iota^v \ &:= \ \operatorname{conv}\left(\mathcal{V}(\mathsf{P}_\iota) \setminus \{v, v^+\} \cup \{d^+(\iota) e_{r+1}\}\right), \\ \mathsf{P}_i^v \ &:= \ \operatorname{conv}\left(\mathcal{V}(\mathsf{P}_i) \setminus \{v^+\} \cup \{d^+(\iota) e_{r+1}\}\right), \qquad i \neq \iota. \end{aligned}$$

(iii) If $\mathcal{L}$ is of type (e-e) or (p-e) and $v = v_{in_i}$ we set

$$\begin{aligned} \mathsf{P}_\iota^v \ &:= \ \operatorname{conv}\left(\mathcal{V}(\mathsf{P}_\iota) \setminus \{v, v^-\} \cup \{d^-(\iota) e_{r+1}\}\right), \\ \mathsf{P}_i^v \ &:= \ \operatorname{conv}\left(\mathcal{V}(\mathsf{P}_i) \setminus \{v^-\} \cup \{d^-(\iota) e_{r+1}\}\right), \qquad i \neq \iota. \end{aligned}$$

(iv) If $\mathcal{L}$ is not of type (p-e) or $v \neq v^+, v_{i1}, v_{in_i}$, set

$$\mathsf{P}_\iota^v \ := \ \operatorname{conv}\left(\mathcal{V}(\mathsf{P}_\iota) \setminus \{v\}\right), \qquad \mathsf{P}_i^v \ := \ \mathsf{P}_i, \quad i \neq \iota.$$

The LDP complex $\mathcal{L}^v$ *contracted along the vertex* v is the polyhedral complex consisting of the polygons P_i^v for $i = 0, \ldots, r$. If $v = v^+$, the complex is of type (e-e), in all the other cases the types of $\mathcal{L}$ and $\mathcal{L}^v$ coincide.

The following theorem shows that the polyhedral complex $\mathcal{L}^v$ is indeed an LDP complex.

Theorem 7.4. *Let $\mathcal{L}$ be an irredundant LDP complex with contractible vertex v. Then the following is true:*

(i) *The polyhedral complex $\mathcal{L}^v$ is an LDP complex.*
(ii) *We obtain $\mathcal{L}^v \subseteq \mathcal{L}$. In particular, if $\mathcal{L}$ is almost k-hollow, then so is $\mathcal{L}^v$.*

Lemma 7.5. *Let $\mathcal{L}$ be an LDP complex of type (p-e) with v^+ combinatorally contractible. Then the following statements hold:*

(i) *We have $\bar{l}^+ > 0$.*
(ii) *We find $m^+/\bar{l}^+ < 1$, i.e. $m^+/\bar{l}^+ e_{r+1} \in \operatorname{conv}(0, v^+)$.*

Proof. We first note that $m^+ > 0$ by Definition 7.2 (i) since v^+ is combinatorially contractible. Now the statements are immediate consequences of the inequality in condition (iv) of Definition 5.2, namely:

$$\bar{l}^+ \ > \ m^+ \ > \ 0 \qquad \text{and} \qquad \bar{l}^+ - m^+ \ > \ 0 \quad \Leftrightarrow \quad \frac{m^+}{\bar{l}^+} \ < \ 1.$$

The last part of statement (ii) is clear since $v^+ = e_{r+1}$. □

Lemma 7.6. *Let $\mathcal{L}$ be an LDP complex of type (e-e) and a contractible vertex $v_{\iota 1}$ and $n_\iota \geq 2$. Then the following statements hold:*

(i) *We have $\bar{l}^+(\iota) > 0$.*
(ii) *We find $d^+ > d^+(\iota)$, i.e. $d^+(\iota) e_{r+1} \in \operatorname{conv}(0, v^+)$.*

The same statements are true for an LDP complex of type (p-e) or (e-e) and a combinatorially contractible vertex $v_{\iota n_\iota}$, namely $\bar{l}^-(\iota) > 0$ and $d^-(\iota) e_{r+1} \in \operatorname{conv}(0, v^-)$.

Proof. We prove exemplarily the case for the first statements and note that the proof for the latter statements is analagous. First note that since $v_{\iota 1}$ is contractible the inequality $m^+(\iota) > 0$ given in Definition 7.2 (ii) holds, furthermore Lemma 5.14 (iv) (b) yields another inequality which after division on both sides by $l_{\iota 1} l_{\iota 2} \bar{l}^+ > 0$ can be restated as follows:

$$(m_{\iota 2} - m_{\iota 1}) + \frac{m^+}{\bar{l}^+}\left(\frac{1}{l_{\iota 1}} - \frac{1}{l_{\iota 2}}\right) \; > \; 0.$$

Hence we have two inequalities depending on $m_{\iota 1} - m_{\iota 2}$. Restating these two inequalities solving for $m_{\iota 1} - m_{\iota 2}$ we find

$$m^+ > m_{\iota 1} - m_{\iota 2} > \frac{m^+}{\bar{l}^+}\left(\frac{1}{l_{\iota 1}} - \frac{1}{l_{\iota 2}}\right).$$

Using these estimates for $m_{\iota 1} - m_{\iota 2}$ and the fact that m^+ is positive we find:

$$\begin{aligned} \frac{1}{\bar{l}^+}\left(\frac{1}{l_{\iota 1}} - \frac{1}{l_{\iota 2}}\right) \quad &< \quad 1 \\ \Leftrightarrow \qquad \bar{l}^+(\iota) \;=\; \bar{l}^+ + \left(\frac{1}{l_{\iota 2}} - \frac{1}{l_{\iota 1}}\right) \quad &> \quad 0. \end{aligned}$$

Hence we have shown the first statement.

For the second statement we consider $d^+(\iota) = \frac{m^+(\iota)}{\bar{l}^+(\iota)}$:

$$\begin{aligned} \frac{m^+(\iota)}{\bar{l}(\iota)} &= \frac{m^+}{\bar{l}^+} + \frac{m^+(\iota)\bar{l}^+ - m^+\bar{l}^+(\iota)}{\bar{l}^+\,\bar{l}^+(\iota)} \\ &= \frac{m^+}{\bar{l}^+} + \frac{(\bar{l}^+ - \bar{l}^+(\iota))m^+ + (m^+(\iota) - m^+)\bar{l}^+}{\bar{l}^+\,\bar{l}^+(\iota)} \\ &= \frac{m^+}{\bar{l}^+} + \frac{\left(\frac{1}{l_{\iota 1}} - \frac{1}{l_{\iota 2}}\right) d^+ + (m_{\iota 2} - m_{\iota 1})}{\bar{l}^+(\iota)} \\ &< \frac{m^+}{\bar{l}^+}. \end{aligned}$$

In the third step we used the definitions of $\bar{l}(\iota)$ and $m^+(\iota)$ and the inequality follows directly from the inequality of Lemma 5.14 (iv) (b) and statement (i), i.e. $\bar{l}^+(\iota) > 0$.

The second part of the statement follows immediately since $v^+ = d^+ e_{r+1}$ and $d^+(\iota) < d^+$. □

Proof of Theorem 7.4. We want show that $\mathcal{L}^v$ is a well-defined LDP complex, i.e. that conditions (i) to (v) of Definition 5.2 hold. We do this casewise using the definitions of $\mathcal{L}^v$ in Construction 7.3.

We start with case (i). By Lemma 7.5 we obtain that $\bar{l}^+$ does not vanish, i.e. $m^+/\bar{l}^+$ is a well defined rational number. Since it is positive we also conclude that the origin lies in the relative interior of $\mathcal{L}^v$. Note that conditions (ii) and (iii) are fulfilled by definition of $\mathcal{L}^v$.

We turn to conditions (iv) and (v). Note that by definition $\mathcal{L}^v$ is of type (e-e), furthermore $\bar{l}^- > 0$ and $m^- = \bar{l}^- d^-$ since the vertices v_{in_i} and v^- are not affected by the contraction. Last note that $\bar{l}^+ > 0$ by Lemma 7.5 (i) and $m^+ = \bar{l}^+ d^+$ by definition of $\mathcal{L}^v$. Last observe that $\mathsf{P}_i^v \subseteq \mathsf{P}_i$ for every $i = 0, \ldots, r$ since $m^+/\bar{l}^+ e_{r+1} \in \operatorname{conv}(0, v^+)$. Thus we have $\mathcal{L}^v \subseteq \mathcal{L}$.

Case (ii) and (iii) are proven analagously; We exemplarily show (ii): First observe that $n_\iota > 1$ since $v_{\iota n_\iota}$ is contractible, furthermore for $\mathcal{L}^v$ the upper slope sum and the sum of inverse integers l_{i1} are given by $m^+(\iota)$ and $\bar{l}^+(\iota)$. By Lemma 7.6 we know that $\bar{l}^+(\iota) > 0$, i.e. $d^+(\iota)$ is well defined. Moreover, since $d^+(\iota)$ is positive, we conclude $0 \in \mathcal{L}^v$, i.e. condition (i) of the definition of an LDP-complex is fulfilled. Observe that conditions (ii) and (iii) of are fulfilled by the construction of $\mathcal{L}^v$.

To see that conditions (iv) and (v) hold note that $\bar{l}^- > 0$ and $m^- = \bar{l}^- d^-$ remain unchanged since the vertices v_{in_i} and v^- are not affected by the contraction, moreover $\bar{l}(\iota)^+ > 0$ by Lemma 7.6 (i) and $m^+ = \bar{l}^+ d^+$ by definition of $\mathcal{L}^v$. Last observe that $\mathsf{P}_i^v \subseteq \mathsf{P}_i$ for every $i = 0, \ldots, r$ since $m^+/\bar{l}^+ e_{r+1} \in \operatorname{conv}(0, v^+)$. Thus we have $\mathcal{L}^v \subseteq \mathcal{L}$.

For case (iv) note that the vertices of P_i and P_i^v contained in λ and the vertices v_{i1} and v_{in_i} coincide. Therefore all conditions of Definition 5.2 hold. □

Recall that an irreducible curve D on a normal projective surface X is called *contractible* if there is a morphism $\pi\colon X \to X'$ onto a normal surface X' mapping D to a point $x' \in X'$ and inducing an isomorphism from $X \setminus D$ onto $X' \setminus \{x'\}$.

Theorem 7.7. *Let $X(A,P)$ be a rational projective $\mathbb{K}^*$-surface with a contraction $\pi\colon X \to X'$ of an irreducible curve D. Then the following holds:*

(i) *If X is log del Pezzo, then so is X'.*
(ii) *If X is $1/k$-log canonical, then so is X'.*

Remark 7.8. Consider a log del Pezzo $\mathbb{K}^*$-surface $X = X(A,P)$. Given a column v of P, let $D \subseteq X$ be the corresponding curve and P' the matix obtained from P by removing v. Then the following statements are equivalent.

(i) The curve $D \subseteq X$ is contractible.
(ii) The matrices A and P' define a $\mathbb{K}^*$-surface $X' = X(A,P')$.

If one of these statements holds, then D is contracted by the $\mathbb{K}^*$-equivariant morphism $X \to X'$ induced by the map of fans $\Sigma \to \Sigma'$ and there is a unique cone $\sigma' \in \Sigma'$ containing v in its relative interior.

Proposition 7.9. *Let $X(A,P)$ be a log del Pezzo $\mathbb{K}^*$-surface with corresponding LDP complex $\mathcal{L}$. Given a column v of P, let $D \subseteq X$ be the corresponding curve and P' the matix obtained from P by removing v. Then the following statements are equivalent:*

(i) *The divisor D is contractible.*
(ii) *The vertex $v \in \mathcal{L}$ is contractible.*

In particular, we obtain $X(A,P') = X(A,P(\mathcal{L}^v))$.

Lemma 7.10. *Let P be a matrix of the following type as seen in Construction 4.1 and Remark 4.13:*

$$P = \begin{bmatrix} L & 0 \\ d & d' \end{bmatrix} \in \operatorname{Mat}(r+1, n+m; \mathbb{Z}),$$

where $m \leq 2$ and for the matrix $L \in \mathrm{Mat}(r, n; \mathbb{Z})$ we have

$$L = \begin{bmatrix} -l_0 & l_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \dots & l_r \end{bmatrix}, \qquad l_i = (l_{i1}, \dots, l_{in_i}) \in \mathbb{Z}^{n_i}_{\geq 1}.$$

As above, we denote the columns as v_{ij} and v^+ or v^-, if present. Then we have $\mathrm{cone}(v_{ij}, v^+, v^-) = \mathbb{Q}^{r+1}$ if and only if one of the following two statements holds:

(i) *We have $m = 0$ and $m^+ > 0$ and $m^- < 0$.*
(ii) *We have $m = 1$ and $m^- < 0$.*
(iii) *We have $m = 2$.*

Proof. We start by showing the following equivalence:

$$\pm e_{r+1} \in \mathrm{cone}(v_{ij}) \qquad \Leftrightarrow \qquad \pm m^{\pm} > 0.$$

Thus, consider the following two equalities:

$$m^+ e_{r+1} = \sum_{i=0}^{r} \frac{1}{l_{i1}} v_{i1}, \qquad m^- e_{r+1} = \sum_{i=0}^{r} \frac{1}{l_{in_i}} v_{in_i}.$$

Hence if $m^+ > 0$, then $e_{r+1} \in \mathrm{cone}(v_{ij}, v^+)$ and $m^- < 0$, then $-e_{r+1} \in \mathrm{cone}(v_{ij}, v^+)$.

Now consider any positive combination of e_{r+1}, i.e.

$$e_{r+1} = \sum_{ij} a_{ij} v_{ij}, \qquad a_{ij} \in \mathbb{Z}_{\geq 0}.$$

Note that the shape of P implies that for every $0 \leq i \leq r$ there is an $1 \leq j \leq n_i$ such that a_{ij} does not vanish. Moreover, we have

$$v_{ij} = l_{ij} \left(\frac{1}{l_{i1}} v_{i1} + (m_{ij} - m_{i1}) e_{r+1} \right) \quad \text{for all } i = 0, \dots, r, \quad j = 1, \dots, n_i.$$

Plugging this equality into the positive combination and solving for e_{r+1} yields:

$$\left(1 + \sum_{ij} a_{ij} l_{ij} (m_{i1} - m_{ij}) \right) e_{r+1} = \sum_{ij} a_{ij} \frac{l_{ij}}{l_{i1}} v_{i1} = \sum_{i} \left(\sum_{j} a_{ij} \frac{l_{ij}}{l_{i1}} \right) v_{i1}.$$

Note that for $i = 1, \dots, r$ the i-th component on the right side vanishes, i.e. we have:

$$-l_{01} \left(\sum_{j} a_{0j} \frac{l_{0j}}{l_{01}} \right) + l_{i1} \left(\sum_{j} a_{ij} \frac{l_{ij}}{l_{i1}} \right) = 0.$$

Now plugging this expression into the equation above and considering the $(r+1)$-st component, we obtain

$$\left(1 + \sum_{ij} a_{ij} l_{ij} (m_{i1} - m_{ij}) \right) = \left(\sum_{j} a_{0j} l_{0j} \right) \sum_{i} \frac{1}{l_{i1}} d_{i1} = \left(\sum_{j} a_{0j} l_{0j} \right) m^+.$$

Note that both brackets are positive, i.e. we have $m^+ > 0$. The same argument yields the statement that $-e_{r+1} \in \mathrm{cone}(v_{ij})$ implies $m^- < 0$. This shows the equivalence above.

We go on to show the statement. By the shape of P it is clear that $\pm e_{r+1} \in \operatorname{cone}(v_{ij}, v^+, v^-)$ implies $e_i \in \operatorname{cone}(v_{ij}, v^+, v^-)$ for all $i = 1, \dots, r$ since the following holds:

$$e_i \;=\; \frac{1}{l_{i1}} v_{i1} - m_{i1} e_{r+1}.$$

Hence to see that the columns of P generate $\mathbb{Q}^{r+1}$ as cone it suffices to show that $\pm e_{r+1} \in \operatorname{cone}(v_{ij}, v^+, v^-)$. Last, observe that in the situation of statements (i) to (iii) this is exactly the case, when the conditions there hold.

We exemplarily show (ii): Since $m = 1$ there is a parabolic fixed point curve, i.e. e_{r+1} is a column of P. Moreover we have $-e_{r+1} \in \operatorname{cone}(v_{ij}, v^+)$ if and only if $-e_{r+1} \in \operatorname{cone}(v_{ij})$. The latter one is equivalent to $m^- < 0$ by the considerations above. □

Proof of Proposition 7.9. By Remark 6.6 it suffices to show that the data (P', A') is defining data for a rational projective $\mathbb{K}^*$-surface, i.e. it suffices to show that the columns of P' generate $\mathbb{Q}^{r+1}$ as a cone.

By Lemma 7.10 it suffices to consider the following conditions, where $(m')^{\pm}$ are the slope sums for P':

$$(m')^+ > 0 \text{ and } (m')^- < 0 \quad \text{type (e-e)}, \qquad (m')^- < 0 \quad \text{type (p-e)}.$$

Casewise we consider the matrices $P' := P(\mathcal{L}^v)$ for a contractible vertex v as in Definition 7.2. We have:

(i) For type (p-e), v^+ is contractible if and only if $m^+ > 0$. Note that we have $(m')^+ = m^+$ and $(m')^- = m^- < 0$.
(ii) For $v = v_{\iota 1}$ we have $m^+(\iota) = (m')^+ > 0$ and $m^- = (m')^- < 0$.
(iii) If $v = v_{\iota j}$, where $j \neq 1, n_\iota$ the slope sums do not change, i.e. $m^{\pm} = (m')^{\pm}$.
(iv) In the case $v = v_{\iota n_\iota}$ we have $m^+ = (m')^+$ and $(m')^- = m^-(\iota) < 0$.

Thus, the statement follows with Lemma 7.10. □

Proof of Theorem 7.7. Consider a log del Pezzo $\mathbb{K}^*$-surface with contractible divisor D. By Proposition 7.9 the corresponding LDP complex $\mathcal{L}$ possesses a contractible vertex $v \in \mathcal{L}$ and we have $X' = X(A, P(\mathcal{L}^v))$. Now the statement follows by Theorem 5.10 since $\mathcal{L}^v$ is an almost k-hollow LDP complex by Theorem 7.4. This shows statement (i).

For the second statement note that by Remark 6.5 X and X' are $1/k$-log canonical if and only if $\mathcal{L}$ and $\mathcal{L}' = \mathcal{L}^v$ are almost k-hollow, respectively. Hence the statement follows since $\mathcal{L}^v \subseteq \mathcal{L}$ by Theorem 7.4. □

Remark 7.11. Recall from Construction 2.4 that the Cox ring of every $\mathbb{K}^*$-surface is given as $R := R(A, P)$ for defining matrices A and P. Furthermore the following holds, compare [**2**, Theorem 3.4.3.7]:

$$\dim(X) \;=\; \dim(R) - \dim(K_{\mathbb{Q}}).$$

Since R is a normal complete intersection, we find $\dim(R) = n + m - (r+1)$, where $n + m$ is the number of variables of R and $r + 1$ is the number of relations. Hence for surfaces, the Picard number $\rho(X)$ can be expressed as follows:

$$\rho(X) \;=\; \dim(K_{\mathbb{Q}}) \;=\; n + m - (r+1).$$

It is clear that with every contracion the Picard number decreases by one, moreover for $\rho(X) = 1$ the surface does not admit any contractions.

Remark 7.12. Let $\mathcal{L}$ be an LDP complex with vertex v. Consider the corresponding $\mathbb{K}^*$-surface X and the curve D with weight $[D] \in \mathrm{Cl}(X)$.

By [**2**, Remark 4.1.3.4] the toric prime divisor D is contractible if and only if the following holds:

(i) The weight $[D] \in \mathrm{Cl}(X)$ sits on an extremal ray of the effective cone $\mathrm{Eff}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$.
(ii) No other weight $[D']$ lies on $\mathbb{Q}[D] \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$.

Such a weight is called *exceptional*. Moreover, there are no contractible curves in X if and only if there are no exceptional weights, i.e. if and only if there are at least two weights of curves on every extremal ray of $\mathrm{Eff}(X)$.

Lemma 7.13. *Let $\mathcal{L}$ be an LDP complex and consider the corresponding log del Pezzo surface $X(A(\mathcal{L}), P(\mathcal{L}))$. Then the following statements are equivalent:*

(i) *The vertex v is contractible.*
(ii) *The selfintersection number D^2 is negative.*

Proof. In order to show the statement we reformulate the self-intersection numbers as given in Remark 5.13. For $n_i = 1$ we have:

$$(D_{i1})^2 = \begin{cases} \frac{1}{l_{i1}^2}\left(\frac{1}{m^+} - \frac{1}{m^-}\right), & \text{(e-e)},\\ 0, & \text{(p-p)},\\ \frac{1}{l_{i1}^2 m^+}, & \text{(e-p)},\\ \frac{-1}{l_{i1}^2 m^-}, & \text{(e-p)}. \end{cases}$$

Observe that in all cases the intersection numbers are positive or vanish since $m^+ > 0$ and $m^- < 0$. Furthermore by Definition 7.2 the corresponding vertices are not contractible.

For $n_i > 1$ we first consider the cases, where $j = 1$ or $j = n_i$ and the types (e-e) or (e-p) and (e-e) or (p-e), respectively. The self-intersection numbers are given as follows:

$$\begin{aligned} (D_{i1})^2 &= \frac{1}{l_{i1}^2}\left(\frac{1}{m^+} - \frac{1}{m_{i1} - m_{i2}}\right)\\ &= \frac{1}{l_{i1}^2}\left(\frac{m_{i1} - m_{i2} - m^+}{(m_{i1} - m_{i2})m^+}\right)\\ &= \frac{-m^+(i)}{l_{i1}^2(m_{i1} - m_{i2})m^+},\\ D_{in_i} &= \frac{m^-(i)}{-l_{i1}^2(m_{in_i-1} - m_{in_i})m^-}. \end{aligned}$$

Note that these expressions are a product of a positive rational number and $-m^+(i)$ or $m^-(i)$, respectively. Hence we find that the self-intersection numbers are negative if and only if $m^+(i) > 0$ or $m^-(i) < 0$, respectively, which is exactly the conditions given in Definition 7.2 (ii) and (iv), respectively.

For all other cases we restate the self-intersection numbers:

$$(D_{ij})^2 \quad = \quad \begin{cases} -\frac{1}{l_{i1}^2(m_{i1}-m_{i2})}, & \text{(p-p) or (p-e)}, \\ \frac{-(m_{ij-1}-m_{ij+1})}{l_{ij}^2(m_{ij-1}-m_{ij})(m_{ij}-m_{ij+1})}, & 1 < j < n_i, \\ -\frac{1}{l_{i1}^2(m_{in_i-1}-m_{in_i})}, & \text{(p-p) or (e-p)}. \end{cases}$$

By slopeorderedness, all given self-intersection numbers are positive, which are exactly statements (ii), (iii), and (iv) of Definition 7.2.

For statement (i) we note the following self-intersection numbers, which completes the proof:

$$(D^+)^2 \;=\; -m^+ < 0 \quad \Leftrightarrow \quad m^+ \;>\; 0.$$

□

8. Combinatorially minimal LDP complexes

In this section we are concerned with non-toric LDP complexes that do not admit contractions to a non-toric LDP complex. In general there are two possibilites: There are no contractible vertices or a contraction yields a toric LDP complex.

We start by considering the first possibility, so called *combinatorially minimal* LDP complexes, and achieve conditions for non-toric LDP complexes, see Proposition 8.2. This is an improvement on similar conditions in [**32**, Proposition 4.18] since Proposition 8.2 yields an equivalent description of non-toric combinatorially minimal LDP complexes. For convenience, we show the full proofs.

Definition 8.1. An LDP complex $\mathcal{L}$ is *combinatorially minimal* if no vertex v_{ij} and possibly v^+ is combinatorially contractible.

Proposition 8.2. *Let $\mathcal{L}$ be a non-toric LDP complex. Then $\mathcal{L}$ is combinatorially minimal if and only if one of the following holds:*

(i) *The LDP complex is of type (p-e) and we have $n_i = 1$ for all $0 \le i \le r$.*
(ii) *The LDP complex is of type (e-e) and there is exactly one index $0 \le \iota \le r$ such that $n_\iota = 2$ and $n_i = 1$ for all $\iota \ne i$.*
(iii) *The LDP complex is of type (e-e) and exactly two indices $0 \le \iota < \kappa \le r$ such that $n_\iota, n_\kappa = 2$ and $n_i = 1$ for all $i \ne \iota, \kappa$ and we have*

$$m^+(\iota) \le 0, \quad m^-(\iota) \ge 0, \qquad m^+(\kappa) \le 0, \quad m^-(\kappa) \ge 0.$$

Lemma 8.3. *Let $\mathcal{L}$ be an irredundant LDP complex.*

(i) *If $\mathcal{L}$ is of type (p-e) and there is an index $0 \le i \le r$ such that $n_i = 2$ then $\mathcal{L}$ is not combinatorially minimal.*
(ii) *If there is an index $1 \le i \le r$ such that $n_i > 2$ then $\mathcal{L}$ is not combinatorially minimal.*

Proof. The statements (i) and (ii) are immediate consequences of Definition 7.2 (ii), (v) and (iii), respectively, namely the vertices v_{i1} and v_{i2} are contractible. □

Lemma 8.4. *Consider an LDP complex $\mathcal{L}$ of type (e-e) such that $n_i = 2$ for all $i = 0, \dots, \bar{r}$ and $n_i = 1$ for all $i = \bar{r}+1, \dots r$. If $\mathcal{L}$ is combinatorially minimal, then $\bar{r} \leq 1$.*

Proof. We first consider the Picard-number $\rho(X)$ of the $\mathbb{K}^*$-surface $X := X(A(\mathcal{L}), P(\mathcal{L}))$. We find:

$$\begin{aligned} \rho(X) &= \sum_{i=0}^{r} n_i - (r-1) - \dim(X) \\ &= \sum_{i=0}^{\bar{r}} 2 + \sum_{i=\bar{r}+1}^{r} 1 - (r-1) - 2 \\ &= 2(\bar{r}+1) + (r-\bar{r}) - (r-1) - 2 \\ &= \bar{r}+1. \end{aligned}$$

Since the cone of effective divisors is full dimensional, there are at least $\bar{r}+1$ extremal rays, which by Remark 7.12 all host at least two weights of the form w_{ij} since $X(A, P)$ is combinatorially minimal.

Now we consider two weights, say w_{i1}, w_{i2} for $0 \leq i \leq \bar{l}$, and the weight $\mu = \deg(g)$. Note that both of the weights mentioned first lie on an extremal ray of $\mathrm{Eff}(X)$ since D_{i1}, D_{i2} are not combinatorially contractible, respectively.

Since $\mu = l_{i1}w_{i1} + l_{i2}w_{i2}$ and $\mu \in \mathrm{Eff}(X)^\circ$ we find that w_{i1}, w_{i2} lie on two distinct extremal rays of $\mathrm{Eff}(X)$.

Consider two weight $w_{\iota 1}, w_{\iota 2}$ with $1 \leq \iota \leq \bar{r}$ such that one of the weights lies on an extremal ray of $\mathrm{Eff}(X)$ that hosts w_{i1} or w_{i2}, say $w_{\iota 1}$ lies on the ray $\mathrm{cone}(w_{i1})$, i.e. $w_{\iota 1} = \lambda w_{i1}$. Then we find:

$$w_{\iota 2} = \frac{\mu - l_{\iota 1}w_{\iota 1}}{l_{\iota 2}} = \frac{l_{i1}w_{i1} + l_{i2}w_{i2} - l_{\iota 1}w_{\iota 1}}{l_{\iota 2}} = \frac{l_{i1} - \lambda l_{\iota 1}}{l_{\iota 2}} w_{i1} + \frac{l_{i1}}{l_{\iota 2}}.$$

Since $\mathrm{cone}(w_{i2}), \mathrm{cone}(w_{\iota 2})$ are extremal rays of $\mathrm{Eff}(X)$, we have $\mathrm{cone}(w_{i2}) = \mathrm{cone}(w_{\iota 2})$.

We conclude that the extremal rays of $\mathrm{Eff}(X)$ come in pairs and for every such pair we find indices $0 \leq i_1, \dots, i_s \leq \bar{r}$ such that $w_{i_k 1}, w_{i_k 2}$ lie on one of the extremal rays of this pair, respectively. Hence the maximal possible number of extremal rays is achieved when every such pair of extremal rays hosts exactly four weight $w_{i1}, w_{i2}, w_{\iota 1}, w_{\iota 2}$, if the number of weight pairs w_{i1}, w_{i2}, namely $\bar{r}+1$, is even, or if every such pair hosts exactly four weights and one pair does host six weights, if $\bar{r}+1$ is odd.

Hence we find the following bounds for the number of extremal rays:

$$2 \cdot \frac{2(\bar{r}+1)}{4} = \bar{r}+1, \quad \text{if } \bar{r} \text{ is odd}, \qquad 2 \cdot \frac{2\bar{r}}{4} = \bar{r}, \quad \text{if } \bar{r} \text{ is even}.$$

We conclude $\bar{r}$ is odd and we have $\bar{r}+1$ many extremal rays. Now after possibly relabeling the weights we can assume that the indices are given such that for $0 \leq i \leq \frac{\bar{r}+1}{2}$ we have:

$$\mathrm{cone}(w_{(2i)1}) = \mathrm{cone}(w_{(2i+1)1}), \quad \mathrm{cone}(w_{(2i)2}) = \mathrm{cone}(w_{(2i+1)2}).$$

Note that for $1 \leq i \leq \frac{\bar{r}+1}{2}$ we find $w_{(2i)2}, w_{(2i+1)1}, w_{(2i+1)2} \in \mathrm{Lin}_{\mathbb{Q}}(\mu, w_{(2i)1})$, since $\mu = l_{i1}w_{i1} + l_{i2}w_{i2}$ and the fact that the weights share extremal rays, furthermore $w_{ij} \in \mathrm{Lin}_{\mathbb{Q}}(\mu)$ for all $i \geq \bar{r}+1$. Therefore we find:

$$\begin{aligned} \rho(X) \;=\; \dim \mathrm{Lin}_{\mathbb{Q}} \mathrm{Eff}(X) \;&=\; \dim \mathrm{Lin}_{\mathbb{Q}}(w_{ij};\, 0 \leq i \leq r,\, 1 \leq j \leq n_i) \\ &=\; \dim \mathrm{Lin}_{\mathbb{Q}} \left(\mu, w_{(2i)1};\, 0 \leq i \leq \frac{\bar{r}+1}{2} \right) \\ &\leq\; \frac{\bar{r}}{2} + \frac{3}{2}. \end{aligned}$$

Since $\rho(X) = \bar{r}+1$, this inequality yields $\bar{r} \leq 1$. □

Proof of Proposition 8.2. We start by showing that every combinatorially minimal LDP complex is of one of the stated forms.

If the LDP complex is of type (p-e) we find by statement (i) of Lemma 8.3 that $n_i = 1$ for all $0 \leq i \leq r$, which exactly yields the first shape stated.

For the type (e-e) we find by Lemma 8.3 (ii) that $n_i \leq 2$ for all $0 \leq i \leq r$. Now Lemma 8.4 tells us that the number of indices i with $n_i = 2$ is bounded by 2. Note that $n_i = 1$ for all $0 \leq i \leq r$ contradicts the following for $X := X(A(\mathcal{L}), P(\mathcal{L}))$:

$$1 \;\leq\; \rho(X) \;=\; n - (r-1) - \dim(X) \;=\; (r+1) - (r-1) - 2 \;=\; 0.$$

Hence we achieve the stated shapes in (ii) and (iii). Furthermore note that for the shape (iii) the inequalities are exactly the inequalities given in Definition 7.2 (i) and (iv).

To see that all stated shapes of P are indeed combinatorially minimal we first note that $\rho(X) = 1$ for shapes (i) and (ii), so there are no combinatorially contractible vertices by Remark 7.11. For case (iii) we compare the inequalities stated in Definition 7.2 (i) and (iv). □

In the remaining part of the section we are concerned with finding all toric LDP complexes that are contractions of non-toric LDP complexes. The main result is Proposition 8.5 characterizing these toric LDP complexes. Construction 8.9 uses these results to give an algorithm to find these LDP complexes.

Note that these toric LDP complexes are not necessarily combinatorially minimal.

Proposition 8.5. *Let P be a Fano polygon with n vertices and let $\mathcal{L}$ be a non-toric LDP complex with a contractible vertex v such that $\mathcal{L}^v = \mathsf{P}$. Then one of the following two statements holds:*

(i) *There is a primitive vector $v \in \mathsf{P}$.*
(ii) *There is a Fano polygon $\tilde{\mathsf{P}}$ with $n+1$ vertices such that $\mathsf{P} \subseteq \tilde{\mathsf{P}}$.*

Lemma 8.6. *Let $\mathcal{L}$ be an irredundant non-toric LDP complex with contractible vertex v such that $\mathcal{L}' = \mathcal{L}^v$ is toric.*

Then $v = v_{\iota j}$ for a pair indices $0 \leq \iota \leq r$ and $1 \leq j \leq n_\iota$ and we have the following constraints:

(i) *The number of polygons P_i in $\mathcal{L}$ is $r+1 = 3$.*
(ii) *For the index ι we have $n_\iota = 2$ and setting $\{j, j'\} = \{1, 2\}$ we have $l_{\iota j'} = 1$.*

Proof. First note that since $\mathcal{L}$ is a non-toric LDP complex we have $r \geq 2$, moreover since $\mathcal{L}'$ is toric we find $r' = 1$. Observe that after contracting

$v = v^+$ we have $\mathcal{L}' = \mathcal{L}^v$ is irredundant, since $\mathcal{L}$ is, thus $r' = r$. Therefore, we have $v \neq v^+$.

Now consider a series of redundant extensions on $\mathcal{L}'$ as described in Construction 6.3 to obtain an LDP complex $\tilde{\mathcal{L}}$ such that $\tilde{r} = r$. Now since $\mathcal{L} = \mathcal{L}^v$ we obtain that, after possibly applying an LDP preserving unimodular transformation to $\mathcal{L}$, the two LDP complexes $\mathcal{L}$ and $\tilde{\mathcal{L}}$ only differ by one vertex, say $v_{\iota j}$. This yields $r = 2$ since otherwise there is an index $i \neq 0, 1, \iota$ such that $n_\iota l_{in_i} = 1$, which contradicts the irredundancy of $\mathcal{L}$.

Since $v_{\iota j}$ is contractible, we have $n_\iota \geq 2$. Moreover, since $r' < r$ a redundancy elimination at ι has been applied, i.e. $l_{\iota j'} = 1$. □

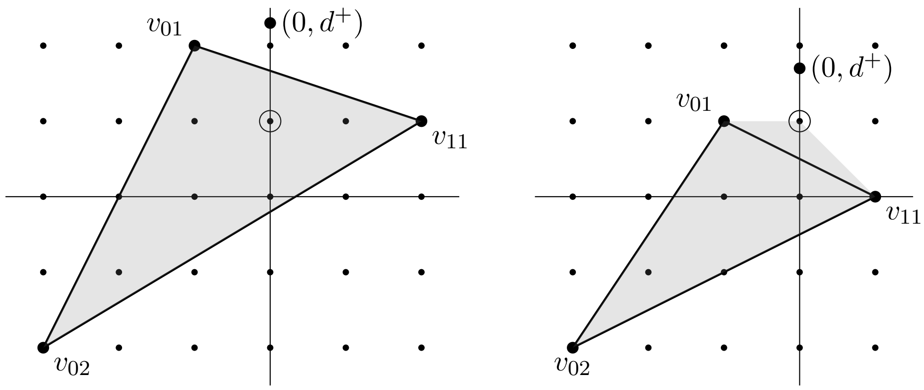


Figure 5. The two cases of Proposition 8.5, i.e. $e_2 \in \mathsf{P}$ on the left and $e_2 \notin \mathsf{P}$ on the right and the Fano polygon $\tilde{\mathsf{P}}$ in gray.

Proof of Proposition 8.5. For the LDP complex $\mathcal{L}$, Lemma 8.6 implies that $r = 2$ and wlog we can take the contracted vertex to be v_{21} and $l_{22} = 1$, furthermore we set $d_{22} = 0$ and $0 \leq d_{1n_1} < l_{1n_1}$.

By Proposition 9.5 we have $d^+ \geq 1$ or $d^- \leq -1$. We assume that $d^+ \geq 1$ and note that the statement follows in the same way for $d^- \leq -1$. Consider the primitive vector $e_2 = (0, 1) \in \mathbb{Z}^2$. We define the following convex polygon and the set of its vertices:

$$\tilde{\mathsf{P}} \;:=\; \mathrm{conv}(\mathcal{V}(\mathsf{P}) \cup \{e_2\}), \qquad \mathcal{V}(\tilde{\mathsf{P}}) \;\subseteq\; \mathcal{V}(\mathsf{P}) \cup \{e_2\}.$$

Note that this polygon is Fano and the following holds since $d^+ \geq 1$:

$$\tilde{\mathsf{P}} \;\subseteq\; \mathrm{conv}(\mathcal{V}(\mathsf{P}) \cup \{d^+ e_2\}).$$

Consider the case that $e_2 \notin \mathsf{P}$, then e_2 is a vertex of $\tilde{\mathsf{P}}$, i.e. $\tilde{\mathsf{P}}$ is a Fano polygon with $n+1$ vertices containing P. Thus the statement (ii) holds. When $e_2 \in \mathsf{P}$ we have a primitive vector in P, namely e_2, hence statement (i) holds. □

Remark 8.7. Let $\mathsf{P} \subseteq \mathbb{Q}^2$ be a Fano polgon and consider the situation of Proposition 8.5. Then we note the following:

(i) For the polygon $\tilde{\mathsf{P}} \subseteq \mathbb{Q}^3$ defined in the proof of Proposition 8.5 we find

$$\tilde{\mathsf{P}} \ \subseteq\ \operatorname{conv}(\mathcal{V}(\mathsf{P}) \cup d^+ e_2) \ =\ \pi(\mathsf{P}_0 \cup \mathsf{P}_1),$$

where π is the projection onto $\mathbb{Q}^2$ as in Lemma 6.8 and $\mathsf{P}_0, \mathsf{P}_1$ the polygons of the LDP complex.

(ii) The proof yields a method to find all polygons $A(\mathsf{P})$, $A \in \mathrm{GL}_2(\mathbb{Z})$, such that there is an LDP complex with a contractible vertex v sufficing $A(\mathsf{P}) \ =\ \mathcal{L}^v$, see Construction 8.9.

Definition 8.8. Let $\mathcal{L}$ be an LDP complex.

(i) We call P *adapted to the source* if it satisfies
   (a) $-l_{i1} < d_{i1} \leq 0$ for $i = 1, \dots, r$,
   (b) $l_{01}, l_{11} \geq l_{21} \geq \dots \geq l_{r1}$.

(ii) We call P *adapted to the sink* if it satisfies
   (a) $0 \leq d_{in_i} < l_{in_i}$ for $i = 1, \dots, r$,
   (b) $l_{01}, l_{11} \geq l_{21} \geq \dots \geq l_{rn_r}$.

Construction 8.9. Let P be a Fano polygon with n vertices and let $\mathcal{L}$ be an LDP complex adapted to the sink with contractible vertex $v_{21} \in \mathcal{L}$.

If $\mathcal{L}^v$ is unimodular equivalent to P, then the polygon $\mathcal{L}^v$ occurs in the set S defined by the following steps:

(i) For all primitive vectors $v \in \mathsf{P}$ do the following:
   (a) Choose a vector $w \in \mathbb{Z}^2$ such that v, w form a $\mathbb{Z}$-linear basis with the following property:
   There is a vertex $\bar{v} \in \mathcal{V}(\mathsf{P})$ such that:
   $$\bar{v} \ =\ lw + dv, \quad l \in \mathbb{Z}_{>0},\, d \in \mathbb{Z}, \quad 0 \leq d < l.$$
   (b) Consider the unimodular transformation $A \in \mathrm{GL}_2(\mathbb{Z})$ with
   $$A\colon \mathbb{Z}^2 \to \mathbb{Z}^2, \qquad e_1 \mapsto w, \quad e_2 \mapsto v.$$
   (c) Add $A^{-1}(\mathsf{P})$ to the set S.

(ii) For all Fano polygons $\tilde{\mathsf{P}}$ with $n+1$ vertices do the following:
   (a) For every vertex $v \in \mathcal{V}(\tilde{\mathsf{P}})$ choose a vector $w \in \mathbb{Z}^2$ such that v, w form a $\mathbb{Z}$-linear basis with the following property:
   There is a vertex $\bar{v} \in \mathcal{V}(\mathsf{P})$ such that:
   $$\bar{v} \ =\ lw + dv, \quad l \in \mathbb{Z}_{>0},\, d \in \mathbb{Z}, \quad 0 \leq d < l.$$
   (b) Consider the unimodular transformation $A \in \mathrm{GL}_2(\mathbb{Z})$ with
   $$A\colon \mathbb{Z}^2 \to \mathbb{Z}^2, \qquad e_1 \mapsto w, \quad e_2 \mapsto v.$$
   (c) If $\mathcal{V}(\mathsf{P}) \subseteq \mathcal{V}(\tilde{\mathsf{P}})$, add $A^{-1}(\mathsf{P})$ to the set S.

If there are only finitely many polygons $\tilde{\mathsf{P}}$ up to unimodular equivalence, the algorithm stops after finitely many steps, in particular the set S is finite.

Proof. Let $\mathcal{L}^v$ be unimodular equivalent to P. By Lemma 8.6 we find $l_{22} = 1$, and thus $d_{22} = 0$ since $\mathcal{L}$ is adapted to the sink.

By Proof of Proposition 8.5 we obtain $e_2 \in \mathcal{L}^v$ or there is a Fano polygon $\tilde{\mathsf{P}}$ containing $\mathcal{L}^v$ with vertex e_2. Note that the first case is treated in part (i), the second case is exactly part (ii). In both cases we have $A(\mathcal{L}^v) = \mathsf{P}$ by the construction.

For the last statement, note that part (i) stops after finitely many steps since every polygon P only contains fintely many lattice points in its interior. For the second part note that two unimodular equivalent polygons yield the same polygon after applying A^{-1} in step (c). This is due to the fact that the conditions in (a) uniquely fix the basis chosen there. Hence part (ii) stops after finitely many steps if there are only finitely many polygons $\tilde{P}$, up to unimodular equivalence. □

9. Almost homogeneous combinatorially minimal LDP complexes

Remark 9.1. We recall the notion of almost homogeneous $\mathbb{K}^*$-surfaces. A $\mathbb{K}^*$-surface $X(A,P)$ is called *almost homogeneous* if there is a horizontal Demazure P-root, i.e. a linear form $u \in M$ and two indices $0 \le i_0, i_1 \le r$ such that $i_0 \neq i_1$ that fulfills the (in)equalities given in Definition 3.4.

By Proposition 9.3 we know that there is a quasismooth elliptic fixed point x with exceptional indices i_0, i_1. We can assume $x = x^-$ i.e. the following is true

(i) For all $i \neq i_0, i_1$ we have $l_{in_i} = 1$.

After a series of admissible operation on P we can assume $i_0, i_1 \in \{0,1\}$ as well. We called such a Demazure P-root a horizontal P-root at (x^-, i_0, i_1) and found out that all horizontal Demazure P-roots are of this type (see Section 9 of the last chapter). Furthermore in Construction 9.7 we define the following rational numbers:

$$\eta_k := -\frac{1}{l_{kn_k} m^-}, \qquad \xi_i := \begin{cases} 0, & n_i = 1, \\ \frac{1}{l_{in_i}(m_{in_i-1} - m_{in_i})}, & n_i \ge 2. \end{cases}$$

Setting $\Delta(i_0, i_1) = \bigcap_{i \neq i_0} [\xi_i, \eta_{i_1}]$ we found that there is a horizontal P-root if and only if the following statement is true:

(ii) There is an integer $\gamma \in \Delta(i_0, i_1)$ such that $\gamma d_{i_1 n_{i_1}} \equiv -1 \bmod l_{i_1 n_{i_1}}$.

In particular, we infered that $u = u(i_0, i_1, \gamma)$, where $u(i_0, i_1, \gamma)$ is as in Definition 9.9. Furthermore we found the following propositions in the last chapter:

(iii) If $l_{0n_0} \le l_{1n_1}$ we find x^- to be smooth.
(iv) If $\gamma = \eta_{i_0}$ for some horizontal P-root, then the elliptic fixed point x^- is smooth.
(v) For all $i \neq i_0$ the divisors D_{in_i} have positive self-intersection number.
(vi) We have $1 \le \gamma \le \eta_{i_1}$, i.e. $l_{i_1 n_{i_1}} m^- \ge -1$.

Definition 9.2. An *almost homogeneous* LDP complex $\mathcal{L}$ is an LDP complex with a linear form $u \in \mathrm{Hom}(\mathbb{Z}^{r+1}, \mathbb{Z})$ with two indices $0 \le i_0, i_1 \le r$ such that the following holds:

$$\langle u, v_{1n_1} \rangle = -1, \quad \langle u, v_{in_i} \rangle = 0,\ i \neq 0,1, \quad l_{in_i} = 1,\ i \neq 0,1,$$

$$\langle u, v_{0n_0} \rangle \ge 0,\ \langle u, v_{1n_1-1} \rangle \ge 0,\ n_1 > 1,\ \langle u, v_{in_i-1} \rangle \ge l_{in_i-1},\ i \neq 0,1.$$

Remark 9.3. Let $\mathcal{L}$ be an almost homogeneous LDP complex with linear form $u \in \mathrm{Hom}(\mathbb{Z}^{r+1}, \mathbb{Z})$. It follows immediately that $u = u(0,1,\gamma)$ for an integer $\gamma \in \Delta(i_0, i_1)$.

THEOREM 9.4. *There are only finitely many combinatorially minimal almost homogeneous almost k-hollow LDP complexes.*

After a set of admissible operations their defining matrices $P(\mathcal{L})$ have the following forms:

$$\begin{bmatrix} -l_{01} & l_{11} & 0 & 0 \\ -l_{01} & 0 & l_{21} & 1 \\ d_{01} & d_{11} & d_{21} & 0 \end{bmatrix}, \qquad \begin{bmatrix} -2 & 2 & 1 & 0 & 0 \\ -2 & 0 & 0 & 2 & 1 \\ -1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

PROPOSITION 9.5. *Let $\mathcal{L}$ be an almost homogeneous LDP complex. Then the following statements hold:*

(i) *We have $m^- > -\bar{l}^-$ and $m^+ \geq \bar{l}^+$.*
(ii) *The LDP complex $\mathcal{L}$ is of type (e-e).*

In particular, there is a one to one correspondence between almost homogeneous log del Pezzo $\mathbb{K}^*$-surfaces and almost homogeneous LDP complexes.

PROOF. We start by proving statement (i). Since there is a horizontal P-root at x^- we find indices i_0, i_1 such that $l_{i_1 n_{i_1}} m^- \geq -1$ and $l_{in_i} = 1$ for all $i \neq i_0, i_1$. Now the following is true:

$$-1 \;\leq\; l_{i_1 n_{i_1}} m^- \;=\; \frac{m^-}{\frac{1}{l_{i_1 n_{i_1}}}} \;<\; \frac{m^-}{\frac{1}{l_{i_0 n_{i_0}}} + \frac{1}{l_{i_1 n_{i_1}}}} \;=\; \frac{m^-}{\bar{l}^-}.$$

Now note that by Proposition 5.15 we have $m^+ \geq \bar{l}^+$, which proves statement (i).

For the second statement note that for an LDP complex of type (p-e) statement (i) contradict the fourth condition of the definition of an LDP complex.

The correspondence follows with Remark 9.1: Applying suitable admissible operations, every log del Pezzo surface $X(A, P)$ with horizontal Demazure P-root u can be brought to a form such that u is a horizontal P-root at $(x^-, 0, 1)$. □

PROPOSITION 9.6. *Consider an almost k-hollow LDP complex $\mathcal{L}$ such that $P(\mathcal{L})$ is of the following form:*

$$P(\mathcal{L}) \;:=\; \begin{bmatrix} -l_{01} & l_{11} & 0 & 0 \\ -l_{01} & 0 & l_{21} & 1 \\ d_{01} & d_{11} & d_{21} & 0 \end{bmatrix}, \qquad 0 \;\leq\; m_{11} \;<\; 1, \qquad 1 \;\leq\; -\frac{1}{l_{11} m^-}.$$

Set c to be the maximal volume of k-hollow Fano polygons with three vertices. Then the integers l_{01}, l_{11}, l_{21} are bounded as follows:

$$2 \;\leq\; l_{01}, l_{11} \;\leq\; 2c - 2, \qquad l_{21} \;\leq\; \frac{l_{01} l_{11}}{l_{01} l_{11} - l_{01} - l_{11}}.$$

Furthermore we find the following constraints on d_{ij}:

(i) *For $i = 0$ we have $0 > d_{i0} > l_{01}$.*
(ii) *For $i = 1$ we have $0 < d_{i1} < l_{11}$.*
(iii) *For $i = 2$ we have $0 < d_{21} \leq l_{21} \left(k\bar{l}^+ - m_{01} - m_{11} \right)$.*

In particular, there are only finitely many such $\mathbb{K}^*$-surfaces.

PROOF. We consider the bound on the integers $l_{ij} \in \mathbb{Z}_{\geq 1}$. First note that $l_{01} = 1$ or $l_{11} = 1$ yield irredundancy of P, therefore we find $l_{01}, l_{11} \geq 2$.

For the upper bounds we first note that since $\mathcal{L}$ is almost k-hollow, we find by Lemma 6.6 that P_0 and P_1 are k-hollow and that $(d^-, d^+) \cap k\mathbb{Z} = \{0\}$.

Now observe that $\mathcal{L}$ satisfies the conditions of Lemma 6.8, hence we find that the convex polytope $\pi(\mathsf{P}_0 \cup \mathsf{P}_1)$ is almost k-hollow. By Proposition 9.5 (i) we know that $d^+ \geq 1$ which yields the containment of the following polygon:

$$\mathsf{P} \;:=\; \operatorname{conv}((-l_{01}, d_{01}), (l_{11}, d_{11}), e_2) \;\subseteq\; \pi(\mathsf{P}_0 \cup \mathsf{P}_1).$$

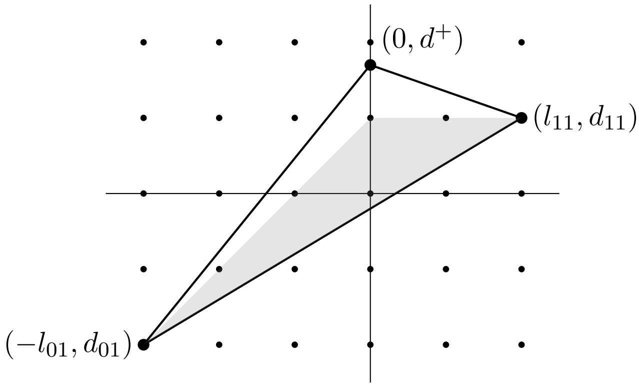


FIGURE 6. The convex polygon $\pi(\mathsf{P}_0 \cup \mathsf{P}_1)$ and the Fano polygon P contained (gray).

Since $\gcd(l_{01}, d_{01})$, $\gcd(l_{11}, d_{11}) = 1$ the polygon P is Fano, hence its volume is bounded by c. Now we find the following:

$$c \;\geq\; \operatorname{vol}(\mathsf{P}) \;=\; \frac{-m^- l_{01} l_{11} + l_{01} + l_{11}}{2}.$$

Bringing l_{01} to one side of the inequality yields the following statement, where we use that $m^- < 0$:

$$l_{01} \;\leq\; 2c - l_{11} + m^- l_{01} l_{11} \leq 2c - l_{11} \leq 2c - 2.$$

The same calculation can be done for l_{11}, i.e. $l_{11} \leq 2c - 2$.

We turn to the integer l_{21}. First observe that $\bar{l}^+ > 0$ by definition of an LDP complex, which implies the following inequality:

$$\frac{1}{l_{21}} > 1 - \frac{1}{l_{01}} - \frac{1}{l_{11}} \quad \Leftrightarrow \quad l_{21} \;<\; \frac{l_{01} l_{11}}{l_{01} l_{11} - l_{01} - l_{11}}.$$

Last we consider the bounds for the integers d_{ij}. First note that $m_{11} = 0$ implies $l_{11} = 1$, which is a contradiction to $\mathcal{L}$ being irredundant. Therefore the integer d_{11} is bounded by the condition $0 < m_{11} < 1$. For the constraint on d_{01} first note that $m_{01} < 0$ since $m_{01} < m_{01} + m_{11} = m^- < 0$. Now using $m^- \geq -\frac{1}{l_{11}}$ we find the following:

$$m_{01} \;\geq\; -m_{11} - \frac{1}{l_{11}} \;=\; -\frac{d_{11} + 1}{l_{11}} \;\geq\; -1.$$

In the last estimate we used that $d_{11} \le l_{11} - 1$. The constraint on d_{21} follows from $(d^-, d^+) \cap k\mathbb{Z} = \{0\}$, which translates to $d^+ \le k$, therefore we have:

$$m_{01} + m_{11} + m_{21} \;=\; m^+ \;\le\; k\bar{l}^+ \quad\Leftrightarrow\quad d_{21} \;\le\; l_{21}\left(k\bar{l}^+ - m_{01} - m_{11}\right).$$

This is exactly the bound stated.

To see that there are only finitely many such $\mathbb{K}^*$-surfaces note that by Proposition 2.5 it is clear that the volume of Fano polygons with three vertices is bounded since there are only finitely many such polygons up to unimodular equivalence. This binds the possible integers l_{01}, l_{11} to a finite number.

Last we note that the bounds for l_{21}, d_{01} and d_{11} only depend on the integers l_{01} and l_{11}. The bound on d_{21} only depends on the integers l_{21}, d_{01} and d_{11}. Hence there are only finitely many possibilities for these integers, which ends the proof. □

PROPOSITION 9.7. *Consider a combinatorially minimal LDP complex $\mathcal{L}$ as in (iii) of Proposition 8.2. Furthermore the following holds:*

(i) *The number polygons P_i is bound by $r - 1 \le 2$.*
(ii) *For the numbers of primitive ray generators we have $n_i = 1$ for all $i = 0, \dots, r-2$ and $n_i = 2$ for $i = r-1, r$.*
(iii) *$\mathcal{L}$ is adapted to the sink, i.e. $0 \le m_{in_i} < 1$ for all $i = 1, \dots, r$.*
(iv) *There is a horizontal P-root at (x^-, i_0, i_1) with $i_0, i_1 \in \{0, 1\}$, i.e.*
   (a) *There is an integer $\gamma \in \Delta(i_0, i_1)$ with $\gamma d_{i_1 n_{i_1}} \equiv -1 \bmod l_{i_1 n_{i_1}}$.*
   (b) *For the integers l_{ij} we have $l_{in_i} = 1$ for all $i \ne 0, 1$.*

Then $r = 2$ and the defining matrix $P(\mathcal{L})$ has the following form:

$$P(\mathcal{L}) \;=\; \begin{bmatrix} -2 & 2 & 1 & 0 & 0 \\ -2 & 0 & 0 & 2 & 1 \\ -1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

PROOF. We first note that since this LDP complex is of the type (iii) of Proposition 8.2, we have

$$m^+(r-1),\, m^+(r) \;\le\; 0 \qquad\text{and}\qquad m^-(r-1),\, m^-(r) \;\ge\; 0.$$

We calculate these expressions, where we note that $m_{i2} = 0$ for $i = r-1, r$ by the conditions (iii) and (iv) (b) and $m_{in_i} = m_{i1}$ for all $i < r-1$. We have:

$$m^+(r-1) \;=\; \sum_{i=1}^{r-2} m_{i1} + m_{r-1\,2} + m_{r1} \;=\; \sum_{i=1}^{r-2} m_{i1} + m_{r1} \;=\; m^-(r),$$

$$m^-(r-1) \;=\; \sum_{i=1}^{r-2} m_{i1} + m_{r-1\,1} + m_{r2} \;=\; \sum_{i=1}^{r-2} m_{i1} + m_{r-1\,1} \;=\; m^+(r).$$

With the estimates to zero above we find that all given equalities vanish which yields $m_{r-1\,1} \;=\; m_{r1}$, furthermore we can deduce the following

$$\sum_{i=2}^{r-2} m_{i1} + m_{r1} = m^- + m_{r1} = 0 \quad\Rightarrow\quad m^- \;=\; -m_{r1}.$$

Now note that since there is a horizontal P-root at (x^-, i_0, i_1) we find an integer γ that fulfills the following inequalities:

$$\frac{1}{m_{r1}} \;=\; \xi_r \;\leq\; \gamma \;\leq\; \eta_{i_1} \;=\; -\frac{1}{l_{i_1 n_{i_1}} m^-} = \frac{1}{l_{i_1 n_{i_1}} m_{r1}}$$

This yields $l_{i_1 n_{i_1}} = 1$, therefore $n_{i_1} = 2$ by irredundancy of P. We conclude that $r = 2$ and $i_1 = 1$ since otherwise condition (ii) implies redundancy, i.e. the matrix $P(\mathcal{L})$ has the following form:

$$P(\mathcal{L}) \;=\; \begin{bmatrix} -l_{01} & l_{11} & 1 & 0 & 0 \\ -l_{01} & 0 & 0 & l_{21} & 1 \\ d_{01} & d_{11} & 0 & d_{21} & 0 \end{bmatrix}.$$

Last we note that by the series of inequalities above we have $\gamma = (-l_{i_1 n_{i_1}} m^-)^{-1}$ and therefore x^- is smooth by Remark 9.1 (iv), i.e. $l_{01} l_{11} m^- = -1$ and the following is true:

$$\frac{d_{01}}{l_{01}} \;=\; m_{01} \;=\; m_{01} + m_{21} \;=\; m^- \;=\; -\frac{1}{l_{01} l_{11}} \;=\; -\frac{1}{l_{01}}.$$

This yields $d_{01} = -1$.

The remaining values for the integers l_{01}, d_{21}, l_{21} follow by the proven equalities $m_{11} = m_{21}$ and $m_{01} = m^- = -m_{21}$. We find $l_{01} = l_{11} = l_{21}$ and $-d_{01} = d_{11} = d_{21} = 1$.

To see the concrete value of l_{01} we consider the condition (iv) of the definition of an LDP complex:

$$0 \;<\; \bar{l}^+ \;=\; \frac{1}{l_{01}} + \frac{1}{l_{01}} + \frac{1}{l_{01}} - 1 \qquad \Leftrightarrow \qquad l_{01} < 3.$$

Since $l_{01} \neq 1$ by irredundancy of P we find $l_{01} = 2$, hence the stated form of P. $\square$

Proof of Theorem 9.4. Since $\mathcal{L}$ is almost homogeneous, by statement (i) of Remark 9.1 we know that $l_{in_i} = 1$ for all $i \geq 2$. By admissible operations we can assume $\mathcal{L}$ to be adapted to the sink, i.e. $0 \leq m_{in_i} < 1$, furthermore $r \geq 2$ since $\mathcal{L}$ is non-toric.

We first turn to the condition of combinatorially minimality which yields three different types of defining matrices P as given in Proposition 8.2. Note that (i) is of type (p-e) and hence not almost homogeneous by Proposition 9.5.

Note that for an LDP complex of the shape (ii) of Proposition 8.2 there is a positive integer $\gamma \in \Delta(0,1)$ such that

$$1 \;\leq\; \gamma \;\leq\; \eta_1 \;=\; -\frac{1}{l_{11} m^-}.$$

Therefore $\mathcal{L}$ satisfies the conditions of Proposition 9.6, hence there are only finitely many such LDP complexes.

We turn to the case (iv) and first consider the integers n_i. Note that there are exactly two indices ι, κ with $n_\iota = n_\kappa = 2$. Since by Remark 9.1 (i) we know that $l_{in_i} = 1$ for all $i \neq 0, 1$, we find that $r \leq 3$ by irredundancy since otherwise there is an index i such that $i \neq 0, 1, \iota, \kappa$, i.e. $l_{in_i} n_i = 1$. Furthermore note that for $r = 3$ we have $\{0,1\} \cap \{\iota, \kappa\} = \emptyset$ since otherewise there is an index $0 \leq i \leq 3$ such that $l_{in_i} = 1$ and $n_i = 1$. Therefore we can swap the blocks such that $\iota, \kappa \in \{r-1, r\}$.

This yields an LDP complex $\mathcal{L}$ which satisfies the conditions of Proposition 9.7, which yields the statement. □

PROPOSITION 9.8. *There are only finitely many polygons* P *such that* $\mathsf{P} = \mathcal{L}^v$ *for an almost* k*-hollow LDP complex* $\mathcal{L}$ *with a contractible vertex* $v \in \mathcal{L}$.

PROOF. Since $\mathcal{L}$ is almost k-hollow note that $\mathcal{L}^v$ is almost k-hollow as well, see Theorem 7.4 (ii). Thus by Remark 5.9 we obtain that $\mathcal{L}^v$ is an almost k-hollow Fano polygon. Since by Proposition 2.5 there are only finitely many up to unimodular transformation, it suffices to show the statement for all polygons that are unimodular equivalent to a specific polygon P.

In this case, Construction 8.9 yields an algorithm to find all LDP complexes contracting onto P. Note that the polygon $\tilde{\mathsf{P}}$ in the second step coincides with the one described in Remark 8.7 (i). i.e. it is contained in $\pi_1(\mathsf{P}_0 \cup \mathsf{P}_1)$. Thus it is almost k-hollow. Since there are only finitely many almost k-hollow polygons, up to unimodular equivalence, we have shown the statement using the last statement in Construction 8.9. □

10. Building almost homogeneous LDP complexes

In this section we want to show following theorem:

THEOREM 10.1. *There are only finitely many almost homogeneous almost* k*-hollow LDP complexes.*

As mentioned in the introduction, finiteness of $1/k$-log canonical del Pezzo surfaces has already been shown in [**6**]. Our approach is to show this statement using an explicit construction (see Construction 10.3) to contract LDP complexes in an orderly manner which ends in non-toric combinatorially minimal LDP complex or a toric LDP complex, both of which we have been studying in the previous section. Reversing this construction not only proves the theorem, furthermore it yields an algorithmic approach to classify all LDP complexes that are almost k-hollow and almost homogeneous.

DEFINITION 10.2. Let $\mathcal{L}$ be an LDP complex of type (e-e). Then we define the following:

(i) A *boundary vertex* is a vertex $v_{ij} \in \mathcal{V}(\mathcal{L})$, where $j = 1$ or $j = n_i$.
(ii) An *interior vertex* is a vertex $v_{ij} \in \mathcal{V}(\mathcal{L})$, where $j \neq 1, n_i$.

For an LDP complex of type (p-e) we define:

(i) A *boundary vertex* is a vertex $v_{ij} \in \mathcal{V}(\mathcal{L})$, where $j = n_i$.
(ii) An *interior vertex* is a vertex $v_{ij} \in \mathcal{V}(\mathcal{L})$, where $j \neq n_i$.

CONSTRUCTION 10.3. Let $\mathcal{L}$ be an almost k-hollow LDP complex.

We consider the following two steps:

(i) Successively contract every interior vertex of $\mathcal{L}$. This yields an LDP complex $\mathcal{L}'$ with no interior vertices.
(ii) For every $0 \leq i \leq r$, successively contract all boundary vertices of $\mathcal{L}$ if contractible. This yields an LDP complex $\mathcal{L}''$.

The Construction ends in a non-toric combinatorially minimal almost homogeneous almost k-hollow LDP complex or a toric LDP complex as in Proposition 8.5.

PROPOSITION 10.4. *Let $\mathcal{L}$ be an almost homogeneous LDP complex with contractible vertex v. Then $\mathcal{L}^v$ is almost homogeneous.*

PROOF. We show the following containment for the intervals $\Delta(i_0, i_1), \Delta'(i_0, i_1)$ which we will do by proving the equivalent statement below:

$$\Delta(i_0, i_1) \subseteq \Delta'(i_0, i_1) \qquad \Leftrightarrow \qquad \xi_i' \leq \xi_i \text{ for all } i \neq 0, \quad \eta_1 \leq \eta_1'.$$

Observe that it suffices to show the containment to prove that $\mathcal{L}'$ is almost homogeneous since then the horizontal P-root corresponding to some $\gamma \in \Delta(i_0, i_1)$ also corresponds to a horizontal P-root for $\mathcal{L}'$.

We start with the case of a contraction of a divisor D_{0j} with $j \neq n_i$, D_{ij} with $i \geq 1$ or $j \neq n_i - 1$ or D^+. All of these cases do not change the rational numbers ξ_i, η_1, i.e. $\xi_i = \xi_i'$, and $\eta_1 = \eta_1'$.

Now consider $D = D_{0n_0}$. The rational numbers ξ_i are not affected by the contraction, i.e. $\xi_i = \xi_i'$. Since D_{0n_0} is contractible we have $m^-(0) < 0$, therefore the following holds

$$\begin{aligned}
\eta_1' \;=\; -\frac{1}{l_{1n_1} m^-(0)} &= -\frac{1}{l_{1n_1} m^-} + \frac{m^-(0) - m^-}{l_{1n_1} m^- m^-(0)} \\
&= -\frac{1}{l_{1n_1} m^-} + \frac{m_{1n_1-1} - m_{1n_1}}{l_{1n_1} m^- m^-(0)} \\
&> -\frac{1}{l_{1n_1} m^-} \;=\; \eta_1.
\end{aligned}$$

The inequalitiy follows by slopeorderedness, i.e. $m_{1n_1-1} > m_{1n_1}$. This ends the prove for this case.

If $n_i \geq 2$ for some $i \neq 0$ and $D = D_{in_i-1}$ is contractible, we note that $\eta_1 = \eta_1'$. For $n_i > 2$ the rational numbers ξ_i, ξ_i' we find:

$$\begin{aligned}
\xi_i &= \frac{1}{l_{in_i}(m_{in_i-1} - m_{in_i})} \\
&= \frac{1}{l_{in_i}(m_{in_i-2} - m_{in_i})} - \frac{(m_{in_i-1} - m_{in_i}) - (m_{in_i-2} - m_{in_i})}{l_{in_i}(m_{in_i-1} - m_{in_i})(m_{in_i-2} - m_{in_i})} \\
&= \frac{1}{l_{in_i}(m_{in_i-2} - m_{in_i})} - \frac{m_{in_i-1} - m_{in_i-2}}{l_{in_i}(m_{in_i-1} - m_{in_i})(m_{in_i-2} - m_{in_i})} \\
&> \frac{1}{l_{in_i}(m_{in_i-2} - m_{in_i})} \;=\; \xi_i'.
\end{aligned}$$

Again, slopeorderedness yields the inequality, i.e. $m_{in_i-2} \geq m_{in_i-1} \geq m_{in_i}$. For $n_i = 2$ note that $\xi_i' = 0 \leq \xi_i$. This ends the proof. □

PROOF OF CONSTRUCTION 10.3. We remark that every interior vertex is contractible by Definition 7.2 (iii), thus the construction is well defined. Furthermore note that $\mathcal{L}''$ is a well-defined, almost homogeneous and almost k-hollow LDP complex by Construction 7.3, Proposition 10.4 and Theorem 7.4, respectively. □

DEFINITION 10.5. Let $\mathcal{L}$ be an LDP complex with vertices $v_1, \ldots, v_k \in \mathcal{V}(\mathcal{L})$. Inductively we define:

$$\mathcal{L}^{v_1,\ldots,v_\kappa} \;:=\; \left(\mathcal{L}^{v_1,\ldots,v_{\kappa-1}}\right)^{v_\kappa} \qquad \kappa = 1, \ldots, k.$$

Moreover, a series of *contractible vertices* are vertices $v_1, \ldots, v_k$ admitting this construction, i.e. the vertex v_κ is a contractible vertex of the LDP complex $\mathcal{L}^{v_1,\ldots,v_{\kappa-1}}$.

PROPOSITION 10.6. *Let $\mathcal{L}'$ be an LDP complex. Then the following two statements hold:*

(i) *There are only finitely many almost k-hollow LDP complexes $\mathcal{L}$ with contractible interior vertices $v_1, \ldots, v_k$ such that*
$$\mathcal{L}^{v_1,\ldots,v_k} \;=\; \mathcal{L}'.$$

(ii) *There are only finitely many almost k-hollow LDP complexes $\mathcal{L}$ with $n_i \leq 2$ for all $0 \leq i \leq r$ and contractible boundary vertices $v_1, \ldots, v_k$ such that*
$$\mathcal{L}^{v_1,\ldots,v_k} \;=\; \mathcal{L}'.$$

PROPOSITION 10.7 (Compare Proposition 5.12 in [**32**]). *Let $\mathcal{L}$ be an almost k-hollow LDP complex. Then the number of poylgons $\mathsf{P}_i \subseteq \mathcal{L}$ is bounded by*

$$r + 1 \;\leq\; 4k.$$

PROOF. Since the LDP complex is almost k-hollow the following two inequalities hold by Lemma 6.6:

$$d^+ \;=\; \frac{m^+}{\bar{l}^+} \;\leq\; k, \qquad d^- \;=\; \frac{m^-}{\bar{l}^-} \;\geq\; -k.$$

Now by Lemma 5.17 (iii) we have the following inequality:

$$\begin{aligned} r+1 \;&\leq\; \left(m^+ - \bar{l}^+\right) - \left(m^- + \bar{l}^-\right) + 4 \\ &\leq\; (k-1)\bar{l}^+ + (k-1)\bar{l}^- + 4. \end{aligned}$$

Here we used the inequalities above, i.e. $d^+ \leq k$, $d^- \geq -k$. Since the maximal value for $\bar{l}^+, \bar{l}^-$ is given by 2 we have proven the statement. $\square$

LEMMA 10.8. *Let $a, b, l \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}$ be integers and consider the convex polytope of the following form:*

$$\mathsf{P}_{a,b} \;:=\; \operatorname{conv}(1/a\, e_2, -1/b\, e_2, (l,d)) \;\subseteq\; \mathbb{Q}^2.$$

If this polytope is k-hollow, we have $l < 2k^2 \cdot \max(a,b)$.

PROOF. Wlog we set $a \leq b$ and assume that $\mathsf{P}_{a,b}$ is k-hollow. Note that $\mathsf{P}_{b,b} \subseteq \mathsf{P}_{a,b}$. We now consider the convex polytope $\mathsf{P}_{b,b} \cup -\mathsf{P}_{b,b} \subseteq \mathbb{Q}^2$. Observe that if there is a k-fold point in $\mathsf{P}_{b,b} \cup -\mathsf{P}_{b,b}$ then there is a k-fold point in $\mathsf{P}_{b,b}$ and therefore in $\mathsf{P}_{a,b}$. We conclude that $\mathsf{P}_{b,b} \cup -\mathsf{P}_{b,b}$ is k-hollow.

By Minkowski's Theorem 2.13 we find that the volume of the polytope is bounded and therefore the statement follows:

$$\operatorname{vol}(\mathsf{P}_{b,b} \cup -\mathsf{P}_{b,b}) \;=\; 2b^{-1}l \;<\; (2k)^2 \qquad \Leftrightarrow \qquad l \;<\; 2bk^2.$$

$\square$

Corollary 10.9. *Let $\mathcal{L}$ be an almost k-hollow LDP complex. For any $0 \le \iota \le r$ we define the following numbers:*

$$\begin{array}{rclrcl} a_\iota & := & \min\left(a; 1/a \le d^+(\iota)\right), & a & := & \min\left(a; 1/a \le d^+\right), \\ b_\iota & := & \min\left(b; 1/b \le -d^-(\iota)\right), & b & := & \min\left(b; 1/b \le -d^-\right). \end{array}$$

Then we find the following constraints on $l_{\iota j}$:

(i) *If $\mathcal{L}$ is of type (p-e) and $j = 1$ we have $l_{\iota 1} < 2k^2 \cdot b$.*
(ii) *If $\mathcal{L}$ is of type (e-e) and $j = 1$ we have $l_{\iota 1} < 2k^2 \cdot max(a_\iota, b)$.*
(iii) *For any $2 \le j \le n_\iota - 1$ we have $l_{\iota j} < 2k^2 \cdot max(a, b)$.*
(iv) *For $j = n_\iota$ we have $l_{\iota n_\iota} < 2k^2 \cdot max(a, b_\iota)$.*

Proof. We first prove the third statement. The following containments hold:

$$\mathrm{conv}(1/a\, e_{r+1}, -1/b\, e_{r+1}, v_{\iota j}) \;\subseteq\; \mathrm{conv}(v^+, v^-, v_{\iota j}) \;\subseteq\; \mathsf{P}_\iota.$$

The latter polygon is k-hollow, hence the same is true for first polygon above. Now Lemma 10.8 yields the statement.

For the statements (ii) and (iv) we note that by Lemma 7.6 (ii) we have $v^+(\iota) \in \mathrm{conv}(0, v^+)$ and $v^-(\iota) \in \mathrm{conv}(0, v^-)$, respectively. Therefore the following containments hold:

$$\begin{array}{rcl} \mathrm{conv}(1/a_\iota\, e_{r+1}, -1/b\, e_{r+1}, v_{\iota 1}) & \subseteq & \mathrm{conv}(v^+(\iota), v^-, v_{\iota 1}) \\ & \subseteq & \mathrm{conv}(v^+, v^-, v_{\iota 1}) \;\subseteq\; \mathsf{P}_\iota, \\ \mathrm{conv}(1/a\, e_{r+1}, -1/b_\iota\, e_{r+1}, v_{\iota 1}) & \subseteq & \mathrm{conv}(v^+, v^-(\iota), v_{\iota 1}) \\ & \subseteq & \mathrm{conv}(v^+, v^-, v_{\iota 1}) \;\subseteq\; \mathsf{P}_\iota. \end{array}$$

Again we conclude that the convex polytopes contained in P_ι are k-hollow since P_ι is k-hollow and therefore we find the statements by Lemma 10.8. The first statement is shown in the same way as statements (iv). □

Remark 10.10. Let $\mathcal{L}$ be an LDP complex comprising of polygons P_i for $0 \le i \le r$. For indices $1 \le j_1 < j_2 < j_3 \le n_\iota$ consider the following convex polygon:

$$\mathrm{conv}(v_{\iota j_1}, v_{\iota j_2}, v_{\iota j_3}, v^+, v^-) \;\subseteq\; \mathbb{Q}^{r+1}.$$

Note that $v_{\iota j}$ is a vertex of this polygon, since it is a vertex of P_ι. Thus Lemma 5.18 yields that the following inequality holds:

$$(l_{\iota j_2} - l_{\iota j_3})(d_{\iota j_1} - d_{\iota j_2}) - (d_{\iota j_2} - d_{\iota j_3})(l_{\iota j_1} - l_{\iota j_2}) \;>\; 0.$$

Lemma 10.11. *Let $\mathcal{L}$ be an almost k-hollow LDP complex. For any index $0 \le \iota \le r$ we have the following constraints on $d_{\iota j}$:*

(i) *If $\mathcal{L}$ is of type (p-e) and $n_\iota = 2$, we set $l := l_{\iota 1}$ and find the following inequalities*

$$\begin{array}{rcl} d_{\iota 1} & < & l m_{\iota 2} + l \bar{l}^+(\iota)\left(1 - d^+(\iota)\right) + l\left(\dfrac{1}{l} - \dfrac{1}{l_{\iota 2}}\right), \\ d_{\iota 1} & > & l m_{\iota 2}, \\ d_{\iota 1} & > & l m_{\iota 2} + l\left(\dfrac{1}{l} - \dfrac{1}{l_{\iota 2}}\right). \end{array}$$

(ii) *If $\mathcal{L}$ is of type (e-e) and $n_\iota = 2$, the following inequalities are true, where $l := l_{\iota 1}$:*

$$\begin{aligned} d_{\iota 1} &\leq lm_{\iota 2} + l\bar{l}^+(\iota)(k - d^+(\iota)) + kl\left(\frac{1}{l} - \frac{1}{l_{\iota 2}}\right), \\ d_{\iota 1} &> lm_{\iota 2}, \\ d_{\iota 1} &> lm_{\iota 2} + \left(\frac{1}{l} - \frac{1}{l_{\iota 2}}\right) ld^+(\iota). \end{aligned}$$

(iii) *If $n_\iota \geq 3$, then for any $2 \leq \kappa \leq n_\iota - 1$ we set $l := l_{\iota\kappa}$ to find:*

$$\begin{aligned} d_{\iota\kappa} &< lm_{\iota 1}, \\ d_{\iota\kappa} &< lm_{i1} + ld^+\left(\frac{1}{l} - \frac{1}{l_{\iota 1}}\right), \\ d_{\iota\kappa} &> lm_{\iota n_\iota} + ld^-\left(\frac{1}{l} - \frac{1}{l_{\iota n_\iota}}\right), \\ d_{\iota\kappa} &> lm_{\iota n_\iota}. \end{aligned}$$

(iv) *If $n_\iota = 2$, the following inequalities are true, where $l := l_{\iota 2}$:*

$$\begin{aligned} d_{\iota 2} &\geq lm_{\iota 1} + l\bar{l}^-(\iota)(-k - d^-(\iota)) + kl\left(\frac{1}{l} - \frac{1}{l_{\iota 1}}\right), \\ d_{\iota 2} &< lm_{\iota 1}, \\ d_{\iota 2} &< lm_{\iota 1} + \left(\frac{1}{l} - \frac{1}{l_{\iota 1}}\right) ld^-(\iota). \end{aligned}$$

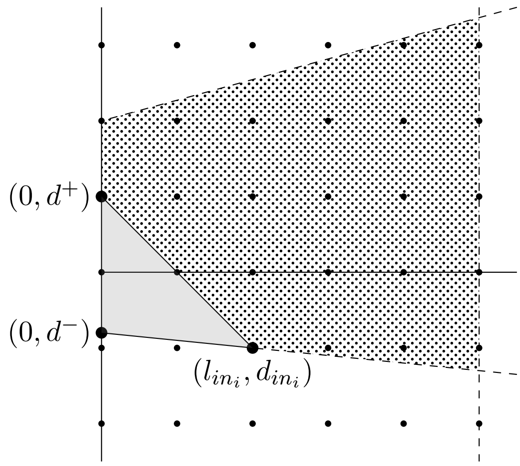


FIGURE 7. The remaining polygon P_i^v of an LDP complex for a boundary vertex v. The hatched space marks all possible positions for v as seen in Lemma 10.11 (ii).

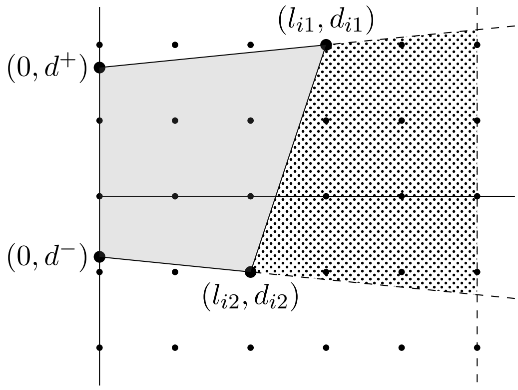


FIGURE 8. The remaining polygon P_i^v of an LDP complex for an interior vertex v. The hatched space marks all possible positions for v as seen in Lemma 10.11 (iii).

PROOF. We consider the statement (i). By Lemma 5.14 (i) we find the following which yields the first inequality:

$$\begin{array}{rrcl} & m^+ & < & \bar{l}^+ \\ \Leftrightarrow & m^+(\iota) - m_{\iota 2} + m_{\iota 1} & < & \bar{l}^+(\iota) - \dfrac{1}{l_{\iota 2}} + \dfrac{1}{l_{\iota 1}} \\ \Leftrightarrow & m_{\iota 1} & < & m_{\iota 2} + \bar{l}^+(\iota) - m^+(\iota) - \dfrac{1}{l_{\iota 2}} + \dfrac{1}{l_{\iota 1}} \\ \Leftrightarrow & d_{\iota 1} & < & l_{\iota 1} m_{\iota 2} + l_{\iota 1}\bar{l}^+(\iota)\left(1 - d^+(\iota)\right) + l_{\iota 1}\left(\dfrac{1}{l_{\iota 1}} - \dfrac{1}{l_{\iota 2}}\right) \end{array}$$

The second inequality follows immediately by slope orderedness, i.e. $m_{\iota 1} > m_{\iota 2}$ and the third one is a restatement of the inequality given in Lemma 5.14 (iv) (a).

We turn to statement (ii). First note that by Lemma 6.6 we have $d^+ \leq k$, which yields the following:

$$\begin{array}{rcl} d^+ \leq k \quad \Leftrightarrow \quad m_{\iota 1} + \sum_{i \neq \iota} m_{i1} & = & m^+ \leq k\bar{l}^+ \\ \Leftrightarrow \quad d_{\iota 1} & \leq & k l_{\iota 1}\bar{l}^+ - l_{\iota 1} \sum_{i \neq \iota} m_{i1} \\ & = & k l_{\iota 1}\bar{l}^+(\iota) + k l_{\iota 1}\left(\dfrac{1}{l_{\iota 1}} - \dfrac{1}{l_{\iota 2}}\right) - l_{\iota 1} m^+(\iota) + l_{\iota 1} m_{\iota 2} \\ & = & l_{\iota 1} m_{\iota 2} + l_{\iota 1}\bar{l}^+(\iota)(k - d^+(\iota)) + k l_{\iota 1}\left(\dfrac{1}{l_{\iota 1}} - \dfrac{1}{l_{\iota 2}}\right). \end{array}$$

The second statement follows immediately by slope orderedness, i.e. $m_{\iota 1} > m_{\iota 2}$ and the third statement follows immediately from Lemma 5.14 (iv) (b) and the following:

$$\begin{aligned}
d_{\iota 1} &> \frac{d_{\iota 2}-d^+}{l_{\iota 2}} l_{\iota 1} + d^+ \\
\Leftrightarrow \quad d_{\iota 1} l_{\iota 2} \bar{l}^+ &> d_{\iota 2} \bar{l}^+ l_{\iota 1} + m^+(l_{\iota 2} - l_{\iota 1}) \\
\Leftrightarrow \quad d_{\iota 1} l_{\iota 2} \bar{l}^+ &> d_{\iota 2} \bar{l}^+ l_{\iota 1} + (m^+(\iota) + m_{\iota 1} - m_{\iota 2})(l_{\iota 2} - l_{\iota 1}) \\
\Leftrightarrow \quad d_{\iota 1} l_{\iota 2} \bar{l}^+ - \frac{d_{\iota 1}}{l_{\iota 1}}(l_{\iota 2} - l_{\iota 1}) &> d_{\iota 2} \bar{l}^+ l_{\iota 1} - \frac{d_{\iota 2}}{l_{\iota 2}}(l_{\iota 2} - l_{\iota 1}) + m^+(\iota)(l_{\iota 2} - l_{\iota 1}) \\
\Leftrightarrow \quad d_{\iota 1} l_{\iota 2} \left(\bar{l}^+ - \left(\frac{1}{l_{\iota 1}} - \frac{1}{l_{\iota 2}}\right)\right) &> d_{\iota 2} l_{\iota 1} \left(\bar{l}^+ - \left(\frac{1}{l_{\iota 1}} - \frac{1}{l_{\iota 2}}\right)\right) + m^+(\iota)(l_{\iota 2} - l_{\iota 1}) \\
\Leftrightarrow \quad d_{\iota 1} l_{\iota 2} \bar{l}^+(\iota) &> d_{\iota 2} l_{\iota 1} \bar{l}^+(\iota) + m^+(\iota)(l_{\iota 2} - l_{\iota 1}) \\
\Leftrightarrow \quad d_{\iota 1} &> l_{\iota 1} m_{\iota 2} + \left(\frac{1}{l_{\iota 1}} - \frac{1}{l_{\iota 2}}\right) l_{\iota 1} d^+(\iota).
\end{aligned}$$

Note that statement (iv) is proven analaguously.

We turn to statement (iii). First note that the first and the last inequalities follow immediately by slopeorderedness.

For the inequalities two and three we consider the inequality given in Remark 10.10. For $j_1 = 0$, $j_2 = 1$ and $j_2 = \kappa$ we find:

$$\begin{aligned}
(l_{i1} - l_{i\kappa})(d_{i0} - d_{i1}) - (d_{i1} - d_{i\kappa})(l_{i0} - l_{i1}) &> 0 \\
\Leftrightarrow \quad (l_{i1} - l_{i\kappa})(d^+ - d_{i1}) + (d_{i1} - d_{i\kappa}) l_{i1} &> 0 \\
\Leftrightarrow \quad \frac{l_{i1} - l_{i\kappa}}{l_{\iota 1}} d^+ + m_{i1} l_{i\kappa} &> d_{i\kappa} \\
\Leftrightarrow \quad m_{i1} l_{i\kappa} + d^+ l_{i\kappa} \left(\frac{1}{l_{\iota\kappa}} - \frac{1}{l_{\iota 1}}\right) &> d_{i\kappa}.
\end{aligned}$$

The third inequality follows in the same manner with $j_1 = \kappa$, $j_2 = n_\iota - 1$ and $j_3 = n_\iota$ and Remark 10.10. ☐

Lemma 10.12. *Let $\mathcal{L}$ be an almost homogeneous, almost k-hollow LDP complex with linear form $u = u(0, 1, \gamma)$, where $\gamma \in \mathbb{Z}_{\geq 0}$. Then, aditionally to the constraint given in Lemma 10.11, we find:*

(ii) *For every $\iota \neq 0$ with $n_\iota = 2$ we have*

$$d_{\iota 1} \;\geq\; l_{\iota 1} m_{\iota 2} + \frac{l_{\iota 1}}{l_{\iota 2}\gamma}.$$

(iii) *For every $\iota \neq 0$ with $n_\iota \geq 3$ and every $2 \leq \kappa \leq n_\iota - 1$ we set $l := l_{\iota\kappa}$ and find*

$$d_{\iota\kappa} \;\geq\; l m_{\iota n_\iota} + \frac{l}{l_{\iota n_\iota}\gamma}.$$

(iv) *For $\iota = 0$ and $n_\iota = 2$ we have*

$$d_{01} \;\leq\; -l_{02} m_{1 n_1} - \frac{l_{02}}{\gamma l_{11}}.$$

Proof. The statements (i), (ii) and (iii) follow from the fact $\xi_\iota \leq \gamma$ and the last statement from the fact that $\gamma \leq \eta_0$. ☐

Remark 10.13. Let $\mathcal{L}$ be an LDP complex with contractible vertex $v \in \mathcal{L}$. Set $\mathcal{L}' := \mathcal{L}^v$.

Then we note the following:

(i) For $v = v_{\iota 1}$ we have $d^+(\iota) = (d')^+$.
(ii) For $v = v_{\iota j}$, with $1 < j < n_\iota$ we have $d^+ = (d')^+$ and $d^- = (d')^-$.
(iii) For $v = v_{\iota n_\iota}$ we have $d^-(\iota) = (d')^-$.

Therefore the bounds for the integer $l_{\iota j}$ given in Corollary 10.9 only depend on $\mathcal{L}'$. Furthermore all bounds on $d_{\iota j}$ stated in Lemma 10.11 and Lemma 10.12 only depend on $\mathcal{L}'$ and the integer $l_{\iota j}$.

PROOF OF PROPOSITION 10.6. We start with statement (i). Note that for every interior vertex $v_{ij} = l_{ij}e_i + d_{ij}e_{r+1}$ of $\mathcal{L}$ we have bounds on l_{ij} by Corollary 10.9 (i) or (iii) and bounds on d_{ij} by Lemma 10.11 (i) or (iii). Both bounds only depend on entries in $\mathcal{L}'$, which yields that there are only finitely many possible interior points for $\mathcal{L}$, i.e. finiteness of almost k-hollow LDP complexes.

We turn to the second statement. Note that by Proposition 10.7 there are at most $4k$ polygons P_i in $\mathcal{L}$, furthermore with every contraction the number of boundary vertices decreases by one since $n_i \leq 2$ for all $0 \leq i \leq r$. Hence the maximal number of successive contractions is given by $4k$.

Thus it suffices to show that for an LDP complex $\mathcal{L}'$ there are only finitely many LDP complexes $\mathcal{L}$ with a contractible boundary vertex v such that $\mathcal{L}^v = \mathcal{L}'$. To see this observe that for $v = l_{ij}e_i + d_{ij}e_{r+1}$ there are bounds for l_{ij} by Corollary 10.9 (ii) or (iv) and bounds on d_{ij} by Lemma 10.11 (ii) or (iv). □

PROOF OF THEOREM 10.1. We consider Construction 10.3 to see that any LDP complex is contracted onto an almost homogeneous, almost k-hollow non-toric combinatorially minimal LDP complex or an almost homogeneous, almost k-hollow toric LDP complex. For the latter case note that there are only finitely many by Proposition 9.8. Moreover, Theorem 9.4 states that there are only finitely many almost k-hollow almost homogeneous combinatorially minimal LDP complexes.

Now since there are only finitely many LDP complexes to contract onto, it suffices to show that for every such LDP complex $\mathcal{L}'$ there are only finitely many LDP complexes $\mathcal{L}$ such that $\mathcal{L}^{v_1,\ldots,v_k} = \mathcal{L}$. This follows with Proposition 10.6 and Construction 10.3. □

11. Classification algorithms for LDP complexes

This section summarizes all observations of the previous ones in algorithmic constructions to find all almost homogeneous, almost k-hollow LDP complexes.

We recall some definitions that will occur in the following constructions. An LDP complex $\mathcal{L}$ is called *irredundant* if for every $i = 0, \ldots, r$ we have $l_{in_i}n_i \neq 1$. It is called *adapted to the sink* if the following two conditions hold:

(i) $0 \leq d_{in_i} < l_{in_i}$ for $i = 1, \ldots, r$.
(ii) $l_{0n_0}, l_{1n_1} \geq l_{2n_2} \geq \cdots \geq l_{rn_r}$.

ALGORITHM 11.1. The following algorithm defines a set S_0 of all polygons P with at most five vertices with a non-toric almost k-hollow, almost homogeneous LDP complex $\mathcal{L}$ such that $\mathsf{P} = \mathcal{L}^v$ for a contractible vertex $v \in \mathcal{V}(\mathcal{L})$ and $\mathcal{L}$ is irredundant and adapted to the sink:

- For every almost k-hollow Fano polygon P with at most five vertices do the following:
- For every vertex $v_1 \in \mathcal{V}(\mathsf{P})$ test whether it admits a Demazure root, i.e. a linear form $u \in M$ such that

$$\langle u, v_1 \rangle \ = \ -1, \qquad \langle u, v \rangle \ \geq \ 0, \quad \text{for all } v \in \mathcal{V}(C) \setminus \{v_1\}.$$

- If the test is true, then do the following steps:
  (i) (a) If $\mathcal{V}(\mathsf{P}) > 3$, then for every vertex $v^+ \in \mathcal{V}(C)$ with $v_1 \neq v^+$ find the basis $(v^+, \bar{v}^+)$ such that the following is true:

$$v_1 \ = \ l\bar{v}^+ + dv^+, \quad \text{where } 0 < l, \quad 0 \leq d < l.$$

  (b) If $\mathcal{V}(\mathsf{P}) < 5$, then for every primitive vector $v^+ \in \mathsf{P}$ with $v_1 \neq v^+$ find the basis $(v^+, \bar{v}^+)$ such that the following is true:

$$v_1 \ = \ l\bar{v}^+ + dv^+, \quad \text{where } 0 < l, \quad 0 \leq d < l.$$

  (ii) For every vertex $v \neq v_1$ write $v = l(v)\bar{v}^+ + d(v)v^+$ and consider the partition of vertices given by

$$\begin{aligned} \mathcal{V}_0 \ &:= \ \{(l(v), d(v));\ v \in \mathcal{V}(C) \text{ and } l(v) < 0\}, \\ \mathcal{V}_1 \ &:= \ \{(l(v), d(v));\ v \in \mathcal{V}(C) \text{ and } l(v) > 0\}, \\ \mathcal{V}_\pm \ &= \ \{(l(v), d(v));\ v \in \mathcal{V}(C) \text{ and } l(v) = 0\}. \end{aligned}$$

  (iii) If we have $|\mathcal{V}_0|, |\mathcal{V}_1| \leq 2$ and $|\mathcal{V}_\pm| = 1$, add the polygon $\mathsf{P}' = \operatorname{conv}((l(v), d(v); v \in \mathcal{V}(\mathsf{P}))$ to the set S_0.

Algorithm 11.2. Let $\mathcal{L}'$ be an almost k-hollow, almost homogeneous LDP complex with $n_i \leq 2$ and linear form $u = u(0, 1, \gamma)$.

The set $S_{int}(\mathcal{L}')$ of all LDP complexes $\mathcal{L}$ with contractible interior points $v_1, \ldots, v_k$ such that $\mathcal{L}^{v_1, \ldots, v_k}$ is determined by the following steps:

(i) For every $0 \leq i \leq r$ find all possible interior points for an LDP complex, i.e. do the following:
  (a) Set $\mathcal{V}_i := \{v_{i1}, \ldots, v_{in_i}\}$.
  (b) For every $0 \leq i \leq r$ with $n_i = 2$ find all pairs of coprime integers (l, d) sufficing conditions (iii) of Corollary 10.9, Lemma 10.11 and Lemma 10.12, respectively.
  (c) If $\operatorname{conv}(0, v_{i1}, v_{in_i}, le_i + de_{r+1})$ is k-hollow add $le_i + de_{r+1}$ to $\mathcal{V}_i$.

(ii) For every $0 \leq i \leq r$ find all possible almost k-hollow polygons P_i with vertices in $\mathcal{V}_i$, i.e. do the following
  (a) Set $\mathcal{P}_i := \{\operatorname{conv}(0, v_{i1}, \ldots, v_{in_i})\}$.
  (b) Take a subset $V \subseteq \mathcal{V}_i$ and define $\tilde{\mathsf{P}}_i(V) := \operatorname{conv}(0, V)$.
  (c) If $\tilde{\mathsf{P}}_i(V)$ is k-hollow, then add $\mathsf{P}_i(V)$ to $\mathcal{P}_i$

(iii) Construct all possible LDP complexes with polgons $\tilde{\mathsf{P}}_i \in \mathcal{P}_i$, i.e. do the following:
  (a) For every $i = 0, \ldots, r$ take a polygon $\mathsf{P}_i \in \tilde{\mathcal{P}}_i$.
  (b) Check the inequalities (iv) of the Definition 5.2 of an LDP complex.

(c) Calculate the vertices v^+ and v^- as in Definition 5.2 (v) and define $\mathsf{P}_i := \operatorname{conv}(v^+, v^-, \tilde{\mathsf{P}}_i)$.
(d) If $\mathcal{L} := \bigcup_i \mathsf{P}_i$ is almost k-hollow, add it to the set $S_{int}(\mathcal{L}')$.

LEMMA 11.3. *Let $\mathcal{L}, \mathcal{L}'$ be almost k-hollow LDP complexes such that $\mathcal{L}^{v_1,\dots,v_k} = \mathcal{L}'$ for contractible boundary vertices $v_1, \dots, v_k \in \mathcal{V}(\mathcal{L})$. We set:*

$$a' := max\left(a;\ 1/a < (d')^+\right), \quad b' := max\left(b;\ 1/b < -(d')^-\right).$$

Then for every $1 \le \kappa \le k$ the following statements is true:

(i) *If $v_\kappa = v_{\iota 1} = l_{\iota 1} e_\iota + d_{\iota 1} e_{r+1}$, then*

$$\begin{aligned} l_{\iota 1} &\le 2k^2 \cdot \max(a', b'), \\ d_{\iota 1} &\le l_{\iota 1} m_{\iota 2} + 2 l_{\iota 1}(k - (d^+)') + k l_{\iota 1} \left(\frac{1}{l_{\iota 1}} - \frac{1}{l_{\iota 2}}\right). \end{aligned}$$

(ii) *If $v_\kappa = v_{\iota n_\iota} = l_{\iota n_\iota} e_\iota + d_{\iota n_\iota} e_{r+1}$, then*

$$\begin{aligned} l_{\iota n_\iota} &\le 2k^2 \cdot \max(a', b'), \\ d_{\iota n_\iota} &\ge l_{\iota n_\iota} m_{\iota 2} + 2 l_{\iota n_\iota}(-k - (d^+)') + k l_{\iota n_\iota} \left(\frac{1}{l_{\iota n_\iota}} - \frac{1}{l_{\iota 1}}\right). \end{aligned}$$

PROOF. This is an immediate consequence of Lemma 7.6, namely the inequalities $d^+ \ge (d')^+$ and $d^- \le (d')^-$ and the fact that the maximal number for $\bar{l}^+$ and $\bar{l}^-$ is 2.

We show the statement for $v_{\iota 1}$ and note that the statement for $v_{\iota n_\iota}$ is shown in the same way. First note that $\mathcal{L}' \subseteq \mathcal{L}$ by Theorem 7.4 (ii), in particular we have $(d')+ \le d^+$ and $(d')^- \ge d^-$. For the bounds for $l_{\iota 1}$ consider the integers defined in Corollary 10.9:

$$a_\iota = \max(a; 1/a \le d^+(\iota)) \le \max(a; 1/a \le (d')^+) = a', \qquad b' \le b.$$

Therefore we find the following:

$$l_{\iota 1} \le 2k^2 \cdot \max(a_\iota, b) \le 2k^2 \cdot \max(a', b').$$

We turn to the upper bound of $d_{\iota 1}$. Here we find:

$$\begin{aligned} d_{\iota 1} &\le l_{\iota 1} m_{\iota 2} + l_{\iota 1} \bar{l}^+(\iota)(k - d^+(\iota)) + k l_{\iota 1} \left(\frac{1}{l_{\iota 1}} - \frac{1}{l_{\iota 2}}\right) \\ &\le l_{\iota 1} m_{\iota 2} + l_{\iota 1} \bar{l}^+(\iota)(k - (d')^+) + k l_{\iota 1} \left(\frac{1}{l_{\iota 1}} - \frac{1}{l_{\iota 2}}\right) \\ &\le l_{\iota 1} m_{\iota 2} + 2 l_{\iota 1}(k - (d')^+) + k l_{\iota 1} \left(\frac{1}{l_{\iota 1}} - \frac{1}{l_{\iota 2}}\right) \end{aligned}$$

For the last estimate we used that $\bar{l}^+ \le 2$ as seen in Remark 6.7. $\square$

REMARK 11.4. Let $\mathcal{L}$ be an almost homogeneous LDP complex. Then the vertices $v_{1n_1}, \dots, v_{rn_r}$ are not contractible since by Proposition 9.6 the self intersection numbers of the corresponding divisors $D_{in_i}^2$ are positive, see Lemma 7.13. In particular, there are no contractible boundary vertices v_{in_i} for every $i = 1, \dots, r$.

ALGORITHM 11.5. Let $\mathcal{L}'$ be an almost k-hollow, almost homogeneous LDP complex with $n_i \le 2$ and linear form $u = u(0, 1, \gamma)$.

The set $S_{bound}(\mathcal{L}')$ of all LDP complexes $\mathcal{L}$ with contractible boundary points $v_1, \ldots, v_k$ such that $\mathcal{L}^{v_1,\ldots,v_k} = \mathcal{L}'$ is determined by the following steps:

(i) Find all possible boundary vertices for an LDP complex $\mathcal{L}$, i.e. find all pairs of coprime integers (l, d) sufficing the conditions of Lemma 11.3 (i) and (ii) and add them to the set $\mathcal{V}^+$ and $\mathcal{V}^-$, respectively.

(ii) Find all possible k-hollow polygons P_i with boundary vertices $\mathcal{V}^+$ and $\mathcal{V}^-$, i.e. do the following:
   (a) For all $i = 0, \ldots, 4k-1$ define $\mathcal{P}_i := \{\operatorname{conv}(v_{i_1}, \ldots, v_{in_i})\}$.
   (b) If $n_0 = 1$, for every $v \in \mathcal{V}^-$ test whether the polygon $\tilde{\mathsf{P}}_0 := \operatorname{conv}(0, v_{01}, v)$ is k-hollow and whether v suffices condition (iv) of Lemma 10.11 and condition (iv) of Lemma 10.12. If so, add $\tilde{\mathsf{P}}_0$ to $\mathcal{P}_0$
   (c) For all $i = 1, \ldots, r$, if $n_i = 1$ then for every $v \in \mathcal{V}^+$ test whether $\tilde{\mathsf{P}}_i := \operatorname{conv}(0, v_{in_i}, v)$ is k-hollow and suffices conditions (ii) of Lemma 10.11 and condition (ii) of Lemma 10.12 if $i \neq 0$. If so, add $\tilde{\mathsf{P}}_i$ to $\mathcal{P}_i$.

(iii) Construct all possible LDP complexes with polygons $\tilde{P}_i \in \mathcal{P}_i$, i.e. do the following:
   (a) For every $i = 0, \ldots, r$ take a polygon $\mathsf{P}_i \in \tilde{\mathcal{P}}_i$.
   (b) Check the inequalities (iv) of the Definition 5.2 of an LDP complex.
   (c) Calculate the vertices v^+ and v^- as in Definition 5.2 (v) and define $\mathsf{P}_i := \operatorname{conv}(v^+, v^-, \tilde{\mathsf{P}}_i)$.
   (d) If $\mathcal{L} := \bigcup_i \mathsf{P}_i$ is almost k-hollow, add it to the set $S_{bound}(\mathcal{L}')$.

Algorithm 11.6. The set S defined with the following steps contains all almost k-hollow, almost homogeneous LDP complexes.

(i) Set S to be the set of all combinatorially minimal non-toric LDP complexes as in Theorem 9.4 and all Fano polygons found with Algorithm 11.1.
(ii) For every LDP complex $\mathcal{L}' \in S$ find all LDP complexes $\mathcal{L}$ with contraction onto $\mathcal{L}'$ with boundary vertices as in Algorithm 11.5. Add the LDP complexes found to the set S.
(iii) For every LDP complex $\mathcal{L}' \in S$ find all LDP complexes $\mathcal{L}$ with contraction onto $\mathcal{L}'$ with interior vertices as in Algorithm 11.2. Add the LDP complexes found to the set S.
(iv) Delete all LDP complexes with coinciding standard form, see Definition 6.21.

12. Almost 3-hollow polygons and LDP complexes

In this section we present the results using the algorithms developed for the case $k = 3$. All algorithms have been implemented in the computer algebra system `Maple`.

With the algorithm described in Construction 12.2 we achieved the following classification.

THEOREM 12.1. *There are exactly 910786 almost* 3*-hollow lattice polygons. Furthermore the maximal volume is given by 48 and the maximal number of vertices by 11. They are distributed as shown in the following table:*

no. of vertices	*no. of polygons*	*maximal volume*
3	*1012*	*48*
4	*18944*	*95/2*
5	*113758*	*47*
6	*280316*	*45*
7	*310587*	*43*
8	*150866*	*39*
9	*32743*	*69/2*
10	*2526*	*61/2*
11	*34*	*28*

COROLLARY 12.2. *There are exactly 47902 many* 3*-hollow Fano polygons. Furthermore the maximal volume is given as 47 and the maximal number of vertices by 11. They are distributed as shown in the following table:*

no. of vertices	*no. of Fano polygons*	*maximal volume*
3	*355*	*44*
4	*3983*	*91/2*
5	*13454*	*47*
6	*17791*	*43*
7	*9651*	*39*
8	*2360*	*34*
9	*280*	*30*
10	*27*	*57/2*
11	*1*	*47/2*

We turn to $\mathbb{K}^*$-surfaces. Using Proposition 9.6 and Proposition 9.7 we found the following combinatorially minimal almost homogeneous 1/3-log canonical del Pezzo surfaces $\mathbb{K}^*$-surfaces. The list is in accordance with the classification result for combinatorially minimal 1/2-log canonical del Pezzo surfaces $\mathbb{K}^*$-surfaces found in [**24**].

PROPOSITION 12.3. *The following list contains all almost homogeneous* 1/3*-log canonical del Pezzo surfaces that are combinatorially minimal. Here* $\iota(X)$ *denotes the Gorenstein index, furthermore the last column marks the numeration in* [**24**] *if the surface is* 1/2*-log canonical.*

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$[w_1, \ldots, w_r]$	$\iota(X)$	*No. in* [**24**]
1	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^{23}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 11 & 81 & 23 & 4 \end{bmatrix}$	*33*	-
2	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^{22}\rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 5 & 39 & 11 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	*15*	-
3	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^{20}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 11 & 49 & 20 & 3 \end{bmatrix}$	*539*	-
4	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^{20}\rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	$\begin{bmatrix} 2 & 18 & 5 & 1 \\ \bar{1} & \bar{3} & \bar{1} & \bar{0} \end{bmatrix}$	*6*	-
5	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^{19}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 10 & 47 & 19 & 3 \end{bmatrix}$	*235*	-

6	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^{19}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}7 & 69 & 19 & 4\end{bmatrix}$	*21*	-
7	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^{17}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}9 & 59 & 17 & 4\end{bmatrix}$	*177*	-
8	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{19}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}11 & 27 & 19 & 2\end{bmatrix}$	*99*	-
9	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{17}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}11 & 23 & 17 & 2\end{bmatrix}$	*253*	-
10	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{17}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}11 & 57 & 34 & 4\end{bmatrix}$	*33*	-
11	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^{17}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}8 & 43 & 17 & 3\end{bmatrix}$	*86*	-
12	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^{17}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}5 & 63 & 17 & 4\end{bmatrix}$	*15*	-
13	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^{15}\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix}3 & 12 & 5 & 1\\ \bar{1} & \bar{2} & \bar{1} & \bar{0}\end{bmatrix}$	*6*	-
14	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^{15}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}7 & 53 & 15 & 4\end{bmatrix}$	*371*	-
15	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{16}\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}\ 2\mathbb{Z}$	$\begin{bmatrix}5 & 11 & 8 & 1\\ 1 & 1 & 1 & 0\end{bmatrix}$	*55*	-
16	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{16}\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}\ 2\mathbb{Z}$	$\begin{bmatrix}5 & 27 & 16 & 2\\ \bar{0} & \bar{0} & \bar{1} & \bar{1}\end{bmatrix}$	*15*	-
17	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^{16}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}7 & 41 & 16 & 3\end{bmatrix}$	*287*	-
18	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^{16}\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$	$\begin{bmatrix}1 & 15 & 4 & 1\\ \bar{1} & \bar{3} & \bar{1} & \bar{0}\end{bmatrix}$	*3*	-
19	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{13}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 15 & 4 & 1\\ \bar{1} & \bar{3} & \bar{1} & \bar{0}\end{bmatrix}$	*51*	-
20	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{13}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}9 & 43 & 26 & 4\end{bmatrix}$	*129*	-
21	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^{13}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}7 & 32 & 13 & 3\end{bmatrix}$	*14*	-
22	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^{13}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}5 & 47 & 13 & 4\end{bmatrix}$	*235*	-
23	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{14}\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}4 & 10 & 7 & 1\\ \bar{1} & \bar{1} & \bar{1} & \bar{0}\end{bmatrix}$	*10*	-
24	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{14}\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$	$\begin{bmatrix}2 & 12 & 7 & 1\\ \bar{3} & \bar{1} & \bar{2} & \bar{0}\end{bmatrix}$	*6*	-
25	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^{14}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}5 & 37 & 14 & 3\end{bmatrix}$	*185*	-
26	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^{14}\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1 & 27 & 7 & 2\\ \bar{0} & \bar{0} & \bar{1} & \bar{1}\end{bmatrix}$	*3*	-
27	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{15}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}7 & 23 & 15 & 2\end{bmatrix}$	*161*	-
28	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{13}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}7 & 19 & 13 & 2\end{bmatrix}$	*133*	-
29	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{13}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}7 & 45 & 26 & 4\end{bmatrix}$	*21*	-
30	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^{13}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}4 & 35 & 13 & 3\end{bmatrix}$	*35*	-
31	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^{13}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 51 & 13 & 4\end{bmatrix}$	*3*	-
32	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^{13}\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}3 & 17 & 5 & 2\\ \bar{1} & \bar{1} & \bar{0} & \bar{1}\end{bmatrix}$	*51*	-
33	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{11}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}7 & 15 & 11 & 2\end{bmatrix}$	*105*	-
34	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{11}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}7 & 37 & 22 & 4\end{bmatrix}$	*259*	-
35	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^{11}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}5 & 28 & 11 & 3\end{bmatrix}$	*10*	*28*
36	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^{11}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}3 & 41 & 11 & 4\end{bmatrix}$	*41*	-
37	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{14}\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}2 & 12 & 7 & 1\\ \bar{1} & \bar{1} & \bar{1} & \bar{0}\end{bmatrix}$	*6*	-
38	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^{9}\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix}2 & 7 & 3 & 1\\ 1 & 2 & 1 & 0\end{bmatrix}$	*21*	-
39	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^{9}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}5 & 31 & 9 & 4\end{bmatrix}$	*155*	-

40	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{11}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}5 & 17 & 11 & 2\end{bmatrix}$	*85*	-
41	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{11}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}5 & 39 & 22 & 4\end{bmatrix}$	*15*	-
42	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^{11}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}2 & 31 & 11 & 3\end{bmatrix}$	*31*	-
43	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^8\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}3 & 5 & 4 & 1\\ \bar{1} & \bar{1} & \bar{1} & \bar{0}\end{bmatrix}$	*3*	-
44	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^8\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}3 & 13 & 8 & 2\\ \bar{1} & \bar{1} & \bar{1} & \bar{1}\end{bmatrix}$	*39*	-
45	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^8\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}5 & 19 & 8 & 3\end{bmatrix}$	*95*	-
46	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^8\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$	$\begin{bmatrix}1 & 7 & 2 & 1\\ \bar{1} & \bar{3} & \bar{1} & \bar{0}\end{bmatrix}$	*14*	-
47	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^9\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}5 & 13 & 9 & 2\end{bmatrix}$	*65*	*21*
48	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^9\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}5 & 31 & 18 & 4\end{bmatrix}$	*155*	-
49	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^9\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix}1 & 8 & 3 & 1\\ \bar{1} & \bar{2} & \bar{1} & \bar{0}\end{bmatrix}$	*2*	*27*
50	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^9\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 35 & 9 & 4\end{bmatrix}$	*35*	-
51	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{10}\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}2 & 8 & 5 & 1\\ \bar{1} & \bar{1} & \bar{1} & \bar{0}\end{bmatrix}$	*8*	-
52	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{10}\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$	$\begin{bmatrix}1 & 9 & 5 & 1\\ \bar{3} & \bar{1} & \bar{2} & \bar{0}\end{bmatrix}$	*3*	-
53	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^{10}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 29 & 10 & 3\end{bmatrix}$	*29*	-
54	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^{11}\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}3 & 19 & 11 & 2\end{bmatrix}$	*57*	-
55	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^7\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}5 & 9 & 7 & 2\end{bmatrix}$	*5*	-
56	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^7\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}5 & 23 & 14 & 4\end{bmatrix}$	*115*	-
57	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^7\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}4 & 17 & 7 & 3\end{bmatrix}$	*34*	*26*
58	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^7\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}3 & 25 & 7 & 4\end{bmatrix}$	*75*	-
59	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^6\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}2 & 4 & 3 & 1\\ \bar{1} & \bar{1} & \bar{1} & \bar{0}\end{bmatrix}$	*2*	*19*
60	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^6\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$	$\begin{bmatrix}1 & 5 & 3 & 1\\ \bar{3} & \bar{1} & \bar{2} & \bar{0}\end{bmatrix}$	*10*	-
61	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^6\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix}1 & 5 & 2 & 1\\ \bar{1} & \bar{2} & \bar{1} & \bar{0}\end{bmatrix}$	*15*	*24*
62	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^4+T_4^6\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1 & 11 & 3 & 2\\ \bar{0} & \bar{0} & \bar{1} & \bar{1}\end{bmatrix}$	*11*	-
63	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^7\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}3 & 11 & 7 & 2\end{bmatrix}$	*11*	*63*
64	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^7\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}3 & 25 & 14 & 4\end{bmatrix}$	*25*	-
65	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^7\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 20 & 7 & 3\end{bmatrix}$	*2*	*25*
66	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^8\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1 & 7 & 4 & 1\\ \bar{1} & \bar{1} & \bar{1} & \bar{0}\end{bmatrix}$	*7*	-
67	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^8\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1 & 15 & 8 & 2\\ \bar{0} & \bar{0} & \bar{1} & \bar{1}\end{bmatrix}$	*3*	-
68	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1^2T_2+T_3^2+T_4^8\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1 & 6 & 4 & 1\\ \bar{1} & \bar{0} & \bar{1} & \bar{0}\end{bmatrix}$	*3*	-
69	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^5+T_4^8\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 39 & 8 & 5\end{bmatrix}$	*3*	-
70	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^5\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}3 & 7 & 5 & 2\end{bmatrix}$	*3*	*18*
71	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^2+T_4^5\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}3 & 17 & 10 & 4\end{bmatrix}$	*51*	-
72	$\frac{\mathbb{C}[T_1,\ldots,T_4]}{\langle T_1T_2+T_3^3+T_4^5\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}2 & 13 & 5 & 3\end{bmatrix}$	*13*	*23*

73	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1T_2+T_3^4+T_4^5\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 19 & 5 & 4\end{bmatrix}$	*19*	-
74	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1T_2+T_3^2+T_4^7\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 13 & 7 & 2\end{bmatrix}$	*13*	-
75	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1T_2+T_3^2+T_4^7\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 27 & 14 & 4\end{bmatrix}$	*3*	-
76	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^2T_2+T_3^2+T_4^7\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 12 & 7 & 2\end{bmatrix}$	*3*	-
77	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1T_2+T_3^9+T_4^3\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix}1 & 8 & 1 & 3\\ \bar{1} & \bar{2} & \bar{0} & \bar{2}\end{bmatrix}$	*6*	-
78	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1T_2+T_3^2+T_4^4\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1 & 3 & 2 & 1\\ \bar{1} & \bar{1} & \bar{1} & \bar{0}\end{bmatrix}$	*1*	*16*
79	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1T_2+T_3^2+T_4^4\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1 & 7 & 4 & 2\\ \bar{0} & \bar{0} & \bar{1} & \bar{1}\end{bmatrix}$	*7*	-
80	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1T_2+T_3^3+T_4^4\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 11 & 4 & 3\end{bmatrix}$	*11*	*22*
81	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1T_2+T_3^3+T_4^4\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 9 & 5 & 2\end{bmatrix}$	*9*	*17*
82	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1T_2+T_3^2+T_4^5\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 19 & 10 & 4\end{bmatrix}$	*19*	-
83	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^2T_2+T_3^2+T_4^5\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 8 & 5 & 2\end{bmatrix}$	*4*	*53*
84	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1T_2+T_3^7+T_4^3\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}2 & 19 & 3 & 7\end{bmatrix}$	*19*	-
85	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1T_2+T_3^2+T_4^3\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 5 & 3 & 2\end{bmatrix}$	*1*	*15*
86	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1T_2+T_3^2+T_4^3\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 11 & 6 & 4\end{bmatrix}$	*11*	-
87	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^2T_2+T_3^2+T_4^3\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 4 & 3 & 2\end{bmatrix}$	*1*	*51*
88	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^3T_2+T_3^2+T_4^3\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 3 & 3 & 2\end{bmatrix}$	*1*	*40*
89	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^3T_2+T_3^2+T_4^3\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 9 & 6 & 4\end{bmatrix}$	*9*	-
90	$\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1T_2+T_3^5+T_4^3\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1 & 14 & 3 & 5\end{bmatrix}$	*7*	-
91	$\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^2T_2+T_3^2T_4+T_5^2\rangle}$	$\mathbb{Z}^2$	$\begin{bmatrix}1 & 2 & 1 & 2 & 2\\ 1 & 0 & 0 & 2 & 1\end{bmatrix}$	*1*	*64*

Last we turn to the classification of almost homogeneous $1/3$-log canonical del Pezzo surfaces. Using Algorithm 11.6 we achieved the following result.

THEOREM 12.4. *There are exactly 21968 almost homogeneous* $1/3$*-log canonical del Pezzo surfaces.*

The unit component of its automorphism group is given as

$$\left(\mathbb{K}^{\rho} \rtimes \mathbb{K}^{\zeta}\right) \rtimes \mathbb{K}^{*}, \qquad \text{where } \rho \leq 5,\ \zeta \leq 1.$$

We find the following distribution among the exponents ρ *and* ζ*:*

(ρ,ζ)	no. of surfaces
$(1,0)$	17274
$(2,0)$	625
$(3,0)$	27
$(4,0)$	1

(ρ,ζ)	no. of surfaces
$(0,1)$	183
$(1,1)$	1002
$(2,1)$	1602
$(3,1)$	884
$(4,1)$	168
$(5,1)$	2

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