Formal Theories of Occurrences and Substitutions

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Zusammenfassung in deutscher Sprache

Während die grundlegenden syntaktischen Gegenstände formaler Sprachen, wie es etwa deren Terme, Formeln und Ableitungen sind, eine wohldurchdachte Behandlung in der Logik und verwandter Wissenschaften erfahren, werden Vorkommen solcher Gegenstände im wissenschaftlichen Diskurs über Gebühr vernachlässigt. An jenen Stellen im Diskurs, an denen die Diskussion der syntaktischen Gegenstände selbst nicht mehr genügt, werden, im besten Fall, rudimentäre Theorien zugrundegelegt oder es wird, im schlechtesten Fall, lediglich auf Intuitionen verwiesen.

Wesentliches Ziel der von uns vorgelegten Untersuchung "Formal Theories of Occurrences and Substitutions" ist, diese Situation zu ändern. Entsprechend werden unsere Intuitionen widerspiegelnde formale Repräsentanten des philosophischen Begriffs von Vorkommen, also die formalen Vorkommen, in verschiedenen Verallgemeinerungen eingeführt und jeweils einer mathematischen Untersuchung unterzogen. Innerhalb dieser Untersuchungen werden grundlegende Methoden und Begrifflichkeiten einer Theorie der Vorkommen erarbeitet und Anwendungen des Begriffs der formalen Vorkommen beispielhaft diskutiert.

Insbesondere wird durch diese Untersuchung eine mathematische Grundlage geschaffen, um in weitergehenden Projekten fundamentale Begriffe der Beweistheorie adäquat einzuführen und um eine formal fundierte Erörterung interessanter Probleme der Philosophie der Mathematik, wie etwa die Reinheit von Beweisen oder auch deren Einfachheit, zu ermöglichen.

Eine erste philosophische Analyse des Begriffs der Vorkommen ergibt, dass diese durch drei Aspekte bestimmt sind. Zunächst ist ein Vorkommen immer schon ein Vorkommen eines syntaktischen Gegenstandes, welcher seine *Form* genannt wird. Weiterhin kommt ein Vorkommen grundsätzlich innerhalb eines syntaktischen Gegenstandes vor, welcher sein *Umfeld* genannt wird. Schließlich ist ein Vorkommen durch seine *Position* innerhalb des Umfelds bestimmt; verschiedene Vorkommen der selben Form im selben Umfeld werden durch ihre Position unterschieden.

Damit wird schon deutlich, dass jede sinnvolle Kombination syntaktischer Gegenstände für Umfeld und Form einen eigenen Typ von Vorkommen nach sich zieht. Die hier vorgelegte Untersuchung konzentriert sich auf den paradigmatischen Fall einer Theorie von Vorkommen von Termen in Termen (einer erststufigen formalen Sprache der Logik); eine Übertragung der hier diskutierten Theorie auf andere Typen von Vorkommen erscheint ohne wesentliche Probleme möglich.

Die von Schütte [29] eingeführten Nominalformen erweisen sich für eine formale Repräsentation der Position von Vorkommen als besonders geeignet und insbesondere den bisher üblichen Methoden überlegen. Diese Nominalformen sind Verallgemeinerungen der üblichen syntaktischen Gegenstände, in denen sogenannte Nominalsymbole als weitere Zeichen des Alphabets Verwendung finden dürfen.

Die Position eines Vorkommens wird durch diejenige Nominalform angegeben, die dadurch entsteht, dass im Umfeld des Vorkommens ihre Form an intendierter Stelle durch ein Nominalsymbol ersetzt ist. Nominalformen, die die Position von Termen in Termen repräsentieren, sind eine Verallgemeinerung der Terme der zugrundeliegenden Sprache, in der die Nominalsymbole für Terme stehen.

Zunächst werden einförmige Vorkommen eingeführt. Für solche Vorkommen ist charakteristisch, dass in der Position lediglich ein bestimmtes Nominalsymbol vorkommen darf, das aber beliebig oft. Entsprechend können sowohl Einzel- als auch Mehrfachvorkommen, in denen die Form an mehreren Stellen im Umfeld simultan intendiert ist, repräsentiert werden. Durch die formale Grundlegung des Begriffs der Vorkommen kann die Anzahl bestimmter Vorkommen (in einem gegebenen Umfeld) formal bestimmt werden; ebenso kann für zwei Vorkommen im gleichen Umfeld bestimmt werden, ob eines der Vorkommen sich innerhalb des anderen befindet oder nicht.

Eine kanonische Verallgemeinerung der einförmigen Vorkommen sind die mehrförmigen Vorkommen. Deren Position kann unterschiedliche Nominalsymbole enthalten, die Form ist durch eine Folge von Termen gegeben. Da verschiedene mehrförmige Vorkommen dasselbe (informell gegebene) Vorkommen repräsentieren können, werden diese durch eine geeignete Äquivalenzrelation identifiziert; kanonische Normalformen werden bestimmt.

Desweiteren wird die Unabhängigkeit von Vorkommen eingeführt. Unabhängige Vorkommen haben ein gemeinsames Umfeld, die intendierten Formen überlappen sich an keiner Stelle. Unabhängige Vorkommen können zu einem gemeinsamen Vorkommen verschmolzen werden, ein einzelnes Vorkommen kann in unabhängige Vorkommen aufgespalten werden.

Eine weitergehende Verallgemeinerung des Begriffs der Vorkommen sind die formalen Substitutionen, welche im Wesentlichen als ein Paar zweier mehrförmiger Vorkommen mit gemeinsamer Position aufgefasst werden können. Das erste Vorkommen gibt das Umfeld und die von der Substitution betroffenen Formen samt ihrer Stellen an. Das zweite Vorkommen beschreibt, welche Formen an den besagten Stellen ersetzt werden und was das Resultat einer

solchen Ersetzung ist. Wie schon im Fall der mehrförmigen Vorkommen wird eine geeignete Äquivalenzrelation für Substitutionen eingeführt; kanonische Normalformen werden bestimmt.

Der Begriff der Substitution erweist sich insbesondere als geeignet, Rechenschritte und Rechnungen, wie sie in der Mathematik und der Informatik alltäglich auftreten, zu repräsentieren und diese so einer formalen Untersuchung zuzuführen.

Wie schon für Vorkommen wird die Unabhängigkeit von Substitutionen eingeführt. Unabhängige Substitutionen können zu einer gemeinsamen Substitution verschmolzen werden, ebenso kann eine Substitution in eine Folge unabhängiger Substitutionen aufgespalten werden.

Es erweist sich, dass Mengen von Substitutionen mengentheoretische Funktionen sind; diese werden als explizite Substitutionsfunktionen bezeichnet. Erwartungsgemäß sind Funktionen, die üblicherweise als Substitutionsfunktion verstanden werden, keine expliziten Substitutionsfunktionen. Um diese dennoch als solche klassifizieren zu können, wird das Konzept impliziter Subsitutionsfunktionen eingeführt.

Implizite Substitutionsfunktionen sind Funktionen, die mithilfe einer Explikationsmethode in explizite Substitutionsfunktionen transformiert werden können. Anhand zentraler Beispiele werden sowohl die Tragfähigkeit dieses Konzepts als auch seine Grenzen aufgezeigt. Es verbleibt als philosophisches Problem, den Begriff der Explikationsmethode soweit einzuschränken, dass nur noch erwünschte Funktionen als implizite Substitutionen herausgestellt werden.

1 Introduction

The central aim of these investigations is to provide a good formal theory of the informally given notions of *occurrences* and *substitutions* representing adequately our informal intuitions about both notions. Consequently, these investigations have a dual character combining a philosophical attitude with a mathematical exploration.

On the one hand, the guiding motivation of these investigations is the conceptual and, therefore, philosophical task of providing a formal representation of apparently clear concepts together with a consideration, whether these formally defined notions correspond with our intuitions.

On the other hand, it is not sufficient to provide only some simple formal definitions; in order to understand the introduced formal notions, they and their properties have to be investigated. The latter means that we have to explore the introduced notions mathematically.

1.1 A Philosophical Analysis

In order to obtain a good intuition about the central notion of these investigations, we provide a brief (philosophical) introduction.

1.1.1 The Notion of Occurrences

In philosophy, the notion of occurrences is usually considered in the context of the distinction between types and tokens.[1]

Paradigmatic Example: We illustrate the informal notion of occurrences on the base of a quote by Schiller:[2]

> Es ist der Geist, der sich den Körper baut.

Counting the words in the quoted sentence may result in two different numbers: the sentence consists of eight *different words*, but we may count nine *occurrences of words*, as the word "der" occurs twice in that sentence.[3]

[1] A brief survey of the philosophical debate about occurrences (in the context of the distinction of types and token) is given by Wetzel in [35] and [36]. Also Quine, who considers in his philosophical lexicon "Quiddities" [26] occurrences from a philosophical point of view, does not treat them separately, but in the entry "Types versus Tokens".

[2] The quote is taken from Schiller's "Wallenstein" and found as an inscription on the Adolphus Busch Hall at Harvard University, Cambridge, Massachusetts. The text of the inscription is considered by Quine in his analysis in Quiddities [26].

[3] Here, we are not interested in grammatical subtleties: even if there is reason to understand the occurrences of "der" in the example sentence as occurrences of two different,

Basic Analysis: Analysing this natural language example, we identify the following three aspects (already mentioned by Wetzel)[4] determining an occurrence:

1. *context:* An occurrence is always an occurrence *inside a (broader) context*, which is a syntactic entity. The word "der" occurs, in our example, inside the quoted *sentence*.

2. *shape:* An occurrence is always an occurrence *of a syntactic entity*, which we call the *shape* of that occurrence. In our example, both occurrences of "der" are occurrences of the *word* "der".

   Other occurrences in the same context (in the same sentence) have different shapes; we find, for example, an occurrence of the word "Geist" in the quoted sentence.

3. *position:* Two different occurrences of the same shape in the same context can be distinguished by their *positions*. In our example, one occurrence of the word "der" is to the left of the other occurrence of that word; both occurrences can be distinguished, as they have different positions.

The first two aspects are, usually, well-known syntactic entities; the crucial aspect of a theory of occurrences is to find a good representation of the third aspect, the position of an occurrence.

Dependence between the Aspects: As a consequence of the third aspect, context and shape of an occurrence cannot determine, in general, the position of an occurrence.

Unfortunately, it is not so clear, whether context and position together should determine the shape or shape and position together the context. The survey of the literature (given below) yields a great variety of approaches to the notion of occurrences and, in particular, to the formal representation of their positions. Some of these approaches satisfy that context and position (or shape and position, respectively) determine the shape (the context), others do not.

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but homonymous words, we count (for the sake of the argument) both occurrences as occurrences of the *same* word (having possibly different meanings).

[4]Wetzel writes: "The notion of *an occurrence of* x *in* y involves not only x and y, but also how x is situated in y." Cf. [36, §8]; emphasis by Wetzel.

As context and shape are syntactic entities, they are usually understood independently of their positions. This means that the considered determinations between the different aspects depend only on the choice, how to represent the position of an occurrence.

In any case: if the shape (the context, respectively) is not determined by the other two aspects, then it becomes necessary to define the formal representatives of occurrences in a way that the shape (the context, respectively) is explicitly represented. Otherwise, we would be in the uncomfortable situation that the same formal object is intended to represent different informally given occurrences. Consequently, such an approach has to be considered as inaccurate.[5]

A central advantage of our own approach to occurrences (as discussed below) is that even both determinations hold: context and position together determine the shape of an occurrence as well as shape and position together determine the context.

Types and Tokens: A common misconception of occurrences is to interpret them as tokens in contrast to types. But the philosophical distinction between types and tokens is independent of the notion of occurrences.[6]

It is possible to understand the example sentence as a type (as the abstract idea of this sentence) or as a token (as the symbols actually written on the page). Independently of the ontological status of the example sentence, we find two distinct occurrences of the word "der". In the first interpretation, we have to understand them as occurrences of abstract types (namely as occurrences of the abstract word "der" in the abstract example sentence), in the second case as occurrences of concrete tokens (namely as occurrences of concrete symbols on the page forming the word "der" inside the concrete example sentence). Quine provides a distinct analysis of this situation:[7]

> Tokens occur in tokens, types in types.

A more detailed discussion of the philosophical problem of types and tokens as well as that of occurrences is interesting, but beyond the needs of our investigations.

[5] In fact, the reasonable, but inaccurate approaches to occurrences found in the literature have to be rejected for exactly this reason.

[6] Cf. also Quine [26] or Wetzel [36, §1.2].

[7] Cf. Quine [26, p. 218].

1.1.2 The Notion of Substitutions

We complement our philosophical analysis of the notion of occurrences by a brief analysis of the (informal) concept of substitutions.

Traditional Associations: In the literature, the notion of a substitution is usually associated with the idea of a (recursively definable) function and distinguished from the more general concept of replacement (which is described in terms of an arbitrary relation). Another idea related with substitutions is that of an (informal) process.

The distinction between substitutions as a process and as a function is emphasised by Negri and von Plato in their logic textbook "Structural Proof Theory" [22]. They understand a substitution as a performative act (without explaining what it means that a substitution is performed) and contrast this with the result of a substitution. This distinction is reflected in their notation: $[s/x]$ denotes the performative act of replacing a variable x by a term s, while $t(s/x)$ refers to the result of that substitution for a term t. Nevertheless, this distinction does not seem to have much relevance: the substitution $[s/x]$ is defined via the usual recursive clauses for $t(s/x)$.[8]

As another example, we mention Leitsch's textbook "The Resolution Calculus" [21]. There, a substitution is explicitly defined as a specific function from the set of variables into the set of terms.[9] Nevertheless, when discussing the composition of substitutions, he writes:[10]

> [...] the action of substitution canonically extends to all terms; there they represent specific endomorphisms of the term algebra.

Besides some obscurities,[11] the association between substitutions and actions is clearly present in the quote.

Substitution as Process: Similarly to Negri and von Plato, we understand a substitution as a process.[12] As in the case of occurrences, we can identify some aspects determining this concept:

1. *context:* A substitution takes place *in* a syntactic entity, which is called the *context* of a substitution.

[8] Cf. Negri and von Plato [22, p. 62f.].

[9] Cf. Leitsch [21, p. 10].

[10] Cf. Leitsch [21, p. 60].

[11] We do not understand, what an "action of substitution" is (according to Leitsch and in contrast to a substitution). The word "they" seems to mean "the extended substitutions", but the grammatical reference is not so clear.

[12] But in contrast to them, we provide a mathematical object, the so called *(formal) substitutions*, representing the informal process of a substitution.

2. *affected entities:* A substitution *affects* some syntactic entities in the context, namely those syntactic entities in the context, which are intended to be replaced. These affected syntactic entities correspond to the shape of an occurrence.

3. *position:* The affected (occurrences of) terms are determined by their position in the context, exactly as in the case of occurrences.

4. *inserted entities:* In a substitution, the affected terms are eliminated in the context at their position. Then, some (new) syntactic entities are *inserted* at the same position.

5. *result:* Finally, a substitution results (as described) in a syntactic entity, which we call the *result* of that substitution.

The three aspects context, affected entities and position as well as the three aspects result, inserted entities and position, respectively, are occurrence as discussed above; in particular, both occurrences have a common position. Under this perspective, the notion of a substitution is a quite natural generalisation of the notion of an occurrence.

Our Terminology: In the informal parts of our investigations, we do not follow the traditional distinction between substitution and replacement. The informal expressions "substitute" and "replace" are used almost synonymously; as long as we do not refer to formally defined *substitutions*, the informal expressions "substitution" and "replacement" are also used synonymously.

Additionally, we introduce the notion of an *explicit substitution function*, which is based on our formal notion of substitutions, but different from them (and also different from the substitution functions defined as usual); more precisely, the explicit substitution functions are defined as sets of substitutions and, therefore, mapping ordered pairs of occurrences and suitable sequences of syntactic entities to the result of the respective formal substitution. The relationship between the explicit and the traditional substitution functions is established by the so called concept of an *explication method*.[13]

1.2 Relevance of Occurrences

We consider the relevance of the notion of occurrences in the formal sciences.

[13]See the subsection about *occurrences in these investigations* for some more details about our concrete definition of substitutions and of (explicit) substitution functions.

Usual Attitude: The (inductive) definitions of the syntactic entities available in a formal language are quite precise; but usually, there is paid little attention to the notion of occurrences. One reason for this ignorance could be that there are a great number of questions concerning occurrences, which can be answered purely on the basis of the inductive definitions of the involved syntactic entities.

Solvable Problems: We provide some examples of problems solvable on the basis of the inductive definition of the involved syntactic entities:

1. *replacing all occurrences:* A prominent problem related to occurrences is the replacement of *all* (free) occurrences of a variable (in a term or a formula) by an arbitrary term. A *substitution function* solving this problem is easily defined recursively.

2. *counting all occurrences:* Another problem related with the notion of occurrences is to count the number of *all* occurrences of a given term in a term. Again, such a *multiplicity function* is defined easily along the inductive structure of the second term.

Common aspect of these solvable problems is that they deal with *all* occurrences and not with an *arbitrary* occurrence. As a consequence, we have access to the intended occurrences via the inductive structure of the underlying syntactic entities.

Limitations: Nevertheless, there are problems not solvable on this inductive base. Situations, in which we have to deal with such problems, can be recognised by the use of some ad hoc solutions to mark the positions of the intended occurrences. Such ad hoc solutions are, for example: pointing at the intended occurrence (in talks), using unique labels to identify the intended occurrences or highlighting them by using different colours or by underlining them.

Simple and Hard Problems: The need to use ad hoc methods to identify intended occurrences motivates the following (informal) distinction:

- *simple problems:* Simple problems are solvable only via the inductive structure of the underlying syntactic entities without a reference to the position of the occurrences under discussion.

- *hard problems:* Hard problems are not solvable on the base of the inductive definition of the syntactic entities under discussion, but need an explicit reference to the position of the intended occurrences.

It is worth mentioning that most of the hard problems, which we discuss in these investigations, are solvable with reference to the position, without an explicit reference to the full notion of occurrences.

Example Problem: We illustrate the concept of hard problems (with regard to the notion of occurrences) by the following example problem formulated in the formal language $\mathfrak{L}_{\mathrm{PA}}$ of arithmetics.

1. *intended occurrences:* As common context of the occurrences under discussion, we choose the following standard term of $\mathfrak{L}_{\mathrm{PA}}$:

   $$t \;\eqsim\; (0+1)+(0+1)$$

   There are two single occurrences of the term 0 in t as well as there are two single occurrences of the term $0+1$ in t. Let us choose now one occurrence of 0 and another occurrence $0+1$ in t.

2. *hard question:* A natural question regarding the chosen occurrences is, whether the intended occurrence of 0 lies within the intended occurrence of $0+1$ or not. If we have chosen both times the left occurrence (the right ones, respectively), then the answer is "yes"; otherwise, the answer is "no".

   Obviously, the answer to the question is independent of the inductive structure of the involved terms, as we are discussing the same terms, whether the answer is "yes" or "no". Consequently, the question under discussion has to be counted as hard.

The Discharge Function: Another interesting problem related with the notion of occurrences is the definition of the *discharge function* for derivations in Gentzen's calculus of Natural Deduction.[14] This function marks some previously undischarged assumptions as discharged.

From the perspective of a theory of occurrences, it seems suitable to represent inference steps by occurrences of arbitrary derivations and assumptions by occurrences of atomic derivations;[15] in particular, it is reasonable to define assumptions as multiple occurrences, as we are entitled by the inference

[14]The calculus of Natural Deduction is introduced by Gentzen in his "Untersuchungen über das logische Schließen" [13]; Peter Schroeder-Heister mentioned in a discussion that the term "discharge function" is due to Prawitz [23].

[15]Motivated by the analogous terminology for terms and formulae, we use the expression "atomic derivation" to denote derivations, which are generated in one step (having no proper subderivations); this means that atomic derivations are syntactically equal to formulae.

rules to discharge arbitrary many assumptions of the right form.[16] In other words: the discharge function maps occurrences to multiple occurrences.

In order to define such a discharge function formally (and, therefore, to define the notion of a derivation adequately), it is necessary to introduce first a suitable theory of occurrences (capable of representing multiple occurrences).[17] As a consequence, we have to understand the adequate introduction of derivations in the calculus of Natural Deduction as a hard problem in the sense discussed above.

Relevance: The hard problems discussed so far seem to be intuitively clear and it seems to be without great impact to solve them formally. Nevertheless, even simple problems deserve to be treated adequately, in particular, in a foundational discipline such as logic. Furthermore, an elaborate formal treatment of occurrences allows to address formal problems with a greater precision; in the conclusion of these investigations, we sketch some more problems benefitting from an adequate treatment of occurrences.

1.3 Occurrences in these Investigations

We provide a brief survey of the central formal notions of these investigations, in particular of the formal occurrences and their generalisations.[18]

Principal Approach: In contrast to the example sentence formulated in a natural language, we are interested in occurrences in and of syntactic entities of formal languages, as terms, formulae and derivations. A formal *occurrence* will be defined as an ordered triple of the following kind:

$$\langle \texttt{context}, \texttt{shape}, \texttt{position} \rangle$$

Context and shape are a meaningful combination of the usual standard syntactic entities (for example, the terms, the formulae or the derivations of a

[16] Prawitz [23, pp. 25-29] suggests to define assumptions as specific formula occurrences in a derivation. A detailed analysis of derivations and of occurrences in derivations suggests not to follow Prawitz's intuitions. Otherwise, we would have to distinguish formula occurrences at the leaves of formula trees from other formula occurrences. The latter can be achieved by checking, whether the formula occurrences are also subderivations. It seems reasonable to avoid this complication and to define assumptions directly as occurrences of atomic subderivations.

[17] As we do not discuss in these investigations occurrences in derivations, this problem is not addressed here. Consequently, we mention only in the section about future work some more details.

[18] A more detailed survey of the concepts discussed in this section and of our results is given in section §17 *Conclusion: Results*.

formal language). The position will be given by so called *nominal forms*, as introduced by Schütte [29], which are generalisations of the standard syntactic entities capable of representing positions.

Informal Notation of Occurrences: Occasionally, if we have to refer to informally given occurrences, we use the method of highlighting (by underlining) to indicate the intended occurrences. This method is needed in these investigations, in particular, for the illustration of the correspondence between informally given occurrences and their formal representatives.[19]

Generalisations of Occurrences: The following generalisations of the notion of occurrences are considered in these investigations:

1. *single occurrence:* The most intuitive examples of occurrences are the *single* occurrences; these are occurrences, in which the shape of an occurrence is intended exactly once in its context.

   In the term $t \doteq 0 + (0 + 1)$, there are exactly two single occurrences of the term 0:

   $$\underline{0} + (0 + 1) \quad ; \quad 0 + (\underline{0} + 1)$$

2. *multiple occurrence:* A slight generalisation of the single occurrences is the notion of a *multiple* occurrence; these are occurrences, in which the shape is simultaneously intended more than once.

   Recalling the example above, we identify a third occurrence in t having the shape 0, namely the following multiple occurrence:

   $$\underline{0} + (\underline{0} + 1)$$

   Observe that we have to underline both single occurrences of 0 separately to indicate the multiple occurrence of 0, as the symbols "+" and "(" are not contained in that occurrence.

   As this notion of occurrences is the most general which is equipped with a trivial identity, it is chosen in these investigations as the *standard* notion; single occurrences are subsumed under this concept.

3. *multi-shape occurrence:* Another quite natural generalisation of the concept of occurrences is to drop the limitation that only a single shape can be intended; in such a *multi-shape* occurrence, the shape of an occurrence is not given as a simple syntactic entity, but as a finite

[19] This is another example of the inevitable gap between informally given concepts and their formalisations, as discussed by Robinson [27] with respect to proofs.

sequence of such entities. More formally, a multi-shape occurrence will be given as the following kind of ordered triple:

$$\langle \texttt{context}, \texttt{sequence of shapes}, \texttt{position} \rangle$$

We provide an example of such a multi-shape occurrence, again in the example term $t \mathrel{\widehat{=}} 0 + (0 + 1)$:

$$\underline{0} + (\underline{0} + \underline{1})$$

Here, we already observe the limitations of the method of highlighting: the sequence of shapes and, therefore, the formal representative of the indicated occurrence is not uniquely determined. The sequence of shapes (and also the position) depends on the way, *how* we intend the intended occurrence. We provide some suitable sequences and their consequence on the way of intending:

(a) $\langle 0, 1 \rangle \mathbin{/\!\!/} \langle 1, 0 \rangle$*:* The indicated multi-shape occurrence subsumes a multiple occurrence of 0 and a single occurrence of 1. Depending on the chosen sequence of shapes, the subsumed occurrences are indicated from the left to the right or vice versa.

(b) $\langle 0, 0, 1 \rangle$*:* In contrast to both sequences above, a sequence containing three entries indicates that three single occurrences are subsumed. The latter means, in particular, that both occurrences of 0 are indicated separately (and not together, as in the example above).

These observations motivate the identification of the formal multi-shape occurrences representing the *same* informal occurrence (differing only in the way of marking the intended positions) via a suitable equivalence relation.

4. *substitution:* The last generalisation of the notion of occurrences discussed in these investigations are the *(formal) substitutions*, which are the formal representatives of the process of a substitution (as discussed above). A substitution is introduced as a generalisation of the multi-shape occurrences and is given by the following kind of quintuple:

$$\langle \texttt{context}, \texttt{sequence of affected terms}, \texttt{position}, \texttt{sequence of inserted terms}, \texttt{result} \rangle$$

The first three entries are a multi-shape occurrence as well as the last three entries (in inverse order). Additionally, it is demanded that both sequences have the same length.

For convenience, we also introduce so called *simplified* substitutions, which are the analogous generalisation of our standard occurrences. In other words, in a simplified substitution, affected and inserted shapes are simple syntactic entities.

Presupposing that the intended terms are intended in their natural order, we can illustrate informally a substitution by providing both multi-shape occurrences constituting the substitution:

$$\underline{0} + (\underline{0} + \underline{1}) \blacktriangleright \underline{1} + (\underline{2} + \underline{3})$$

This illustration can be read as follows: in the context $0 + (0 + 1)$ the substitution affects the sequence $\langle 0, 0, 1\rangle$ of terms at the indicated positions. These terms are replaced by the terms given in the sequence $\langle 1, 2, 3\rangle$, which results in the term $1 + (2 + 3)$.

As in the case of multi-shape occurrences and for similar reasons, we have to introduce a non-trivial identity relation for substitutions.

Substitution Functions: Substitution functions (according to our terminology) are closely related with formal substitutions. We provide a sketch of their introduction.

1. *explicit substitution functions:* We can understand a quintuple as an ordered pair of a quadruple and of the last entry of that quintuple. Recalling the set theoretical definition of a function as a specific set of ordered pairs, it is canonical to understand a set of substitutions as a function. Such a function is what we call an *explicit substitution function.*

   The argument of an explicit substitution function can be understood as an ordered pair of a multi-shape occurrence (the first three entries) and of a suitable sequence of syntactic entities. The result of an application of an explicit substitution function on a multi-shape occurrence and on a suitable sequence is, indeed, the result of the respective substitution.

2. *traditional substitution functions:* The traditional (recursively defined) substitution functions are described in these investigations as *implicit substitution functions.* These are functions such that there is an implicitly given method, the *explication method*, transforming the function into an explicit substitution function.

Types of Occurrences: These investigations focus on a simple case of occurrences, namely on occurrences of terms in terms of a first order formal language. Context and shape of such occurrences are standard terms (or sequences of them) of a first order language, the position is given by *nominal terms*, which is the mentioned generalisation with respect to standard terms.

The formal theory of occurrences and substitutions of terms in terms is sufficiently general to be understood as a blueprint for the (basic) treatment of various other kinds of occurrences.

2 Occurrences in the Literature

We provide a brief survey of the development of the notion of occurrences as found in the literature with special focus on the logic literature, but also including some examples from philosophy and computer science.

2.1 Unreflected Use of Occurrences

It is not surprising that in the beginning of modern logic, occurrences and substitutions are treated naively: the focus was on the development of the formal language itself.

Frege: As an example of such a naive treatment of occurrences and substitutions, we mention Frege's discussion of bounded substitution in his "Begriffsschrift":[20]

> Es ist natürlich gestattet, einen deutschen Buchstaben überall in seinem Gebiete durch einen bestimmten anderen zu ersetzen, wenn nur an Stellen, wo vorher verschiedene Buchstaben standen, auch nachher verschiedene stehen.

Frege permits here the uniform replacement of a bound variable (notated with German letters and referred to by "deutsche Buchstaben") by another variable everywhere inside the scope of the quantifier binding this variable. Frege formulates a side condition: the variables have to be different in the result of a substitution *at all places* (in German,"an allen Stellen"), where they have been different before substitution.

The side condition is non-trivial: we deal with occurrences of variables in formulae, and we have to relate such occurrences in different formulae, namely in the formulae before and after substitution. Frege provides neither an explanation nor a critical consideration of the concept of position (the places) nor a definition of a substitution function specifying formally the intended substitution; these meta-lingual concepts are used naively as primitive concepts.

Hilbert and Ackermann: Besides technical improvements, we can still observe such an unreflected treatment of occurrences and substitutions in the logic textbook "Grundzüge der theoretischen Logik" [16] by Hilbert and Ackermann.

[20] Cf. Frege [7, p. 21].

When discussing the substitution of variables, for example, they formulate the demand of replacing uniformly the variables via a reference to *all places*, where they occur. In their own words:[21]

> [...] aber immer so, daß die gleiche Variable an allen Stellen, an denen sie vorkommt, auch immer in gleicher Weise ersetzt wird [...]

A priori, substitution is not understood here as a function, but as an arbitrary replacement, as the possibility of a non-uniform replacement is (implicitly) considered. Only by the demand of uniformity substitution becomes a function. The formulation of this demand is necessary, as the concept of substitution is used here naively and without a (recursive) definition.[22] Also the concept of "all places of occurrence" is used naively, without a formal explication.

We have to mention a subtle improvement with regard to Frege: while Frege refers in his discussion to German letters (which is an accentuation chosen by Frege to denote variables), Hilbert and Ackerman refer to the variables themselves (which are a type of symbols having a special role).

We summarise: Hilbert and Ackermann deal with the concept of occurrences and their positions, but these concepts are naively understood as primitive notions of the metalanguage. They are neither aware of the non-triviality of these phenomena nor providing formal representatives.

2.2 Awareness of Occurrences

The situation slightly improves, for example, with Gentzen and Prawitz. They are aware of the distinction between syntactic entities and their occurrences, but their explanations and definitions are only rephrasing our informal intuitions; no (mathematical) object is introduced to represent formally occurrences or, at least, their positions.

Gentzen: In his "Untersuchungen über das logische Schließen"[23], Gentzen introduces a number of terms intended to denote different types of occurrences: the *S-formulae* are occurrences of formulae in a sequent (of the Sequent Calculus), the *D-formulae* occurrences of formulae in a derivation of

[21] Cf. Hilbert and Ackermann [16, p. 75]

[22] If substitution would be defined as a function, as usual, then the uniformity would be a provable property of the function.

[23] All quotes in this paragraph are taken from the English translation [14] of Gentzen's paper by Szabo; due to some subtle differences, it is worth to pay additional attention to the German original [13]. Subsequently, we mention some of these subtleties.

Natural Deduction, the *D-sequents* occurrences of sequents in a derivation of the Sequent Calculus and, finally, the *D-S-formulae* occurrences of formulae in a sequent occurring in a derivation (again of the Sequent Calculus). Gentzen abstains explicitly from introducing terminology for occurrences of variables.

The terminology is meant to denote occurrences, but the introduction is apodictic and does not provide any method to deal with occurrences on a formal base. The *D*-formulae, for example, are introduced as follows:[24]

> The formulae which compose a derivation so defined are called *D-formulae* (i.e., derivation formulae). By this we wish to indicate that we are not considering merely the formula as such, but also its position in the derivation.[25]

Instead of providing a formal representative of occurrences or, at least, a formal object representing their positions, Gentzen only rephrases some (good) intuitions. He explains, for example, in the case of *D*-formulae the difference between such occurrences and the underlying syntactic entities as follows:[26]

> Thus by '$\mathfrak{A}$ is the *same* *D*-formula as $\mathfrak{B}$' we mean that $\mathfrak{A}$ and $\mathfrak{B}$ are not only formally identical, but occur also in the same place in the derivation. We shall use the words 'formally identical' to indicate identity of form regardless of place.[27]

[24] Cf. Gentzen [14, p. 73].

[25] Considering *also the position* of a formula in a derivation is the translation of considering the formula *in connection with its position* ("verbunden mit ihrer Stellung", cf. Gentzen [13, p. 181f.]). In the English translation, a consideration of a formula is accompanied by a second consideration, namely by the consideration of the position. In the German original, the consideration of a formula as such is replaced by an alternative consideration of formula and position together. The latter means that Gentzen understands an occurrence as a kind of a composition of shape and position (without providing a formal method of establishing this connection, as, for example, by an ordered pair of shape and position). In particular, Gentzen neglects in his conception of an occurrence the context. The latter is remarkable, as Gentzen's terminology reflects the type of context.

[26] Cf. Gentzen [14, p. 73]; the accentuation in the quote is due to Szabo. Gentzen uses, in the German original [13, p. 182], letter-spacing as accentuation, in contrast to an accentuation by italics used at other places.

[27] There are two identical occurrences of the phrase "formally identical" in the English quote. This is different in the German original. The first occurrence of "formally identical" is the translation of an informal description of formal identity (in our terminology, of syntactic equality); Gentzen writes "der Form nach gleich", which is, in English, "equal according to the form". The second occurrence of this phrase is the introduction of a terminus technicus, namely of the expression "formally identical", which is in German expressed by "formal gleich". Cf. Gentzen [13, p. 182].

We conclude that Gentzen is aware of the distinction between the syntactic entities and their occurrences, but he only deals with occurrences on the base of (good) intuitions, instead of treating them formally.

Prawitz: Similarly to Gentzen's approach to occurrences, Prawitz intends in "Natural Deduction" [23] to "take for granted the notion of an *occurrence of a formula* or (synonymously) a *formula occurrences* in a formula-tree."[28] Nevertheless, Prawitz explains subsequently the concept of an occurrence by recalling our intuitions; as in Gentzen, no formal object is introduced representing these occurrences or their positions.

Due to this intuitive account to the notion of occurrences, it is not surprising that Prawitz uses the method of highlighting to indicate occurrences. For example, the correspondence between informal argumentations and derivations (of Natural Deduction) is illustrated as follows:[29]

1. *occurrences of sentences:* First, Prawitz provides an informal argumentation and enumerates the sentences (accompanied by corresponding formulae representing the sentences).

   This means that occurrences of sentences in an informal argumentation are marked (and this way identified) by individual labels.

2. *occurrences of formulae:* In the derivation representing the informal argumentation, he labels formula occurrences corresponding with sentences (with their formalisation) in the informal argumentation by the respective labels. In particular, two different occurrences of the same formula are labeled by the label "(4)" to indicate that these specific two occurrences both correspond with the previously marked occurrence of sentence (4) in the informal argumentation.

   This means that the labels in the derivation are not used to identify occurrences, but to relate them with occurrences of sentences in the informal argumentation.

In the course of his investigations, Prawitz reuses, in another example, the labels in the derivation to refer to the marked formula occurrences; this way, the role of the labels in the derivation changes, as they now individuate occurrences.[30]

This example illustrates that rephrasing the intuitive properties of occurrences improves only apparently the situation. Rephrasing does not provide

[28] Cf. Prawitz [23, p. 25]; the accentuation is due to Prawitz.

[29] Cf. Prawitz [23, pp. 18-19].

[30] More precisely, Prawitz distinguishes even between occurrences of the label (4) in the derivation by referring to its left and to its right occurrence; cf. Prawitz [23, p. 52].

any methods for the treatment of occurrences; when actually dealing with occurrences, we have to step back to highlighting.

2.3 Inaccurate Theories of Occurrences

The next step in the evolution of the theories of occurrences is, what we call an inaccurate theory of occurrences: a basic theory of occurrences is provided (together with a mathematical object intended to represent occurrences formally), but this theory does not capture our intuitions about occurrences adequately.

Quine: Such an inaccurate theory of occurrences is proposed by Quine in his textbook "Mathematical Logic".[31] Presupposing that a syntactic entity is a sequence of symbols,[32] Quine suggests to identify "an occurrence of a word or sign in an expression as the initial segment of the expression up to and including that word or sign."[33]

As already observed by Simons,[34] Quine's approach is not sufficient. In order to illustrate this, we contrast our example sentence by Schiller with the following quote taken from Goethe's ballade "Der Erlkönig":

> Es ist der Vater mit seinem Kind.

We can observe two kinds of fallacies:[35]

1. *fallacy of the context:* The leftmost occurrence of the word "der" in the sentence by Goethe is different from the leftmost occurrence of the same word in the sentence by Schiller; nevertheless, their representatives according to Quine are equal, namely the sequence "Es ist der".

2. *fallacy of the shape:* The leftmost occurrence of the word "der" in the sentence by Schiller (or, equivalently, by Goethe) is different of the leftmost occurrence of the letter "r" in the same sentence; nevertheless, their representatives according to Quine are equal, namely the sequence "Es ist der".

[31] Cf. Quine [25, p. 297f.]; in Quiddities [26, pp. 216ff.], Quine recalls his conception of occurrences and provides an informal explanation.

[32] This presupposition is unproblematic; using, for example, a prefix notation, we can easily transform trees into linear sequences of symbols (as long as the underlying language is well-behaved).

[33] Cf. Quine [26, pp. 218f.].

[34] Cf. Simons [30, p. 196].

[35] Both kinds of fallacies are considered by Simons [30] and recalled by Wetzel [35].

Both fallacies have the same source: the context (the shape, respectively) is neither given explicitly in the formal representative of the intended occurrence (the initial segment) nor determined by the formal representative. As a consequence, the same representative (according to Quine's approach) represents different (informally given) occurrences. The latter means that Quine's approach to the notion of occurrences is inaccurate.

There are strategies to avoid the fallacies of Quine's approach:

1. *encoding:* Instead of identifying occurrences with sequences representing the position of the informally given occurrences, we could define the formal representative as a triple containing context, shape and position.

   This is the strategy suggested by Wetzel to improve Quine's approach.[36]

2. *determination:* Choosing a better representation of the position, we can simplify the representation of an occurrence as follows: if the context and the position determine the shape, then we do not need to encode the shape of an occurrence; it is sufficient to define an occurrence using only context and position. (Analogously, if shape and position determine the context.)[37]

   For example, as suggested by Simons:[38] instead of using only an initial segment of the context to represent the position, where the shape begins, we could use additionally a final segment to represent the position where the shape ends. The shape is determined, if we know additionally the context, and the context is determined, if we know additionally the shape. Simons considers to define an occurrence via the context and the two segments representing the position, but for philosophical reasons, he is not satisfied with the result of this improvement.

Improving Quine's approach, as discussed, is sufficient. Nevertheless, the resulting theories would be so called *weak* theories of occurrences (as discussed below).

[36] For some details see our subsequent discussion of Wetzel's account to occurrences.

[37] In fact, our own choice to represent the position of occurrences by nominal forms has the advantage that both context and position determine the shape as well as shape and position determine the context. Nevertheless, we decided to encode all three relevant aspects of an occurrence.

[38] Cf. Simons [30, p. 197].

2.4 Weak Theories of Occurrences

A reasonable theory of occurrences provides a sufficient formal representative of informally given occurrences. Such a theory of occurrences is called *weak*, if the method of representing the position of an occurrence is limited to single occurrences. The representation of the different generalisations of single occurrences (namely the standard occurrences, the multi-shape occurrences and the substitutions) is beyond the capacity of these weak theories. We comment some of these weak theories found in the literature.

Wetzel: We mentioned already Wetzel's attempt of improving Quine's approach to occurrences in her article "What are occurrences of expressions?" [35]. She presupposes (as Quine) that expressions are finite sequences of symbols as, for example, the following two sequences of symbols x_i and y_i:

$$x \;=\; \langle x_0, \ldots x_n \rangle \quad ; \quad y \;=\; \langle y_0, \ldots y_m \rangle$$

In a first step, Wetzel defines, whether "x *starts* in y at i" by the demand that the entries x_k and y_{k+i} have to be equal for all $0 \leq k \leq n$.[39] Then she defines, whether "x occurs in y at least n times" via the condition that the set of starting positions contains at least n elements.[40] More formally:

$$|\{i;\ x \text{ starts in } y \text{ at } i\}| \;\geq\; n$$

Without providing a definition, she presupposes the notion of *the nth occurrence of an expression x in an expression y*. These nth occurrences are her notion of occurrences.[41] Due to philosophical concerns, she finally suggests to "adopt as the nth occurrence of x in y the ordered triple $\langle n, x, y \rangle$."[42]

As Wetzel does not tell us her definition, we have to figure out the missing details. We suggest: a triple $\langle n, x, y \rangle$ is called the *nth occurrence of x in y*, if the following both conditions are satisfied:

1. There is i such that x starts in y at i.
2. The cardinality of the set of starting positions up to i equals to n. More formally:
$$|\{k;\ k \leq i \text{ and } x \text{ starts in } y \text{ at } k\}| \;=\; n$$

[39] Cf. Wetzel [35, p. 219]; accentuation by Wetzel.
[40] Cf. Wetzel [35, p. 219].
[41] Whether occurrences (according to Wetzel) are defined only on the base of the nth occurrences or the nth occurrences themselves, is clarified, when Wetzel [35, p. 217] writes: "Every occurrence of x in y has these parameters [the number n and the expressions x and y], and they uniquely individuate the occurrence."
[42] Cf. Wetzel [35, p.219].

Observe that we do not use the set of all starting positions, as introduced by Wetzel, but subsets of this set. The full set is only needed, if we introduce (as Wetzel) the property that x *occurs in* y *at least* n *many times*.[43]

We analyse Wetzel's (improved) account to occurrences:

1. Wetzel's account of occurrences is capable of representing occurrences, as she encodes the context, the shape and the position of an occurrence. As the representative of the position is only able to mark single positions, she provides a weak theory of occurrences.

2. Neither position and context determine the shape nor position and shape determine the context.

3. We do neither see the advantage of the nth occurrence over an occurrence starting in i nor do we see the advantage of encoding the starting position by natural numbers in contrast to initial segments.

   Presupposing context and shape, we are able to determine the starting position i on the base of the number n as well as the corresponding initial segment of y on the base of the starting position i and vice versa. All of these approaches are equivalent, provided context and shape are encoded.[44]

4. According to our own intuitions, it seems that the property of an occurrence, namely to be the nth occurrence of a given shape in a given context, is not essential to the notion of occurrences, but a contingent fact. As a consequence, we would prefer Quine's (improved) definition, because an initial segment represents more directly the position of an occurrence than the number of preceding occurrences with the same shape.

   The advantage of Quine's approach becomes more visible, if we try to provide the definition of the notion of an occurrence (in contrast to the nth occurrence): an occurrence (according to Wetzel) has to be a triple

[43] Wetzel [35, p. 217] uses this property in an identity criterion for the nth occurrence of x in y; there, she demands besides the usual condition for the identity of ordered triples that the expression x indeed occurs sufficiently often. The latter condition is not superfluous, as an occurrence not satisfying this condition would not be an occurrence at all.

[44] It seems that Wetzel does not recognise, why her approach to occurrences is an improvement on Quine's approach. Instead of using Quine's method of representing a position, she develops her own method and discusses this in detail; but the reason, why her approach to occurrences is better than Quine's is not her way of representing positions, but the fact that she encodes all necessary aspects of an occurrence.

> $\langle \xi, x, y \rangle$, where (besides other conditions) ξ is the number of preceding occurrences of the same context and the same shape. We have to refer in a recursive clause to previously defined occurrences; this is not the case, if ξ is the starting position i or the corresponding initial segment of y, the latter as suggested by Quine.

Van Dalen: Van Dalen provides in his textbook "Logic & Structure" a brief excursus on the notion of occurrences (of formulae in formulae).[45] The distinction between syntactic entities and their occurrences is motivated only by convenience. This is contrasted by his concluding remark that he "will not be overly formal in handling occurrences [...], but it is important that it can be done."[46]

Referring to trees of formulae (a recursively defined representation of formulae in tree form), van Dalen first defines:

> [A]n occurrence of a formula $[\phi]$ in a given formula ψ is a pair (ϕ, k), where k is a node in the [parsing] tree of ψ.

We analyse the situation:

1. *circularity:* The representation of the position of an occurrence by a node k is problematic: a parsing tree is not defined as a set of distinct nodes equipped with a suitable relation, but recursively on the base of subtrees. (The nodes of this tree are labelled with full formulae and not only with the main logical symbol and atomic formulae.) Consequently, a node in such a tree can only be understood as an occurrence. Occurrences are introduced in terms of occurrences. This means that the definition, as stated, is circular.[47]

2. *fallacy of the context:* Van Dalen discusses only occurrences in a given formulae;[48] this way, he avoids trivially the fallacy of the context.

[45] Cf. van Dalen [33, p. 12f.]; all quotes are taken from this excursus.

[46] We agree with van Dalen with respect to the importance of the notion of occurrences; therefore, we reject that occurrences should be understood as a matter of convenience. Having developed an adequate theory of occurrences, we may deal with them informally for convenience reasons.

[47] Due to this circularity, we could call van Dalen's conception of occurrences an *apparent* theory of occurrences.

[48] This is, at least, what he claims first. In the next sentence, where he provides his circular definition, he only speaks about occurrences without a restriction to a given context. Cf. van Dalen [33, p. 12].

Van Dalen himself improves immediately his circular definition of an occurrence by discussing an alternative (and better) account to the representation of the position. A position (a node) in a tree of formula is represented by a sequence of natural numbers encoding the path from the top node of the formula tree to the top node of the intended occurrence: an entry "0" means the left direct subtree, an entry "1" means the right direct subtree.[49]

We communicate some observations:

1. *limited notion:* Even if we assume that the fallacy of the context is avoided trivially by restricting the notion of an occurrence to a previously fixed context: having a concrete definition of the position of an occurrence at hands, we observe that shape and position do not determine the context.

   Therefore, van Dalen discusses only a limited notion of occurrences. Of course, encoding additionally the context would change the situation.

2. *lies-within relation:* The hard problem of deciding, whether an occurrence lies within another occurrence, has a quite natural solution: we have to check only, whether the respective sequences representing the positions of the occurrences are initial segments of each other or not.

The improved version of van Dalen's approach to occurrences captures adequately the concept of single occurrences, but it is limited to this special case. For the latter reason, van Dalen's suggestion of a theory of occurrences (its improved version) is a weak theory.

Huet: Huet provides in his article "Confluent Reductions: Abstract Properties and Applications to Term Rewriting Systems" [18, p. 807] a variation of a weak theory of occurrences based on sequences of natural numbers.[50]

These finite sequences are used as a position of what we call an occurrence, but are called *occurrences* by Huet.[51] As we sketched above, this kind of position is sufficient to define a lies-within relation for occurrences; additionally, Huet introduces the concept of *disjoint* occurrences describing that

[49]This account of representing positions in a tree is easily generalised to the case of trees having finitely, but arbitrary many direct subtrees by using the remaining natural numbers. It seems that even the infinite case can be represented via infinite ordinal numbers.

[50]In particular, this article is worth mentioning, as the proposed treatment of occurrences became standard in the term-rewriting community.

[51]This means that Huet does not distinguish between the concept of a position and of an occurrence.

two single occurrences do not lie within each other.[52]

Based on this abstract notion of occurrences (of positions), Huet defines recursively and in parallel the *set of occurrences* of a standard term (defined, essentially, via the usual inductive clauses), which is the set of suitable positions, and the *subterm* M/u of a standard term M at such an occurrence u (at such a position u). The definition of occurrences is concluded by introducing the way of speaking that a (suitable) sequence u of natural numbers "is *an occurrence of* M/u *in* M"[53] and by defining a special substitution function for single occurrences.

We communicate some observations:

1. Huet does not introduce a formal object, which we would call an occurrence, but only (abstract) positions.

2. As a consequence of his abstract approach, the same occurrence (in Huet's terminology) can have different contexts as well as different shapes. But this principle problem has no impact on Huet's theory of occurrences, as he relates the standard terms with their set of suitable positions and as he considers only occurrences, where the context is (implicitly) clear.

3. Huet sketches an elaborate theory of occurrences (of positions), but this theory is restricted to single occurrences, which means that the theory under discussion is a weak theory.

Avenhaus: As an example of the influence of Huet, we discuss briefly the notion of occurrences as introduced by Avenhaus in his text book "Reduktionssysteme" [2]. As in van Dalen's account of occurrences (and in contrast to Huet), he presupposes a representation of the standard syntactic entities in tree form (so called parsing trees) and defines the position of an occurrence via sequences describing the position of the intended occurrence in the parsing tree.

Avenhaus' approach is an improvement in comparison to van Dalen, as he provides a basic theory of occurrences:[54] he defines recursively the set $\mathcal{O}(t)$ of all positions in a term t (given as the discussed sequences of natural numbers).

[52]This is what we call the independence of occurrences, but restricted to single occurrences. See section 8.2.4 for a brief discussion, why this restricted relation is not sufficient to capture the concept of independence in the general case.

[53]Cf. Huet [18, p. 807].

[54]Cf. Avenhaus [2, pp. 81f.].

Furthermore, he introduces the notation t/p to denote an occurrence in a term t at a position $p \in \mathcal{O}(t)$ (without calling the introduced notion explicitly an occurrence). As context and position determine the shape, this definition of an occurrence is adequate; in particular, he introduces a function mapping an occurrence to its shape.

Nevertheless, the occurrences as introduced by Avenhaus are only single occurrences; therefore, the resulting theory is a weak theory of occurrences.

2.5 Flat Notion of Occurrences

A special case of an inaccurate theory of occurrences is a theory in which only a *flat notion* of occurrences is used. The latter means that the central aspect of an occurrence, its position, is not subsumed in the conception of an occurrence. In other words: the expression "occurrence" belongs to the terminology of such a theory, but this expression refers only to the fact, whether a syntactic entity occurs *somewhere*, without specifying, *where* this syntactic entity occurs.

Leitsch: An example of such a flat notion of occurrences is found in Leitsch [21]. Leitsch defines, *whether* a term (a formula, respectively) *occurs* in a term (in a formula) in terms of that recursive definition, which we would use to define the concept of *being a subterm* of a term (of *being a subformula* of a formula).

Worth mentioning here: this flat notion of an occurrence is, indeed, used by Leitsch to define in the next step the subformula relation: "A is called a *subformula* of B if A occurs in B."[55]

Curry: We mention Curry's textbook "Foundations of Mathematical Logic" [4]. There, Curry develops a general and elaborate approach to languages;[56] in particular, he develops a notion of occurrences, which "corresponds to current practise with syntactical systems, where an occurrence is identified with the initial segment which ends at the last letter of the occurrence."[57] This means that Curry introduces, at most, a weak theory of occurrences.

Nevertheless, this does not prevent Curry from introducing also a flat notion of occurrences. In particular, he uses the term "occurrence" to define (according to the expected recursive clauses), whether a variable occurs in a term or in a formula, respectively.[58]

[55] Cf. Leitsch [21, p. 7].
[56] A detailed analysis of Curry's approach to languages is left for future work.
[57] Cf. Curry [4, p. 102].
[58] Cf. Curry [4, p. 318].

2.6 Implicit Theory of Occurrences

We conclude our survey with a brief comment on Schütte.

Schütte: Schütte provides in "Proof Theory" [29] the nominal forms used in these investigations to represent the positions of occurrences.

More precisely, Schütte provides an inductive definition for two kinds of nominal forms depending on each other: the p-forms and the n-forms. These nominal forms are used to mark the position of special occurrences of subformulae in propositional formulae, namely the *p-parts* and the *n-parts*. As these occurrences determine (by their positions) semantic properties of formulae, Schütte is able to discuss semantic properties of propositional formulae based only on syntactic notions.[59]

Even though the central concepts related to the notion of (a specific kind of) occurrences are already present here, Schütte does not make the notion of an occurrence explicit, nor does he develop a general theory of occurrences; an investigation of such a theory seems to be outside of the focus of his investigations. For this reason, we consider Schütte's treatment of nominal forms as an implicit theory of occurrences.

[59] Cf. Schütte [29, pp. 10ff.].

3 Preliminaries

We provide the mathematical and logical concepts and notations used in these investigations.

3.1 General Mathematics

A basic set theory is presupposed; we mention some details.[60]

3.1.1 Basic Concepts of Set Theory

Basic Notions: $\emptyset$ is the *empty set*. $\in$ is the *membership relation*, $\subset$ the strict *subset relation* (or, equivalently, *set inclusion*). $\cup$ and $\cap$ denote the *union* and the *intersubsection* of two sets, respectively. Occasionally, we use generalised union and intersubsection, as usual in set theory:

$$\bigcup X = \bigcup_{x \in X} x = \{z;\ \exists x \in X :\ z \in x\}$$
$$\bigcap Y = \bigcap_{y \in Y} y = \{z;\ \forall y \in Y :\ z \in y\}$$

Thereby, X and Y have to be sets of sets and Y may not be empty.

$X \backslash Y = \{x \in X;\ x \notin Y\}$ is the *difference* of X and Y, $\mathfrak{p}(X)$ is the *power set* (the set of all subsets) of a set X. An *ordered pair* $\langle X, Y \rangle$ can be defined as $\langle X, Y \rangle = \{\{X\}, \{X, Y\}\}$ according to Kuratowski's suggestion.[61] Furthermore, $X \times Y = \{\langle x, y \rangle;\ x \in X, y \in Y\}$ is the *cartesian product* of two sets X and Y. Ordered pairs and cartesian products can be generalised canonically to arbitrary, but finite many arguments.

Definite Descriptions: If we can prove that exactly one element x contained in a given set X satisfies a description ϕ, then we refer occasionally to this element via its *definite description*:

$$!\, x \in X :\ \phi(x)$$

[60] Such a basic set theory can be developed and justified on the base of the axiomatic set theory ZF formulated in a first of language. Most of the mathematical concepts used in these investigations can be found in any good set theory textbook; we suggest Jech "Set Theory" [19] for a distinct presentation of ZF. A detailed and precise development of the set theoretical concepts used in these investigations are beyond the needs of these investigations.

[61] Cf. Kuratowski [20].

Such definite descriptions are useful, for example, for a distinct formulation of suitable definitions.

Ordinal Numbers: We presuppose *ordinal numbers* (or, equivalently, *ordinals*) as defined by von Neumann [34] (as ϵ-transitive sets well-ordered by $\in$).[62] In particular, Ω denotes the proper class of all ordinals, $\alpha' = \alpha \cup \{\alpha\}$ is the *successor* of α for all ordinals $\alpha \in \Omega$.[63] A non-empty ordinal α is called a limit ordinal, if $\alpha = \bigcup \alpha$. We have the following transfinite induction principle for ordinals: If a property holds for $\emptyset$, and if this property is transferred from α to α' and from all $\beta \in \lambda$ to λ (where α is arbitrary ordinal and λ arbitrary limit number λ), then this property holds for all ordinals α.

Natural Numbers: Natural numbers are understood as finite ordinals. This means that a natural number is identified with the set of its predecessors: $0 = \emptyset$ and $n' = n \cup \{n\} = \{0, \dots n\}$. The set of all natural numbers is denoted by ω, which is the first infinite ordinal number and, in particular, the first limit ordinal.

Addition and multiplication can be defined via their canonical recursive definitions. Both operations can be generalised to arbitrary, but finitely many arguments; additionally, we use infinite sums and products, if almost all arguments are 0 or 1, respectively.

It is convenient that set inclusion and the lesser-than relation coincide with respect to the natural numbers. More formally, for all $n, m \in \omega$: $n \in m$, if and only if $n < m$. Equivalently: $n \in m'$, if and only if $n \leq m$.

The natural numbers and the set of natural numbers itself are, essentially, the only ordinals used in these investigations; they can be provided as the elements of the ordinal $\omega' = \omega \cup \{\omega\}$.

3.1.2 Relations

A relation R is a subset of a cartesian product. In the case of a binary relation $R \subseteq X \times X$ on a set X, we use also infix notation, as usual.

[62] A set is ϵ-transitive, if all of the elements of that set are subsets of that set; the concept of well-ordering is introduced below.

[63] Reinhard Kahle mentioned in a discussion that the notation α' to denote the successor of α was introduced by Dedekind [5, p. 21] with respect to the natural numbers. It is worth mentioning here that Dedekind did not coin this notation especially for natural numbers. He introduced this notation to denote the image $\phi(a)$ of an object a under the application of an arbitrary functions ϕ; only in the discussion of the natural numbers, the use of this notation is related to successor function.

Basic Properties (Binary Relations): We provide the definitions of basic properties of a binary relation R on a set X.

1. *reflexive:* R is *reflexive*, if $\langle x, x\rangle \in R$ for all $x \in X$.
2. *anti-reflexive:* R is *anti-reflexive*, if $\langle x, x\rangle \notin R$ for all $x \in X$.
3. *symmetric:* R is *symmetric*, if $\langle x, y\rangle \in R$ implies $\langle y, x\rangle \in R$ for all $x, y \in X$.
4. *anti-symmetric:* R is *anti-symmetric*, if $\langle x, y\rangle \in R$ and $\langle y, x\rangle$ in R implies that $x = y$ for all $x, y \in R$.

   This means: if R is anti-reflexive, then there are no $x, y \in R$ such that $\langle x, y\rangle \in R$ and $\langle y, x\rangle \in R$.
5. *transitive:* R is *transitive*, if $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$ implies that $\langle x, z\rangle \in R$ for all $x, y, z \in X$.

Closure Operations: The closure of a binary relations R on a set X with respect to a (basic) property is the smallest relation $\hat{R}$ extending R having this property; we can construct such a closure as the intersubsection of all suitable relations. Alternatively, we can define the closure operations with respect to the basic properties as follows:

1. *reflexive closure:* $X_{=} = \{\langle x, x\rangle;\ x \in X\}$ is the *identity relation* for X; the relation $R^{\mathbf{r}} = R \cup X_{=}$ is the *reflexive closure* of R.
2. *symmetric closure:* $R^{-1} = \{\langle y, x\rangle;\ \langle x, y\rangle \in R\}$ is the *inverse* relation of R; the relation $R^{\mathbf{s}} = R \cup R^{-1}$ is the *symmetric closure* of R.
3. *transitive closure:* In contrast to reflexivity and symmetry, the transitive closure cannot be defined directly, but only with the help of an ascending chain of relations:

   $$R_0 \ = \ R \ ; \ \ R_{n'} \ = \ R_n \cup \{\langle x, z\rangle;\ \exists y \in X : \langle x, y\rangle \in R_n,\ \langle y, z\rangle \in R\}$$

   The relation $R^{\mathbf{t}} = \bigcup\{R_n;\ n \in \omega\}$ is the *transitive closure* of R.

Equivalence Relation: Let $R \subseteq X \times X$ be a binary relation on a set X. R is an *equivalence relation* on X, if R is reflexive, symmetric and transitive. The set $[x]_R = \{y \in R;\ \langle x, y\rangle \in R\}$ is called the *equivalence class* of x modulo R for all $x \in X$.

Order Relations: Let $< \subseteq X \times X$ be a binary relation on X.

1. *partial order:* The ordered pair $\langle X, < \rangle$ is called a *partial order*, if $<$ is anti-reflexive and transitive.

   As a consequence, $<$ is anti-symmetric.

2. *linear order:* A partial order $\langle X, < \rangle$ is called a *linear order*, if all pairs of elements of X are pairwise comparable (more formally, if $x = y$ or $x < y$ or $y < x$ for all $x, y \in X$).[64]

3. *well-order:* A linear order $\langle X, < \rangle$ is called a *well-order*, if every non-empty set $\emptyset \neq Y \subseteq X$ contains a least element with respect to R. (A definition of the least element is given below.)

We introduced strict order relations (which means anti-reflexive order relations). Occasionally, non-strict order relations are considered, which are the reflexive version of their strict counterparts. As reflexivity and transitivity do not imply anti-symmetry, we have to demand the latter property axiomatically. Usually, we stick to the traditional notational conventions: if "$<$" denotes a strict order relation, then "$\leq$" its non-strict version.[65]

The following terminology is used to denote different kinds of *extreme elements* $x \in X$ in a partial order $\langle X, < \rangle$.

1. *least⁄⁄greatest element:* x is the *least element* (or the *minimum*) with respect to $<$, if x is smaller than all other elements in the partial order (more formally, if $x = y$ or $x < y$ for all $y \in X$); analogously, x is the *greatest element* (or the *maximum*).

2. *minimal⁄⁄maximal element:* x is a *minimal element* with respect to R, if there is no smaller element in the partial order (more formally, if $y < x$ implies $x = y$ for all $y \in X$); analogously, x is a *maximal element*.

Order Relation Modulo Equivalence Relation: Let $R \subseteq X \times X$ be a binary relation on X and $\equiv \subseteq X \times X$ an equivalence relation on X. If R and $\equiv$ are compatible, then the relation R induces canonically a relation $R_{/\equiv}$ on the set $X_{/\equiv} = \{[x]_{\equiv};\ x \in X\}$ of all equivalence classes with respect to $\equiv$.

[64] This property is also called *trichotomy*.

[65] Occasionally, we consider relations, where "$<$" denotes already the non-strict version.

Thereby, compatibility means that the following condition is satisfied for all $x, x', y, y' \in X$:

$$x \equiv x' \ , \ y \equiv y' \ , \ \langle x, y \rangle \in R \quad \Rightarrow \quad \langle x', y' \rangle \in R$$

In the course of our investigations, we come into the situation that the induced relation $R_{/\equiv}$ has some nice properties, which the relation R does not have, but where we are still interested in dealing with the objects $x \in X$ themselves and not with the respective equivalence classes. In order to deal with such a situation, we introduce the following ways of speaking:

1. *basic properties:* We say that R has a basic property with respect to $\equiv$, if the induced relation $R_{/\equiv}$ has this property.

2. *order:* We say that R is a partial order modulo $\equiv$, if the induced relation $R_{/\equiv}$ is a partial order on the set $X_{/\equiv}$.

3. *extreme elements:* An element $x \in X$ is extreme with respect to $\equiv$ (in one of the senses given above), if the respective equivalence class $[x]_R$ is so in $\langle X_{/\equiv}, R_{/\equiv} \rangle$.

   Observe that least and greatest elements are not determined uniquely, but up to the respective equivalence.

This means that a partial order modulo an equivalence relation is a partial order, if we neglect the differences identified by the equivalence relation.

3.1.3 Functions

A subset $F \subseteq X \times Y$ of a cartesian product of two sets X and Y is called a *function* from X to Y, if the following condition is satisfied for all $x \in X$ and $y, y' \in Y$:

1. *functionality:* If $\langle x, y \rangle \in F$ and $\langle x, y' \rangle \in F$, then $y = y'$.

In this case, we also write $F : X \to Y$. The set $\mathbf{dom}(F) = X$ is the *domain* of F and $\mathbf{codom}(F) = Y$ the *codomain* of F. Furthermore, we define:

1. *preimage:* The set $\mathbf{pre}(F) = \{x \in X; \ \exists y \in Y : \langle x, y \rangle \in F\}$ is the *preimage* of F.

2. *image:* The set $\mathbf{img}(F) = \{y \in Y; \ \exists x \in X : \langle x, y \rangle \in F\}$ is the *image* of F

The function F is *total*, if $\mathbf{dom}(F) = \mathbf{pre}(F)$; in the course of these investigations, we presuppose that all functions are total. For $x \in \mathbf{pre}(F)$: we use the usual term notation for functions and we write $F(x) = y$, if $\langle x, y\rangle \in F$, and $F(x) \neq y$ otherwise. Furthermore, $F_{|X'} = F \cap (X' \times Y)$ is the restriction of F to arguments in X' for subsets $X' \subset X$ of the domain.

The set $\mathbf{Func}(X, Y)$ is the set of all functions $F : X \to Y$. If $X = Y$, then we also write $\mathbf{Func}(X)$. Occasionally, we use the set theoretic notation Y^X to denote alternatively the set $\mathbf{Func}(X, Y)$. If the domain X of F is a cartesian product of length n (this means that $X = Z_0 \times \ldots Z_{n-1}$ for some suitable Z_k), then we say that F is n-ary.

Basic Function Properties: Let $F : X \to Y$ be an arbitrary function. We recall the basic function properties:

1. *injective:* F is *injective*, if for all $x, x' \in X$ the following condition is satisfied: if $F(x) = F(x')$, then $x = x'$.

2. *surjective:* F is *surjective*, if for all $y \in Y$ the following condition is satisfied: there is $x \in X$ such that $F(x) = y$.

3. *bijective:* F is *bijective*, if F is injective and surjective.

In order to discuss the standard function properties, the following notions are useful. Let $y \in Y$ be an element of the codomain of F.

1. *isolated:* y is *isolated* by F, if at most one element x of the domain is mapped by F to y (formally, if $|\{x \in X; F(x) = y\}| \leq 1$).

2. *hit:* y is *hit* by F, if there is an element $x \in X$ of the domain which is mapped by F to y (formally, if there is $x \in X$ such that $F(x) = y$).

The relationship to the standard function properties of functions is obvious.

Special Functions:

1. *permutation:* Let X be arbitrary. A bijective function $F : X \to X$ is called a *permutation* of X; the set of all permutations on X is denoted by $\mathbf{Sym}(X)$.

2. *natural function:* A function $F \in \mathbf{Func}(\omega)$ is called a *natural* function.

Functions on Power Sets: Let $F : X \to Y$ be an arbitrary function. The function F induces canonically a function $\hat{F} : \mathfrak{p}(X) \to \mathfrak{p}(Y)$ on the power set of X as follows:

$$\hat{F}(Z) = \{F(x);\ x \in Z\}$$

The induced function is usually denoted by the same symbol as the original function, but this convention is problematic: if there are members $x \in X$ such that $x \subseteq X$ (which is the case for ordinals), then $F(x)$ can be ambiguous. In order to avoid such ambiguities, we introduce the following notational convention:

- The same function symbol is used for the induced function on the power set as for the original function, but squared brackets for the arguments indicate the induced function.

Investigate, for example, the function $F : \omega \to \omega : x \mapsto 2$ on the set of natural numbers. We easily calculate:

$$F[2] \ = \ F[\{0,1\}] \ = \ \{F(x);\ x \in 2\} \ = \ \{2,2\} \ \neq \ \{0,1\} \ = \ 2 \ = \ F(2)$$

Function Spaces: Let X, Y and Z be arbitrary sets.

1. *identity function:* The function $\mathbf{id}_X : X \to X : x \mapsto x$ is the *identity function* for X.[66]

2. *composition:* If $F : X \to Y$ and $G : Y \to Z$ are two functions such that the codomain of the first is the domain of the second, then the composition $F \circ G$ is defined as follows: $F \circ G : X \to Z : x \mapsto G(F(x))$

Let $\mathfrak{X} \subseteq \mathbf{Func}(X)$ be a set of functions. The ordered triple $\langle \mathfrak{X}, \circ, \mathbf{id}_X \rangle$ is a *function space*, if both $\mathbf{Func}(X)$ is closed under the composition $\circ$ of functions and contains the respective identity function $\mathbf{id}_X$. More formally:

$$F \in \mathfrak{X} \text{ and } G \in \mathfrak{X} \quad \Rightarrow \quad F \circ G \in \mathfrak{X} \quad ; \quad \mathbf{id}_X \in \mathfrak{X}$$

The symmetric group $\langle \mathbf{Sym}(X), \circ, \mathbf{id}_X \rangle$ is a function space; in particular, it is also an algebraic group. The latter means that the group axioms are satisfied: the composition is associative, there is a neutral element, the identity function, contained in the set of function and there is an inverse function for all functions in the set of functions.

Sequences: A function $\mathfrak{x} : \alpha \to X$ from an ordinal α into a set X is called a *sequence* with entries in X. The ordinal $\alpha = \mathbf{lng}(\mathfrak{x})$ is the *length* of the sequence F.

[66] Observe that $\mathbf{id}_X = X_=$.

1. *sets of sequences:* $X^\alpha = \mathbf{Func}(\alpha, X)$ is the set of all sequences $\mathfrak{x}$ in X of length α. Additionally, we introduce the following notation:

$$X^{<\alpha} = \bigcup_{\beta<\alpha} X^\beta \quad ; \quad X^{\leq\alpha} = \bigcup_{\beta\leq\alpha} X^\beta$$

2. *notation:* We use the usual notation of sequences, which means that sequences are notated analogously to ordered pairs. In particular, for $\alpha \in \Omega$:

$$\mathfrak{x} = \langle x_\beta;\ \beta \in \alpha\rangle$$

   Thereby, $x_\beta \in X$ for all $\beta \in \alpha$. Ordinals $\beta \in \alpha$ are also called *positions* in the sequence $\mathfrak{x}$ and x_β is the entry of $\mathfrak{x}$ at the position β.

3. *empty sequence:* The empty sequence is denoted by ϵ.

Sequences versus Tuples: There is a small caveat: a finite sequence $\mathfrak{x} = \langle x_0, \dots x_n\rangle$ of length n' and the corresponding ordered n'-tuple $\langle x_0, \dots x_n\rangle$ are different mathematical objects (for $0 \neq n \in \omega$). Nevertheless, as there is a bijective correspondence between both kinds of entities, we identify both without any further comment. Analogously, we identify a sequence of length 1 (an ordered 1-tuple) with its single entry.

Chains: Let $\langle X, <\rangle$ be a partial order, $\alpha \in \omega'$ an ordinal number and $\mathfrak{x} = \langle x_\beta;\ \beta \in \alpha\rangle \in X^\alpha$ a sequence of length α.

1. *ascending chain:* The sequence $\mathfrak{x}$ is an *ascending chain*, if $x_\beta \leq x_{\beta'}$ for all β such that $\beta' \in \alpha$; the sequence is a *proper* ascending chain, if even $x_\beta < x_{\beta'}$ for all β such that $\beta' \in \alpha$.

2. *stationary:* Let $\alpha = \omega$ and $\mathfrak{x}$ an ascending chain. $\mathfrak{x}$ is stationary, if there is $k \in \omega$ such that $x_k = x_l$ for all $l \in \omega$ such that $k \leq l$.

Analogous terminology is presupposed for descending chains.

3.1.4 Axiom of Choice

The following statement is called the *Axiom of Choice* (AC):

- There is a choice function $\mathcal{F} : \mathcal{X} \to \bigcup\mathcal{X}$ on $\mathcal{X}$ for every set $\mathcal{X}$ of sets not containing the empty set; thereby, a choice function on $\mathcal{X}$ satisfies the condition $\mathcal{F}(X) \in X$ for all $X \in \mathcal{X}$.

The Axiom of Choice is independent of the usual axioms of set theory; the Axiom of Choice guarantees the existence of mathematical objects without an explicit construction. For these reasons, the Axiom of Choice is of interest for the philosophy of mathematics. It is good mathematical practise to announce explicitly, if this axiom is used in a proof.

There is a great number of interesting statements (related with various fields of mathematics) equivalent to the Axiom of Choice. In the course of these investigations, we apply the Enumeration Theorem (ET) which is one of the equivalences of the Axiom of Choice:

- There is an ordinal $\alpha \in \Omega$ and a bijection $F : \alpha \to X$ for every set X.

In other words, every set X can be enumerated by a suitable ordinal α. The advantage of the Enumeration Theorem (in contrast to the Axiom of Choice) is that it allows proofs by transfinite induction along the well-order of α for the enumerated set X.

3.2 Formal Languages

We provide our conception of a formal first order language. This introduction is guided by the needs of our investigations; some concepts contained in a complete introduction are omitted without any comment.

This introduction can be seen as a guideline to the definitions of the analogous concepts formulated with respect to the generalisations of the standard syntactic entities, which are discussed in the course of these investigations.

Basic Terminology: We mention some basic terminology.

1. *syntactic entity:* The expression "*syntactic entity*" is used to denote all kinds of objects defined in a (formal) language, as, for example, the symbols of an alphabet, the terms and formulae of the respective language, the derivations in a calculus etc.

2. *syntactic equality:* The symbol "$\doteq$" denotes in the metalanguage the syntactic equality between two syntactic entities; this symbol-by-symbol equality is the strictest possible (as long as we do not distinguish between occurrences or tokens of the same syntactic entities).

3. *standard entity:* We introduce in the subsequent investigations various generalisations of the syntactic entities usually defined in formal languages; to distinguish the usually defined syntactic entities from their generalisations, we call the former kind of entities also "*standard*". For examples, the terns and formulae of a first order language are also called standard terms and standard formulae.

3.2.1 Principle Approach to Formal Languages

We discuss our principle approach to formal languages.

Principle Conception of Languages:

1. *associated sets:* In our understanding, a formal language $\mathfrak{L}$ is different from the various sets of syntactic entities associated with that language. In particular, a formal language is different from its set of formulae, but also different from its alphabet and of its set of terms. The advantage of this philosophical position is the possibility to associate new sets to a given language, in particular, to associate sets containing the generalisations of the standard syntactic entities (which we intend to introduce in our investigations), and still to speak about the same language.

2. *identification:* It seems reasonable to identify a formal language $\mathfrak{L}$ with an extended *signature*, in which not only the non-logical symbols are codified (together with their arities), but all of the available symbols (together with their use).[67]

   Nevertheless, a detailed development of a philosophically justified notion of a formal language is beyond the needs of our investigations. For this reason, we abstain here from the introduction of a mathematical object representing formal languages, and use this expression informally, and according to our intuitions.

3. *signature:* For the purposes of our investigations, we are also not interested in the exact conception of signatures determining the available non logical symbols together with their arities. The underlying formal mechanisms are not relevant; it is sufficient to know that this determination is *somehow* done. For this reason, we also abstain here from a definition of signatures.

[67] For the non-logical symbols, it seems sufficient to codify their arities. But there are more factors determining the use of symbols: for example, it makes a difference, whether we allow nested quantification of the same variable, whether we allow the quantification of a variable which does not occur in the scope of the quantifier, etc. If we intend to distinguish languages also with respect to such differences, we would have to codify all of the generation rules in the extended signature. On the other hand, there may be reasons to subsume different generation rules under the same language: at least, our generalisations of the standard entities are generated to different and new rules, and we still would like to associate them to the same language as the underlying entities. Also, one may argue that it makes no difference, which exact symbol is used in a language; for example, we may identify languages of group theory using "$+$", "$\cdot$", or "$\circ$" as symbol for the binary group operation, we may identify languages using different sets of of symbols as the same sort of variables, etc.

Specific Languages:

1. *logical symbols:* To keep the complexity of our investigations as low as possible, we prefer formal languages with few logical symbols. Usually, we presuppose the falsum ($\bot$), the implication ($\to$), the universal quantifier ($\forall$), and the equality symbol ($=$).[68] The negation ($\neg$) and the biimplication ($\leftrightarrow$) are understood as the usual abbreviations.

   Our investigations are easily carried over to languages, in which negation and biimplication are proper symbols or in which other logical symbols are available, as, for example, the conjunction ($\wedge$), the disjunction ($\vee$), or the existential quantifier ($\exists$).

2. *paradigmatic language:* We discuss in the theoretical parts of our investigations a *paradigmatic* formal language $\mathfrak{L}$. In such a paradigmatic language, we presuppose *some* constant symbols c, *some* n'-ary function symbols f, and *some* m-ary relation symbols R, but without specifying the number of these symbols or their arities.[69] Our investigations are easily carried over to concrete formal languages.

3. *languages of arithmetics:* In most of our examples, we use a first order language $\mathfrak{L}_{\mathrm{PA}}$ of arithmetics. Such a language consists, at least, of a constant symbol 0, a unary function symbol S for the successor function, and two binary function symbols $+$ and $\cdot$ for addition and multiplication, respectively. If necessary, we may extend the alphabet of $\mathfrak{L}_{\mathrm{PA}}$ by constant symbols for every natural number $n \in \omega$, and some relations symbols as $<$ for the smaller-than relation.

4. *other languages:* Occasionally, it is convenient to discuss other formal languages; their non-logical symbols (and the arities of these symbols) will be obvious from the context.

3.2.2 Standard Syntactic Entities

We introduce the standard syntactic entities of a formal language $\mathfrak{L}$, as far as needed in these investigations. As we focus in these investigations on term occurrences in terms, it is sufficient to introduce the alphabet and the terms. The introduction of formulae is omitted.

[68] In classical logic, this set of logical symbols is sufficient.
[69] We mention here the relation symbols only for completeness reason.

3.1 DEF (Alphabet): The alphabet $\mathcal{A}$ of $\mathfrak{L}$ contains the following symbols, sorted by their type.

1. *object variables:* Countable many *variables* v_k (for $k \in \omega$).

   $\mathbf{V}_x = \{v_k;\ k \in \omega\}$ is the *set of all variables.*

   The symbols x, y, z (also with indices $k \in \omega$) etc. are used as metavariable for variables.

2. *logical constants:* $\bot$, $\rightarrow$, $\forall$, $=$

3. *non-logical constants:* some *constant symbols* c, some n'-ary *function symbols* f, and some m-ary *relation symbols* R (where $n, m \in \omega$)

4. *auxiliary symbols:* parentheses "(" and ")", the comma ",", etc.

3.2 DEF (Terms): The set $\mathbf{T}$ of the *terms* of $\mathfrak{L}$ is defined inductively according to the following inductive clauses:

$$c \quad | \quad x \quad | \quad f(t_0, \ldots t_n)$$

Terms generated according to the first two clauses are called *atomic*; otherwise, they are called *complex.*

Notation (Definitions): If it seems convenient, we separate the clauses in our definitions by the symbol "|". This notation is similar to the notation found in theoretical computer science with respect to production rules for formal grammars. Nevertheless, our definition is still to be understood as a traditional inductive definition. This means that, for example, the definition of the terms has to be understood as follows:

1. *constant symbols:* Every constant symbol c is a term.

2. *variables:* Every variable x is a term.

3. *function symbols:* If f is an n'-ary function symbol and if $t_0, \ldots t_n$ are n'-many terms, then $f(t_0, \ldots t_n)$ is a term.

Such an inductive definitions is to be understood finitely: only objects generated in finitely many steps according to the given clauses are contained in the inductively defined domain (here, in the set of all terms).

Remarks (Terms):

1. *metavariables:* The symbols t, s, r etc. are used as metavariables for arbitrary terms.
2. *notation:* Usual notational conventions are presupposed. In particular:
   (a) *binary symbols:* Binary function symbols can be notated infix.
   (b) *parentheses:* Unnecessary parentheses can be omitted (according to the usual conventions).
3. *direct subterms:* The terms t_k (with $k \in n'$) out of which a complex term $t \eqsim f(t_0, \ldots t_n)$ is generated are called the *direct* substerms of the term t; atomic terms have no direct subterms.

   If s is a direct subterm of a term t, then we also write $s \lhd_1 t$.

3.2.3 Subterms of Terms

We introduce the subterm relation for terms and, via these notions, some canonical sets of variables.

3.3 DEF (Subterms of Terms): The *set* $\mathbf{Sub}(t)$ *of (proper) subterms* of a tern t is defined recursively as follows:

1. t *atomic:* $\mathbf{Sub}(t) = \emptyset$
2. $t \eqsim f(t_0, \ldots t_n)$ *complex:* $\mathbf{Sub}(t) = \{t_0, \ldots t_n\} \cup \bigcup_{k \in n'} \mathbf{Sub}(t_k)$

A term s is a *(strict) subterm* of a term t (formally, $s \lhd t$), if it is contained in the set of proper subterms (formally, if $s \in \mathbf{Sub}(t)$).

Remarks (Subterms of Terms):

1. *all subterms:* Sometimes, it is convenient to use the non-strict (or, equivalently, the weak) version of the notion of subterms. This notion can be defined as follows.
   - $\mathbf{Sub}'(t) = \{t\} \cup \mathbf{Sub}(t)$ is the *set of all subterms* of t.

   Furthermore, if s is contained in the set of all subterms of t (formally, if $s \in \mathbf{Sub}'(t)$), then s is called a *subterm* of t (formally, $s \trianglelefteq t$).

It is checked almost immediately that the set of all subterms of a term t can be characterised as follows:

$$\mathbf{Sub}'(t) \;=\; \{t\} \cup \bigcup_{s \lhd_1 t} \mathbf{Sub}'(s)$$

2. *closure operations:* The different versions of the subterm relations on the set $\mathbf{T}$ of all terms are related by closure operations:
   (a) *strict subterm relation:* The strict subterm relation $\lhd$ is the transitive closure of the direct subterm relation $\lhd_1$.
   (b) *subterm relation:* The subterm relation $\trianglelefteq$ is the reflexive closure of the strict subterm relation $\lhd$.
3. *partial order:* The strict subterm relation $\lhd$ is a strict partial order on the set $\mathbf{T}$ of all terms. Atomic terms are the minimal elements with respect to this relation. There are infinite ascending chains with respect to the strict subterm relation, if and only if there are function symbols available in underlying the formal language; otherwise, there are only trivial chains of length 1.

The set of variables occurring in a term is inroduced via the set of all subterms.

3.4 DEF (Variables in Terms and Formula):

1. *set of variables:* $\mathbf{V}_x(t) = \mathbf{Sub}'(t) \cap \mathbf{V}_x$

Notation (Set of Variables): We use the designator $\mathbf{V}_x$ in two different meanings. First, the symbol $\mathbf{V}_x$ denotes a set of symbols, namely the set of all variables of a formal language $\mathfrak{L}$. Second, $\mathbf{V}_x$ is a function symbol for a (related) function. This ambiguity is not problematic, as the meaning of the symbol is clear from the context of its use. We will use similar simplifications of the notation in our investigations.

3.2.4 Multiplicity Function

The multiplicity function counts, how often a subterm occurs in a terms; this function is closely related with the subterm relation, but provides a finer insight in the structure of terms.

3.5 DEF (Multiplicity): The *multiplicity function* $\mathbf{mult} : \mathbf{T} \times \mathbf{T} \to \omega$ is defined recursively (in its second argument) as follows:

1. *t atomic:* $\mathbf{mult}(s,t) = \begin{cases} 1 & \text{if } s \eqsim t \\ 0 & \text{otherwise} \end{cases}$

2. *$t \eqsim f(t_0, \ldots t_n)$ complex:* $\mathbf{mult}(s,t) = \begin{cases} 1 & \text{if } s \eqsim t \\ \sum_{k \in n'} \mathbf{mult}(s,t_k) & \text{otherwise} \end{cases}$

We say that $\mathbf{mult}(s,t)$ is the multiplicity of the term s in the term t.

Remarks (Multiplicity):

1. *subterm relation:* The (weak) subterm relation can be characterised with the help of the multiplicity functions as follows:

$$s \trianglelefteq t \quad \text{if and only if} \quad \mathbf{mult}(s,t) \neq 0$$

4 Basic Theory of Nominal Terms

We provide the basic theory of nominal terms. Besides the introduction of the nominal terms themselves, we carry over the standard auxiliary functions and relations, as given in the preliminaries, and introduce some auxiliary functions and relations especially coined for nominal terms. Finally, we provide a basic categorisation of nominal terms.

4.1 Introduction of Nominal Terms

Nominal terms are generated according to the same clauses as the standard terms together with an additional clause generating so called *nominal symbols* as a new kind of atomic terms. The introduction of nominal terms is guided by our conception of formal languages as presented in the preliminaries.

4.1 DEF (Symbols): The alphabet $\mathcal{A}$ of $\mathfrak{L}$ is extended by a new sort of symbols as follows:

1. *nominal symbols:* Countable many *nominal symbols* $*_k$ (for $k \in \omega$).
   $\mathbf{V}_* = \{*_k;\ k \in \omega\}$ is the *set of all nominal symbols.*

Remarks (Symbols):

1. *notation:* We use the symbol $*$ as an abbreviation for the nominal symbol $*_0$. In particular, the symbol $*$ is not a metavariable for arbitrary nominal symbols.

2. *disjoint types:* We presuppose that symbols of different types are different; in particular, nominal symbols are different from variables and the non-logical symbols of $\mathfrak{L}$.

4.2 DEF (Nominal Terms): The set $\mathtt{T}$ of *nominal terms* of $\mathfrak{L}$ is defined inductively as follows:

$$*_k \quad | \quad x \quad | \quad c \quad | \quad f(\mathtt{t}_0, \ldots \mathtt{t}_n)$$

Nominal terms generated according to the first three clauses are called *atomic*; otherwise, they are called *complex*. The nominal terms $\mathtt{t}_k$ out of which a complex nominal term $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ is generated ($k \in n'$) are called the *direct subterms* of $\mathtt{t}$; atomic nominal terms have no direct subterms. If $\mathtt{s}$ is a direct subterm of $\mathtt{t}$, then we also write $\mathtt{s} \lhd_1 \mathtt{t}$.

Remarks (Nominal Terms):

1. *standard term:* Obviously, a nominal term is a *standard* term, if the first clause of the definition is not applied in its generation. If a nominal term $\mathtt{t}$ is not a standard term, then $\mathtt{t}$ is a *proper* nominal term.

2. *notation:* We agree upon the following conventions:

   (a) *metavariables:* The symbols $\mathtt{t}$, $\mathtt{s}$, $\mathtt{r}$ (also with indices $k \in \omega$) etc. are used as metavariables for nominal terms. In order to emphasise that a nominal term is a standard term, we also use the symbols t, s, r (again also with indices $k \in \omega$) etc. as metavariables.

   (b) *usual conventions:* The usual notational conventions, as sketched in the preliminaries, are presupposed.

   (c) *sequences:* We use the symbols $\mathfrak{t}$, $\mathfrak{s}$, $\mathfrak{r}$ etc. to mean arbitrary sequences (finite or infinite) of nominal terms. Usually, the symbols $\mathtt{t}_k$, $\mathtt{s}_k$, $\mathtt{r}_k$ etc. are used to denote the k-th entries of the respective sequences.

   (d) *constant sequences:* Constant sequences $\langle \mathtt{t}_k;\ k \in \omega \rangle$ satisfying that all entries $\mathtt{t}_k$ are equal to a nominal term $\mathtt{t}$ are occasionally denoted by $\mathfrak{c}_\mathtt{t}$. More formally:

$$\mathfrak{c}_\mathtt{t} = !\langle \mathtt{t}_k;\ k \in \omega \rangle \in \mathtt{T}^\omega :\ \forall k \in \omega . \mathtt{t}_k \mathrel{\widehat{=}} \mathtt{t}$$

Examples (Nominal Terms): We provide some examples of nominal terms of the language $\mathfrak{L}_{\mathrm{PA}}$ of arithmetics.

$$0 \quad ; \quad *_0 \quad ; \quad *_0 + (0 + *_0) \quad ; \quad (*_0 + *_1) + (*_0 + v_0)$$

Observe that we use the convenient infix notation for binary function symbols and that we omit unnecessary parentheses according to the usual conventions.

Nominal Forms by Schütte: Our definition of nominal terms is inspired by Schütte's notion of *nominal forms* as introduced in his logic book "Proof Theory" [29]; also, the denomination of the nominal symbols $*_k$ as "nominal symbols" is already present there. Schütte defines a nominal form (implicitly) as an arbitrary non-empty string of symbols of the extended alphabet;

interesting nominal forms are defined (inductively) as a subset of the set of all strings.[70]

Our approach to nominal forms is slightly different: we extend the usual inductive clauses for specific types of syntactic entity. This way, the specific types of nominal forms are defined directly, without a detour over the set of all strings. As an advantage, we do not have to identify suitable nominal forms in the set of all strings. The term *nominal form* is used in these investigations to subsume all kinds of such generalisations of the standard syntactic entities.

As mentioned in the survey of the literature, Schütte discusses first two kinds of nominal forms, which we would both call specific *propositional nominal formulae*. Additionally, Schütte discusses nominal forms in the context of formulae in first order languages. Due to his general definition of nominal forms (as arbitrary sequences), the nominal symbols may represent there the positions of different kinds of syntactic entities, namely the positions of variables as well as of subformulae.[71] In particular, Schütte uses nominal forms for an elaborate formulation of substitutions.

Simulation of Nominal Terms: We mention that we may simulate nominal symbols and nominal terms inside the usual framework of standard logic by, for example, the following convention: variables v_{2k} with even indices $2k$ have the role of usual standard variables and variables v_{2k+1} with odd indices $2k+1$ are understood as the nominal symbols.

Such an adhoc-convention is found in literature in several contexts (whenever we want to distinguish two different kinds of variables with two different uses); nevertheless, we abstain here from such an simulative account, as this would obscure the special role of nominal symbols in contrast to the standard variables in a formal language.

Another drawback of the simulative account to occurrences is the presupposition of suitable variables in the formal language. A formal first order language without variables seems unusual, but possible: the terms of such a language would only consists of constant symbols and function symbols applied on constant terms. In the theory of occurrences of formulae in formulae of a formal first order language the problems becomes clearer: in the

[70] Schütte provides two definitions of nominal forms (p. 11 and p. 14), but both times of a specific kind having some additional restrictions. For this reasons, we have to extract the exact definition of nominal forms according to Schütte, and we are not able to present a precise definition.

[71] We would define two different kinds of nominal forms, namely nominal formulae, in which the nominal symbols represent subformulae, and a second type, in which the nominal symbols represent variables.

general case, there are no propositional variables (zero-ary relation symbols) available in such a language; for simulating nominal formulae in such a situation, we would have to extend the underlying formal language by introducing suitable variables. Nothing is gained by not using nominal symbols instead.

Similarity of Nominal Terms: We say that two complex nominal terms $\mathtt{t}$ and $\mathtt{s}$ are *similar* (formally, $\mathtt{t} \sim \mathtt{s}$), if they have the same main function symbol. The latter means that there are an n'-ary function symbol f and suitable nominal terms $\mathtt{t}_k$ and $\mathtt{s}_k$ (with $k \in n'$) such that:

$$\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n) \quad \text{and} \quad \mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$$

Obviously, similarity is an equivalence relation on the set of complex nominal terms, but not on the set of all nominal terms.

4.2 Standard Auxiliary Functions and Relations

The definitions of the standard auxiliary functions and relations (as defined in the preliminaries with respect to standard terms) are easily carried over from their canonical versions.

These auxiliary notions are, in particular, the set of (proper) subterms $\mathbf{Sub}(\mathtt{t})$, the (strict) subterm relation $\lhd$, the set $\mathbf{V}_x(\mathtt{t})$ of variables and the multiplicity function $\mathbf{mult}(\mathtt{s}, \mathtt{t})$.

4.3 Specific Auxiliary Functions

Besides the auxiliary functions mentioned above, we introduce additionally some auxiliary functions coined especially for the treatment of nominal terms.

First, we provide the definition of the set of nominal symbols in a nominal term, which is the nominal version of the set of variables in a nominal term.

4.3 DEF (Nominal Symbols in Nominal Terms): The set $\mathbf{V}_*(\mathtt{t})$ of nominal symbols in a nominal term $\mathtt{t} \in \mathtt{T}$ is defined as follows:

$$\mathbf{V}_*(\mathtt{t}) \;=\; \mathbf{Sub}'(\mathtt{t}) \cap \mathbf{V}_*$$

4.3.1 Place Function

Furthermore, we introduce the set of *free places* of a nominal term, which is the set of indices of nominal symbols occurring in that nominal term.

4.4 DEF (Free Places): The function $\mathbf{place} : \mathtt{T} \to \mathfrak{p}(\omega)$ is defined recursively as follows:

1. $\mathtt{t}$ *atomic:*
$$\mathbf{place}(\mathtt{t}) \;=\; \begin{cases} \{k\} & \text{if } \mathtt{t} \doteq *_k \in \mathbf{V}_* \\ \emptyset & \text{otherwise} \end{cases}$$
2. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* $\mathbf{place}(\mathtt{t}) = \bigcup_{k \in n'} \mathbf{place}(\mathtt{t}_k)$

The set $\mathbf{place}(\mathtt{t})$ is the set of *the free places* of a nominal term $\mathtt{t}$.

Remarks (Free Places):

1. *set of nominal symbols:* The sets $\mathbf{V}_*(\mathtt{t})$ and $\mathbf{place}(\mathtt{t})$ are related. We have for all $k \in \omega$:
$$k \in \mathbf{place}(\mathtt{t}) \quad \Leftrightarrow \quad *_k \in \mathbf{V}_*(\mathtt{t})$$
2. *recursive definition:* In order to have a recursive definition at hands, we preferred to define directly the set of free places of a nominal term, instead of using a characterisation via the set of nominal symbols.

4.3.2 The Nominal Rank

The set of free places allows the introduction of a new rank function for nominal terms: the (nominal) *rank*, which is the smallest natural number containing all free places of a nominal term.

4.5 DEF (Rank Function): The *(nominal) rank function* $\mathbf{rank} : \mathtt{T} \to \omega$ is defined as follows:

$$\mathbf{rank}(\mathtt{t}) \;=\; \min(n \in \omega;\; \mathbf{place}(\mathtt{t}) \subseteq n)$$

Remarks (Rank Functions):

1. *subterm relation:* The rank function weakly respects the subterm relation:
   - If $\mathtt{s} \lhd \mathtt{t}$, then $\mathbf{rank}(\mathtt{s}) \leq \mathbf{rank}(\mathtt{t})$.

   Even if the subterm relation is strict, we only obtain, in the general case, a non-strict inequality on the right side; it is impossible to improve this bound. Investigate the following example in the language $\mathfrak{L}_{\mathrm{PA}}$:

   (a) *strict subterms:* $\mathtt{s} \mathrel{\widehat{=}} *_k \lhd \mathtt{t} \mathrel{\widehat{=}} S(*_k)$.

   (b) *rank:* $\mathbf{rank}(\mathtt{s}) = k' = \mathbf{rank}(\mathtt{t})$

2. *standard terms:* We may use the rank function to characterise standard terms.

   - $\mathtt{t}$ is a standard term, if and only if $\mathbf{rank}(\mathtt{t}) = 0$.

3. *n-ary nominal terms:* The n-ary nominal terms $\mathtt{t}$ (for arbitrary $n \in \omega$) can be characterised via the condition $\mathbf{place}(\mathtt{t}) = \mathbf{rank}(\mathtt{t})$. (Observe that the inclusion $\mathbf{place}(\mathtt{t}) \subseteq \mathbf{rank}(\mathtt{t})$ is trivial by the definition of the rank function.)

4.3.3 The Weight Functions

We provide two more rank function especially coined for nominal terms, namely the *weight* of a nominal term, which is the number of nominal symbols occurring in that nominal term, and its dual function counting the standard atomic subterms.

4.6 DEF (Weight Function): The *weight function* $\mathbf{weight} : \mathtt{T} \to \omega$ is defined recursively as follows:

1. $\mathtt{t}$ *atomic:*
$$\mathbf{weight}(\mathtt{t}) \;=\; \begin{cases} 1 & \text{if } \mathtt{t} \in \mathbf{V}_* \\ 0 & \text{otherwise} \end{cases}$$

2. $\mathtt{t} \mathrel{\widehat{=}} f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* $\mathbf{weight}(\mathtt{t}) = \sum_{k \in n'} \mathbf{weight}(\mathtt{t}_k)$

Remarks (Weight):

1. *multiplicity:* We may express the weight of a nominal term via the multiplicity function:
$$\mathbf{weight}(\mathtt{t}) \;=\; \sum_{k \in \omega} \mathbf{mult}(*_k, \mathtt{t}) \;=\; \sum_{k \in \mathbf{place}(\mathtt{t})} \mathbf{mult}(*_k, \mathtt{t})$$

2. *subterm relation:* The weight function is a rank function respecting weakly the subterm relation.

   - *weight:* If $\mathtt{s} \lhd \mathtt{t}$, then $\mathbf{weight}(\mathtt{s}) \leq \mathbf{weight}(\mathtt{t})$.

Even if the subterm relation is strict, we have only, in the general case, a non-strict inequality on the right side. Investigate the following example in $\mathfrak{L}_{\mathrm{PA}}$:

(a) *strict subterms:* $\mathtt{s} \doteq *_k \lhd \mathtt{t} \doteq S(*_k)$.

(b) *weight:* $\mathbf{weight}(\mathtt{s}) = 1 = \mathbf{weight}(\mathtt{t})$

3. *standard terms:* We may use the weight function to characterise standard terms.

   - $\mathtt{t}$ is a standard term, if and only if $\mathbf{weight}(\mathtt{t}) = 0$.

We provide the definition of the *dual weight function* counting the standard atomic subterms of a nominal term.

4.7 DEF (Dual Weight Function): We define recursively the *dual weight function* $\overline{\mathbf{weight}} : \mathtt{T} \to \omega$ as follows:

1. $\mathtt{t}$ *atomic:*

$$\overline{\mathbf{weight}}(\mathtt{t}) \;=\; \begin{cases} 0 & \text{if } \mathtt{t} \in \mathbf{V}_* \\ 1 & \text{otherwise} \end{cases}$$

2. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:*

$$\overline{\mathbf{weight}(\mathtt{t})} \;=\; \sum_{k \in n'} \overline{\mathbf{weight}(\mathtt{t}_k)}$$

Remarks (Dual Weight Function):

1. *multiplicity function:* We can characterise the dual weight function via the multiplicity function:

$$\overline{\mathbf{weight}(\mathtt{t})} \;=\; \sum_{t \in \mathtt{T}_0 \text{ atomic}} \mathbf{mult}(t, \mathtt{t}) \;=\; \sum_{t \in \mathbf{Sub}'(\mathtt{t}) \text{ atomic}} \mathbf{mult}(t, \mathtt{t})$$

2. *proper nominal terms:* We can use the dual weight function only to characterise nominal terms, which are not generated with the help of atomic standard terms (which can be understood as pure nominal terms).

   As a consequence, we obtain the following, slightly weaker condition for proper nominal terms:

- If $\overline{\mathbf{weight}}(\mathtt{t}) = 0$, then $\mathtt{t} \notin \mathtt{T}_0$ is a proper nominal term.

Observe that the other direction does not hold: there are proper nominal terms $\mathtt{t}$ having a dual weight different from 0. Investigate the following example in the language $\mathfrak{L}_{\mathrm{PA}}$:

$$\mathtt{t} \;\widehat{=}\; 0 + *_0 \notin \mathtt{T}_0 \quad ; \quad \overline{\mathbf{weight}}(\mathtt{t}) \;=\; 1 \;\neq\; 0$$

3. *subterm relation:* The dual weight function is a rank function respecting weakly the subterm relation.

   - *weight:* If $\mathtt{s} \lhd \mathtt{t}$, then $\overline{\mathbf{weight}}(\mathtt{s}) \leq \overline{\mathbf{weight}}(\mathtt{t})$.

   Even if the subterm relation is strict, we only obtain, in the general case, a non-strict inequality on the right side. Investigate the following example in $\mathfrak{L}_{\mathrm{PA}}$:

   (a) *strict subterms:* $\mathtt{s} \;\widehat{=}\; *_k \;\lhd\; \mathtt{t} \;\widehat{=}\; S(*_k)$.
   (b) *weight:* $\overline{\mathbf{weight}}(\mathtt{s}) = 0 = \overline{\mathbf{weight}}(\mathtt{t})$

4. *terminology:* The weight function and the dual weight function are insofar dual to each other, as both functions complement each other in the following sense: the sum of weight and dual weight is the multiplicity of all atomic subterms. More formally:

$$\mathbf{weight}(\mathtt{t}) + \overline{\mathbf{weight}}(\mathtt{t}) \;=\; \sum_{t \in \mathbf{Sub}'(\mathtt{t}) \text{ atomic}} \mathbf{mult}(t, \mathtt{t})$$

4.4 Basic Categorisation of Nominal Terms

With the help of the place function and the multiplicity function, we are able to provide a basic categorisation of the nominal terms.

4.8 DEF (Basic Categorisation): Let $\mathtt{t} \in \mathtt{T}$ be a nominal term.

1. *n-ary:* The nominal term $\mathtt{t}$ is called *n-ary* (with $n \in \omega$), if exactly the numbers $k \in n$ are a free places in $\mathtt{t}$ (formally, if $\mathbf{place}(\mathtt{t}) = n$).

   In particular, $\mathtt{t}$ is unary, if $\mathbf{place}(\mathtt{t}) = \{0\} = 1$.

2. *simple // multiple:* The nominal term $\mathtt{t}$ is called *simple*, if no nominal symbol occurs more than once (formally, if $\mathbf{mult}(*_k, \mathtt{t}) \in \{0, 1\}$ for all $k \in \omega$); otherwise, $\mathtt{t}$ is called *multiple*.

3. *single:* The nominal term $\mathtt{t}$ is called *single*, if $\mathtt{t}$ is unary and simple.

Remarks (Basic Categorisation):

1. *zero-ary:* Standard terms are, according to the categorisation above, exactly the 0-ary nominal terms.

2. *arity of proper subterms:* The proper subterms of a complex n-ary nominal term are, in general, not n-ary. Investigate, for example the binary nominal term $*_0 + *_1$ in the language of arithmetics $\mathfrak{L}_{\mathrm{PA}}$. The first direct subterm $*_0$ is unary (and not binary), and the second direct subterm $*_1$ is not n-ary, for no $n \in \omega$.

3. *simplicity of proper subterms:* The proper subterms of a complex simple nominal term are simple; in particular standard terms are simple.

4. *Schütte:* The concept of simple nominal forms is already found in Schütte [29]; he additionally uses the concept of n-place nominal forms (no nominal symbol $*_k$ with $k > n$ may occur).

Notation (Restricted Sets of Nominal Terms): We agree upon the following notational conventions for indicating that a set of nominal terms is restricted to specific nominal terms:

1. *n-ary:* The restriction of a set of nominal terms to n-ary nominal terms (with $n \in \omega$) is indicated by the subscript n. For example:

   - $\mathtt{T}_0$ is the set of all standard terms.
   - $\mathtt{T}_1$ the set of all unary nominal terms.

2. *simple:* The restriction of a set of nominal terms to simple nominal terms is indicated by the subscript s. For example:

   - $\mathtt{T}_s$ is the set of all simple nominal terms.

3. *special sets:* The restriction of a set of nominal terms to standard terms and unary terms is indicated by a superscript $*$.[72] For example:

   - $\mathtt{T}^* = \mathtt{T}_0 \cup \mathtt{T}_1$ is the set of all standard terms or unary nominal terms.

[72] The subscript $*$ is used in a way not consistent with the concept of restrictions. In $\mathbf{V}_*$, for example, the subscript $*$ is used to indicate the type of symbols collected in this set.

Furthermore, the restriction of a set of nominal terms to n-ary nominal terms of arbitrary arity is indicated by the subscript ω. For example:

- $\mathtt{T}_\omega = \bigcup_{k \in \omega} \mathtt{T}_k$ is the set of all n-ary nominal terms of arbitrary arity.

Meaningful combinations of these labels are permitted; these labels are not only used to restrict the set $\mathtt{T}$ of all nominal terms, but also with other sets and functions, which we introduce in the course of these investigations.

5 The General Substitution Function

We introduce the *general substitution function*, which is the central tool for dealing with nominal terms. This functions allows the simultaneous replacement of the nominal symbols in a nominal term according to an infinite sequence of nominal terms. Applications on finite sequences are explained.

Furthermore, the *essential equality* of sequence of nominal terms is introduced identifying sequences such that an application of the general substitution function on a given nominal term and on the sequences have the same result. We conclude the introduction of the general substitution function by considering the subterms of nominal terms under the application of this function.

5.1 Introduction of the General Substitution Function

We provide the formal definition of the general substitution function.

5.1 DEF (General Substitution Function): The binary *general substitution function* $\cdot[\cdot] : \mathtt{T} \times \mathtt{T}^{\omega} \to \mathtt{T}$ is defined recursively (in the first argument) for arbitrary (but infinite) sequences $\mathfrak{s} = \langle \mathtt{s}_k;\ k \in \omega \rangle \in \mathtt{T}^{\omega}$ of nominal terms as follows:

1. $\mathtt{t}$ *atomic:* $\mathtt{t}[\mathfrak{s}] \mathrel{=\!\!\!\wedge} \begin{cases} \mathtt{s}_k & \text{if } \mathtt{t} \mathrel{=\!\!\!\wedge} *_k \\ \mathtt{t} & \text{otherwise} \end{cases}$

2. $\mathtt{t} \mathrel{=\!\!\!\wedge} f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* $\mathtt{t}[\mathfrak{s}] \mathrel{=\!\!\!\wedge} f(\mathtt{t}_0[\mathfrak{s}], \ldots \mathtt{t}_n[\mathfrak{s}])$

An application of the general substitution function results in the simultaneous replacement of all occurrences of nominal symbols in the first argument by the respective entries of the second argument.

Remarks (General Substitution Function):

1. *conception:* The general substitution function is a simple function (according to the distinction discussed in the introduction of these investigations), as we do not replace specific occurrences of nominal symbols, but all such occurrences.

   Nevertheless, the general substitution function is a simple function in the realm of nominal terms, which is a more general realm than that of standard terms. Due to the generalisation of the domain, we are able to address successfully hard problems of the realm of standard terms.

2. *Schütte:* Schütte [29] provides two intensional definitions of the general substitution function (with respect to propositional logic (p. 11) and first order logic (p. 14)) by describing informally the result of an application of the general substitution function.

5.2 Categorisation of the Preimage

The following categorisation of the preimages of the general substitution function (depending on the result of an application of this function) will be useful many times in the course of these investigations.

5.2 Proposition (Categorisation of the Preimage): Let $\mathfrak{r} \in \mathtt{T}^{\omega}$ be an arbitrary sequence of nominal terms. The following statements hold for all nominal terms $\mathtt{t} \in \mathtt{T}$:

1. *nominal symbol:* If $\mathtt{t}[\mathfrak{r}] \in \mathbf{V}_*$, then $\mathtt{t} \in \mathbf{V}_*$.
2. *constant symbol:* If $\mathtt{t}[\mathfrak{r}] \mathrel{=\!\!\!\wedge} c$, then $\mathtt{t} \in \mathbf{V}_* \cup \{c\}$.
3. *variable:* If $\mathtt{t}[\mathfrak{r}] \mathrel{=\!\!\!\wedge} x$, then $\mathtt{t} \in \mathbf{V}_* \cup \{x\}$.
4. *complex term:* If $\mathtt{t}[\mathfrak{r}] \mathrel{=\!\!\!\wedge} f(\mathtt{s}_0, \ldots \mathtt{s}_n) \mathrel{=\!\!\!\wedge} \mathtt{s}$, then $\mathtt{t} \in \mathbf{V}_*$ or $\mathtt{t} \sim \mathtt{s}$.

   The latter means that there are nominal terms $\mathtt{t}_k$ (for all $k \in n'$) such that $\mathtt{t} \mathrel{=\!\!\!\wedge} f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ and, in particular, $\mathtt{t}_k[\mathfrak{r}] \mathrel{=\!\!\!\wedge} \mathtt{s}_k$ for all $k \in n'$.

Proof. We check each statement; let $\mathfrak{r} \in \mathtt{T}^{\omega}$ and $\mathtt{t} \in \mathtt{T}$ be arbitrary.

1. *nominal symbol:* We assume $\mathtt{t}[\mathfrak{r}] \in \mathbf{V}_*$. We exclude according to clause (2) of the definition of the general substitution function that $\mathtt{t}$ is complex (otherwise, $\mathtt{t}[\mathfrak{r}]$ would be complex too). Therefore, $\mathtt{t}$ is atomic. Furthermore, we can exclude that $\mathtt{t}$ is a constant symbol or a variable, as in this case, we would have according to clause (1) of the definition of the general substitution function that $\mathtt{t} \mathrel{=\!\!\!\wedge} \mathtt{t}[\mathfrak{r}] \in \mathbf{V}_*$. The latter is a contradiction. The only remaining possibility is that $\mathtt{t} \in \mathbf{V}_*$ is a nominal symbol.

2. *constant symbol:* We assume $\mathtt{t}[\mathfrak{r}] \mathrel{=\!\!\!\wedge} c$ for a constant symbol c. Additionally, we assume $\mathtt{t} \notin \mathbf{V}_*$ and prove that $\mathtt{t} \mathrel{=\!\!\!\wedge} c$. As above, we conclude that $\mathtt{t}$ must be atomic. As $\mathtt{t} \notin \mathbf{V}_*$, we immediately obtain by clause (1) of the definition of the general substitution function that $c \mathrel{=\!\!\!\wedge} \mathtt{t}[\mathfrak{r}] \mathrel{=\!\!\!\wedge} \mathtt{t}$.

3. *variables:* Analogously to the proof of statement (2).

4. *complex term:* We assume $\mathfrak{t}[\mathfrak{r}] \doteq f(\mathfrak{s}_0, \ldots \mathfrak{s}_n) \doteq \mathfrak{s}$ and $\mathfrak{t} \notin \mathbf{V}_*$. According to clause (1) of the definition of the general substitution function, we can exclude that $\mathfrak{t}$ is atomic, as $\mathfrak{t}[\mathfrak{r}]$ would be atomic, too. Therefore, $\mathfrak{t}$ has to be complex. The latter means that there is an m'-ary function symbol g and m'-many nominal terms $\mathfrak{t}_k$ ($k \in m'$) such that $\mathfrak{t} \doteq g(\mathfrak{t}_0, \ldots \mathfrak{t}_m)$. According to clause (2) of the general substitution function, we obtain:

$$f(\mathfrak{s}_0, \ldots \mathfrak{s}_n) \doteq \mathfrak{t}[\mathfrak{r}] \doteq g(\mathfrak{t}_0[\mathfrak{r}], \ldots \mathfrak{t}_m[\mathfrak{r}])$$

Comparing the leftmost nominal term with the rightmost, we conclude stepwise that $f \doteq g$, then $n = m$ and finally that $\mathfrak{t}_k[\mathfrak{r}] \doteq \mathfrak{s}_k$ for all $k \in n'$. In particular, $\mathfrak{t} \sim \mathfrak{s}$. Q.E.D.

5.3 Application on Finite Sequences

The general substitution function is defined as a function on infinite sequences (as second argument). In contrast to this definition, we are interested in applications of this function on finite sequences or even on single arguments. In order to explain such applications, we have to discuss the notion of *neutrality* and suitable *extension* of finite sequences.

5.3 DEF (Neutrality): Let $\alpha \in \omega'$ arbitrary, and $\mathfrak{t} = \langle \mathfrak{t}_k;\ k \in \alpha \rangle \in \mathfrak{T}^\alpha$ be a (finite or infinite) sequence of nominal terms.

1. *neutral entry:* An entry $\mathfrak{t}_k$ of the sequence $\mathfrak{t}$ is called *neutral*, if $\mathfrak{t}_k$ is the respective nominal symbol $*_k$ (formally, if $\mathfrak{t}_k \doteq *_k$). In this case, k is called a *neutral position* of $\mathfrak{t}$.

2. *neutral sequence:* The sequence $\mathfrak{e}_\alpha = \langle *_k;\ k \in \omega \rangle$ is called the neutral sequence of length α. If $\alpha = \omega$, we abbreviate $\mathfrak{e}_\alpha$ by $\mathfrak{e}$; we also say that $\mathfrak{e}$ is the *neutral sequence* (without reference to its length).

3. *ω-extension:* An infinite sequence $\mathfrak{s} = \langle \mathfrak{s}_k;\ k \in \omega \rangle$ is the *ω-extension* of the sequence $\mathfrak{t}$, if $\mathfrak{t}$ is an initial segment of $\mathfrak{s}$ which is extended by neutral entries. Formally, if the following condition is satisfied for all positions $k \in \omega$:

   - If $k \in \mathbf{lng}(\mathfrak{t})$, then $\mathfrak{s}_k \doteq \mathfrak{t}_k$; otherwise, $\mathfrak{s}_k \doteq *_k$.

   The ω-extension of $\mathfrak{t}$ is denoted by $\omega(\mathfrak{t})$.

Remarks (Neutrality):

1. *neutral sequence:* Applying the general substitution function on a nominal term $\mathfrak{t}$ and the neutral sequence $\mathfrak{e}$ results in the nominal term $\mathfrak{t}$, as the nominal symbols $*_k$ are replaced by themselves. (Simple induction over the structure of $\mathfrak{t}$.) Furthermore, $\mathfrak{e}$ is the uniquely determined (infinite) sequence with this property. (If there is a position k in a sequence $\mathfrak{s}$ such that the entry $\mathfrak{s}_k$ is not neutral, then $*_k[\mathfrak{s}] \doteq \mathfrak{s}_k \neq *_k$.)

2. *neutral positions:* We have the (good) intuition that the neutral positions of a sequence do not affect an application of the general substitution function: if we apply this function on a sequence with neutral positions, then the respective nominal symbols $*_k$ in the first argument are replaced by themselves.[73]

3. *α-extension:* The notion of an ω-extension of arbitrary sequences is easily generalised to α-extensions (for arbitrary ordinal numbers α – as long as we have a sufficient supply of nominal symbols).[74] In particular, we may use such α-extensions to identify proper initial segments of sequences (namely, if we chose an α such that $\alpha < \mathbf{lng}(\mathfrak{t})$).

Using ω-extension, we can explain an application of the general substitution function on a nominal term and a finite sequence.

5.4 DEF (Finite Substitution): Let $\alpha \in \omega$ be a finite ordinal. An application of the general substitution function on a nominal term $\mathtt{t} \in \mathtt{T}$ and a finite sequence $\mathfrak{t} \in \mathtt{T}^\alpha$ of length α is defined as follows:

$$\mathtt{t}[\mathfrak{t}] \doteq \mathtt{t}[\omega(\mathfrak{t})]$$

A single nominal term as second argument is understood as the respective sequence of length one.

[73] In order to prove this observation formally, we would have to identify the occurrences of nominal terms in the result of such an application of the general substitution function, which are the images of a specific nominal symbol $*_k$ in the first argument. We would have to prove that these occurrences of nominal terms are exactly the nominal symbols $*_k$. In order to identify these occurrences of nominal terms in the result, we need a theory of occurrences of nominal terms in nominal terms. Such a theory can be developed analogously to the theory of occurrences of terms in terms, but on the base of nominal terms (and would presuppose a new kind of nominal symbols $\star_k$). We abstain here from doing so, and are satisfied with the intuitive description given above.

[74] As our account to nominal terms only provides nominal symbols $*_k$ for ordinals $k \in \omega$, we have to restrict here α to be finite or ω itself. But this is not a principle restriction.

Remarks (Finite Substitution):

1. *alternative account:* An alternative account to finite substitutions would be to alter the definition of the general substitution function in a way that its second argument could be (also) a finite sequence. In this case, we would replace (in the atomic clause) a nominal symbol $*_k$ only by $\mathfrak{s}_k$, if the condition $k \in \mathbf{lng}(\mathfrak{s})$ is satisfied.

   As "not-replacing a nominal symbol" has the same effect as "replacing that symbol by itself", there is no extensional difference between both accounts.

   Nevertheless, dealing with the general substitution function would be more involved in the alternative account, as we would have to consider the length of the arguments. In particular, we would have to consider many case distinctions with respect to the length of sequences and the nominal symbols present in the first argument.

2. *notation:* When notating an application of the general substitution function on a finite sequence (of length greater than zero), we may omit the parentheses signifying the sequence. Thus, we may write:

$$\mathfrak{t}[\mathfrak{s}_0, \ldots \mathfrak{s}_n] \quad \text{instead of} \quad \mathfrak{t}[\langle \mathfrak{s}_0, \ldots \mathfrak{s}_n \rangle]$$

3. *neutral sequences:* As $\omega(\mathfrak{e}_\alpha) = \mathfrak{e}$ for all $\alpha \in \omega'$, the finite neutral sequences are neutral with respect to an application of the general substitution function.

Generation Forms: We can represent the generation of complex nominal terms $\mathfrak{t} \mathrel{\widehat{=}} f(\mathfrak{t}_0, \ldots \mathfrak{t}_n)$ as an application of the general substitution function on a very simple type of nominal term together with the sequence of its direct subterms as follows:

$$\mathfrak{t} \mathrel{\widehat{=}} f(*_0, \ldots *_n)[\mathfrak{t}_0, \ldots \mathfrak{t}_n]$$

This observation motivates to call the nominal term $\mathfrak{t}_f \mathrel{\widehat{=}} f(*_0, \ldots *_n)$ the *generation form* of the n'-ary function symbol f. Observe that two complex nominal terms $\mathfrak{t}$ and $\mathfrak{s}$ are similar ($\mathfrak{t} \sim \mathfrak{s}$), if and only if they are generated with the help of the same generation form.

5.4 Essential Equality

We introduce *essential equality*, which is a class of equivalence relations on sequences of nominal terms depending on a previously chosen nominal term.

5.5 DEF (Essential Equality): Let $\mathtt{t} \in \mathtt{T}$ arbitrary. Two (finite or infinite) sequences $\mathfrak{s}, \mathfrak{r} \in \mathtt{T}^{\leq\omega}$ of nominal terms are called *essentially equal* with respect to $\mathtt{t}$ (formally, $\mathfrak{s} \equiv_{\mathtt{t}} \mathfrak{r}$), if the following condition is satisfied:

$$\omega(\mathfrak{s})_{|\mathbf{place}(\mathtt{t})} \;=\; \omega(\mathfrak{r})_{|\mathbf{place}(\mathtt{t})}$$

Remarks (Essential Equality):

1. *well-defined:* Recall that an infinite sequence of nominal terms is defined in these investigations as a function from ω into the set $\mathtt{T}$ of nominal terms; such function can be restricted to a subset of ω, namely to the set $\mathbf{place}(\mathtt{t})$. Therefore, essential equality is well-defined.

2. *conception:* Essential idea of essential equality is the pointwise comparison of the entries of both sequences $\mathfrak{t}$ and $\mathfrak{s}$ at the positions contained in the set of free places of the nominal term $\mathtt{t}$.

   Such a comparison becomes problematic, if one or both sequences are too short: if there is a number $k \in \mathbf{place}(\mathtt{t})$ such that $k \notin \mathbf{lng}(\mathfrak{t})$ or $k \notin \mathbf{lng}(\mathfrak{s})$, then there are no entries to compare. In order to avoid such a complication, we compare the ω-extensions of the sequences instead of the sequences themselves.

3. *pointwise characterisation:* Two sequences $\mathfrak{s}$ and $\mathfrak{r}$ are essentially equal with respect to a nominal term $\mathtt{t}$, if and only if the following conditions are all satisfied for all $k \in \mathbf{place}(\mathtt{t})$:

   (a) *common entries:* If $k \in \min(\mathbf{lng}(\mathfrak{s}), \mathbf{lng}(\mathfrak{r}))$, then $\mathfrak{s}_k \mathrel{\widehat{=}} \mathfrak{r}_k$.

   (b) $\mathfrak{s}$ *longer:* If $k \in \mathbf{lng}(\mathfrak{s}) \backslash \mathbf{lng}(\mathfrak{r})$, then $\mathfrak{s}_k \mathrel{\widehat{=}} *_k$.

   (c) $\mathfrak{r}$ *longer:* If $k \in \mathbf{lng}(\mathfrak{r}) \backslash \mathbf{lng}(\mathfrak{s})$, then $\mathfrak{r}_k \mathrel{\widehat{=}} *_k$.

   Observe that at least one of these conditions is trivially satisfied, as it is impossible that both $\mathfrak{s}$ is longer than $\mathfrak{r}$ and vice versa.

Simple Observations (Essential Equality): We communicate some simple observations about essential equality.

1. *standard terms:* If $\mathtt{t}$ is a standard term, then essential equality is trivially given. This means that $\mathfrak{s} \equiv_{\mathtt{t}} \mathfrak{r}$ for all sequences $\mathfrak{s}$ and $\mathfrak{r}$.

2. *complex term:* Let $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ be a complex term. Two sequences $\mathfrak{s}$ and $\mathfrak{r}$ are essentially equal with respect to $\mathtt{t}$, if and only if both sequences are essentially equal with respect to all direct subterms of $\mathtt{t}$. Formally:
$$\mathfrak{s} \equiv_{\mathtt{t}} \mathfrak{r} \quad \Leftrightarrow \quad \mathfrak{s} \equiv_{\mathtt{t}_k} \mathfrak{r} \text{ for all } k \in n'$$

3. *ω-extension:* A sequence $\mathfrak{s}$ and its ω-extension $\omega(\mathfrak{s})$ are essentially equal with respect to all nominal terms $\mathtt{t}$.

Essential Equality as Equivalence Relation: It is easily checked that essential equality $\equiv_{\mathtt{t}}$ with respect to a nominal term $\mathtt{t}$ is an equivalence relation on the set of all sequences $\mathtt{T}^{\leq\omega}$ for all nominal terms $\mathtt{t} \in \mathtt{T}$. We mention some more details.

1. *canonical representatives:* Canonical representatives $\mathfrak{t}$ of an equivalence classes $[\mathfrak{s}]_{\mathtt{t}}$ of a sequence $\mathfrak{s}$ with respect to essential equality $\equiv_{\mathtt{t}}$ (with respect to $\mathtt{t}$) can be obtained via the following demands:

   (a) The sequence $\mathfrak{t}$ is infinite.

   (b) All positions k of $\mathfrak{t}$, which are not contained in $\mathbf{place}(\mathtt{t})$, are neutral.

   (c) $\mathfrak{t}$ and $\mathfrak{s}$ are essentially equal with respect to $\mathtt{t}$.

   More sophisticatedly, the canonical representative $\mathfrak{t}$ of $[\mathfrak{s}]_{\mathtt{t}}$ is given as follows:
   $$\mathfrak{t} \;=\; \mathfrak{s}_{|\mathbf{place}(\mathtt{t})} \;\cup\; \{\langle k, *_k\rangle;\; k \in \omega \backslash \mathbf{place}(\mathtt{t})\}$$

2. *finite representatives:* The special role of n-ary nominal terms $\mathtt{t}$ becomes visible, if we investigate the equivalence classes $[\mathfrak{s}]_{\mathtt{t}}$ of essential equality with respect to such nominal terms: there is another canonical representative, namely the finite initial segment of $\omega(\mathfrak{s})$ of length n.

   We find a similar canonical representative of $[\mathfrak{s}]_{\mathtt{t}}$ with respect to arbitrary nominal terms $\mathtt{t}$, but there we have to investigate sequences with length $\mathbf{rank}(\mathtt{t})$; and we have to replace the entries at positions k with $k \in \mathbf{rank}(\mathtt{t}) \backslash \mathbf{place}(\mathtt{t})$ by neutral entries.

The reason to introduce the concept of essential equality is that we obtain a purely syntactical criterion, whether two applications of the general substitution function on a given nominal term have the same result or not.

5.6 Proposition (Essential Equality): Let $\mathtt{t} \in \mathtt{T}$. The following statement holds for all sequences $\mathfrak{s}, \mathfrak{r} \in \mathtt{T}^{\leq\omega}$: the result of an application of the general substitution function on $\mathtt{t}$ and on one of the sequences equals to an application on that term and the other sequence, if and only if both sequences are essentially equal with respect to $\mathtt{t}$. Formally:

$$\mathfrak{s} \equiv_{\mathtt{t}} \mathfrak{r} \quad \Leftrightarrow \quad \mathtt{t}[\mathfrak{s}] \mathrel{=\!\!\!\wedge} \mathtt{t}[\mathfrak{r}]$$

Proof. By induction over the structure of $\mathtt{t}$. As the substitution with finite sequences is defined in terms of the ω-extensions (which are essentially equal to the original sequences), we may assume without loss of generality that both sequences are infinite.

1. $\mathtt{t} \mathrel{=\!\!\!\wedge} *_k$*:* We have both that $*_k[\mathfrak{s}] \mathrel{=\!\!\!\wedge} \mathtt{s}_k$ and $*_k[\mathfrak{r}] \mathrel{=\!\!\!\wedge} \mathtt{r}_k$.

   Assuming $\mathfrak{s} \equiv_{\mathtt{t}} \mathfrak{r}$, we obtain $\mathtt{s}_k \mathrel{=\!\!\!\wedge} \mathtt{r}_k$, as $\mathbf{place}(\mathtt{t}) = \{k\}$. Therefore:

   $$\mathtt{t}[\mathfrak{s}] \;\mathrel{=\!\!\!\wedge}\; \mathtt{s}_k \;\mathrel{=\!\!\!\wedge}\; \mathtt{r}_k \;\mathrel{=\!\!\!\wedge}\; \mathtt{t}[\mathfrak{r}]$$

   Otherwise, if $\mathfrak{s} \not\equiv_{\mathtt{t}} \mathfrak{r}$, then $\mathtt{s}_k \neq \mathtt{r}_k$ (as we still have $\mathbf{place}(\mathtt{t}) = \{k\}$). And therefore, $\mathtt{t}[\mathfrak{s}] \neq \mathtt{t}[\mathfrak{r}]$. This means that the stated equivalence holds.

2. $\mathtt{t}$ *atomic standard term:* As $\mathtt{t}$ is a standard atomic term, we immediately obtain for all sequences $\mathfrak{s}$ and $\mathfrak{r}$:

   $$\mathfrak{s} \equiv_{\mathtt{t}} \mathfrak{r} \quad \text{and} \quad \mathtt{t}[\mathfrak{s}] \mathrel{=\!\!\!\wedge} \mathtt{t} \mathrel{=\!\!\!\wedge} \mathtt{t}[\mathfrak{r}]$$

   Again, the stated equivalence holds.

3. $\mathtt{t} \mathrel{=\!\!\!\wedge} f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* Let $\mathfrak{s}$ and $\mathfrak{r}$ be two sequences of nominal terms such that $\mathfrak{s} \equiv_{\mathtt{t}} \mathfrak{r}$. This implies that $\mathfrak{s} \equiv_{\mathtt{t}_k} \mathfrak{r}$ for all $k \in n'$. This means that we may apply n'-many times induction hypothesis and obtain $\mathtt{t}_k[\mathfrak{s}] \mathrel{=\!\!\!\wedge} \mathtt{t}_k[\mathfrak{r}]$ for all $k \in n'$. We calculate:

   $$\mathtt{t}[\mathfrak{s}] \;\mathrel{=\!\!\!\wedge}\; f(\mathtt{t}_0[\mathfrak{s}], \ldots \mathtt{t}_n[\mathfrak{s}]) \overset{(IV)}{\mathrel{=\!\!\!\wedge}} f(\mathtt{t}_0[\mathfrak{r}], \ldots \mathtt{t}_n[\mathfrak{r}]) \;\mathrel{=\!\!\!\wedge}\; \mathtt{t}[\mathfrak{r}]$$

   Otherwise, if $\mathfrak{s} \not\equiv_{\mathtt{t}} \mathfrak{r}$, then there is a $k \in n'$ such that $\mathfrak{s} \not\equiv_{\mathtt{t}_k} \mathfrak{r}$. By induction hypothesis, we obtain that $\mathtt{t}_k[\mathfrak{s}] \neq \mathtt{t}_k[\mathfrak{r}]$. Therefore:

   $$\mathtt{t}[\mathfrak{s}] \;\mathrel{=\!\!\!\wedge}\; f(\ldots \mathtt{t}_k[\mathfrak{s}] \ldots) \overset{(IV)}{\neq} f(\ldots \mathtt{t}_k[\mathfrak{r}] \ldots) \;\mathrel{=\!\!\!\wedge}\; \mathtt{t}[\mathfrak{r}]$$

   Again, the stated equivalence holds. Q.E.D.

We communicate some corollaries of the proposition about essential equality. First, we show that standard terms are invariant under the application of the general substitution.

5.7 Corollary (Standard Terms): $t[\mathfrak{t}] \mathrel{-\!\!\!\circ} t$ for all sequences $\mathfrak{t} \in \mathtt{T}^{\leq\omega}$ and all standard terms $t \in \mathtt{T}_0$.
Proof. Let t be a standard term and $\mathfrak{t}$ an arbitrary sequence of nominal terms. We first observe that $\mathfrak{t} \equiv_t \mathfrak{e}$, as $\mathbf{place}(t) = \emptyset$. Therefore:

$$t[\mathfrak{t}] \;\mathrel{-\!\!\!\circ}\; t[\mathfrak{e}] \;\mathrel{-\!\!\!\circ}\; t$$

The first equality is due to the proposition about essential equality, the second due to the neutrality of the neutral sequences $\mathfrak{e}$ (as mentioned before). Q.E.D.

Next, we provide some criteria, whether applications of the general substitution function on a given nominal term results in the same nominal term.

5.8 Corollary (Criteria for Equality): The following statements are equivalent for all nominal terms $\mathtt{t} \in \mathtt{T}$ and all sequences $\mathfrak{t}, \mathfrak{t}' \in \mathtt{T}^{\leq\omega}$:

1. $\mathtt{t}[\mathfrak{t}] \mathrel{-\!\!\!\circ} \mathtt{t}[\mathfrak{t}']$
2. $\mathtt{s}[\mathfrak{t}] \mathrel{-\!\!\!\circ} \mathtt{s}[\mathfrak{t}']$ for all $\mathtt{s} \in \mathbf{Sub}'(\mathtt{t})$.
3. $*_k[\mathfrak{t}] \mathrel{-\!\!\!\circ} *_k[\mathfrak{t}']$ for all $k \in \mathbf{place}(\mathtt{t})$.

Proof.

1. (1) $\Rightarrow$ (2)*:* Let $\mathtt{t}[\mathfrak{t}] \mathrel{-\!\!\!\circ} \mathtt{t}[\mathfrak{t}']$. Therefore, $\mathfrak{t} \equiv_\mathtt{t} \mathfrak{t}'$. As $\mathbf{place}(\mathtt{s}) \subseteq \mathbf{place}(\mathtt{t})$, for all $\mathtt{s} \in \mathbf{Sub}'(\mathtt{t})$, we obtain $\mathfrak{t} \equiv_\mathtt{s} \mathfrak{t}'$. The latter means $\mathtt{s}[\mathfrak{t}] \mathrel{-\!\!\!\circ} \mathtt{s}[\mathfrak{t}']$.
2. (2) $\Rightarrow$ (3)*:* Immediate, as $*_k \in \mathbf{Sub}'(\mathtt{t})$ for all $k \in \mathbf{place}(\mathtt{t})$.
3. (3) $\Rightarrow$ (1)*:* As $*_k[\mathfrak{t}] \mathrel{-\!\!\!\circ} \mathtt{t}_k$, statement (3) is only a slight reformulation of essential equality. The latter implies (1). Q.E.D.

As a last corollary, we show that we may replace substitutions with infinite sequences by substitutions with suitable finite sequences, namely with initial segments of these sequences of sufficient length (greater than the rank of the nominal term on which the general substitution function is applied). Additionally, if the nominal term, on which the general substitution function

is applied, is n-ary, then the initial segment of length n is the uniquely determined sequence of that length with this property.

5.9 Corollary (Initial Segments): Let $\mathtt{t} \in \mathtt{T}$ be an arbitrary nominal term, $n = \mathbf{rank}(\mathtt{t})$ its (finite) rank, and $\alpha \in \omega$ such that $n \leq \alpha$. Furthermore, let $\mathfrak{t} \in \mathtt{T}^{\omega}$ be an infinite sequence and $\mathfrak{s} = \alpha(\mathfrak{t})$ its initial segment of length α. The following statements hold:

1. *initial segment:* $\mathtt{t}[\mathfrak{t}] \mathrel{=\!\!\!\wedge} \mathtt{t}[\alpha(\mathfrak{t})]$
2. *n-ary:* If $\mathtt{t}$ is n-ary and $\alpha = n$, then $\mathfrak{s}$ is uniquely determined sequence of length n such that $\mathtt{t}[\mathfrak{t}] = \mathtt{t}[\mathfrak{s}]$.

Proof.

1. *initial segment:* By definition of the rank, $\mathbf{place}(\mathtt{t}) \subseteq \mathbf{rank}(\mathtt{t})$. As $\mathbf{rank}(\mathtt{t}) = n \leq \alpha$, we have that $\mathtt{t}_k = \mathtt{s}_k$ for all $k \in \mathbf{place}(\mathtt{t})$. Therefore, $\mathfrak{t} \equiv_{\mathtt{t}} \alpha(\mathfrak{t}) = \mathfrak{s}$. As a consequence, $\mathtt{t}[\mathfrak{t}] \mathrel{=\!\!\!\wedge} \mathtt{t}[\alpha(\mathfrak{t})]$.
2. *n-ary:* We presuppose that $\mathtt{t}$ is n-ary and $\alpha = n$. Let $\mathfrak{r} \in \mathtt{T}^n$ such that $\mathtt{t}[\mathfrak{s}] \mathrel{=\!\!\!\wedge} \mathtt{t}[\mathfrak{t}] \mathrel{=\!\!\!\wedge} \mathtt{t}[\mathfrak{r}]$. Due to the proposition about essential equality, $\mathfrak{s} \equiv_{\mathtt{t}} \mathfrak{r}$. As a consequence, $\mathtt{s}_k = \mathtt{r}_k$ for all $k \in \mathbf{place}(\mathtt{t}) = n$. The latter means that $\mathfrak{s} = \mathfrak{r}$. Q.E.D.

Remarks (Initial Segments):

1. *uniqueness:* If $\mathtt{t}$ is not n-ary, then $\mathfrak{s}$ is not uniquely determined: there is $k \in n \backslash \mathbf{place}(\mathtt{t})$. We may exchange $\mathtt{s}_k$ in $\mathfrak{s}$ by arbitrary nominal term $\mathtt{s}$ without loosing essential equality.
2. *minimality:* In general, the sequence $\mathfrak{s}$ as constructed above, is not of minimal length. If the last entries of $\mathfrak{s}$ are neutral, then every proper initial segment of $\mathfrak{s}$ in which only these neutral entries are missing, is still essentially equal to $\mathfrak{t}$. Nevertheless, this possibility to shorten $\mathfrak{s}$ depends on the special choice of $\mathfrak{t}$. If the n-th entry of $\mathfrak{t}$ is not neutral, then $\mathfrak{s}$ is of minimal length.
3. *relevance:* The relevant aspect of the proposition about initial segments is that we may replace any substitution with infinite sequences by a substitution with a finite sequence. Nevertheless, this cannot be done uniformly: the finite initial segment has to be long enough, in the general case as long as the rank of the nominal term on which the substitution is applied.

5.5 Subterms under Substitution

If the general substitution function is applied on a nominal term and a sequence, the subterms of some entries (determined by the free places of the nominal term) become subterms of the result. We investigate this phenomenon in some details.

5.10 Proposition (Subterms under Substitution): Let $\mathtt{t} \in \mathtt{T}$ be a nominal term, $\mathfrak{s} = \langle \mathtt{s}_k;\ k \in \omega \rangle \in \mathtt{T}^\omega$ a sequence of nominal terms. The following statement holds:

$$\bigcup_{k \in \mathbf{place}(\mathtt{t})} \mathbf{Sub}'(\mathtt{s}_k) \ \subseteq\ \mathbf{Sub}'(\mathtt{t}[\mathfrak{s}])$$

Proof. By induction over the structure of $\mathtt{t}$.

1. $\mathtt{t}$ *atomic:* If $\mathtt{t} \in \mathbf{V}_*$ is a nominal terms, then there is $k \in \omega$ such that $\mathtt{t} \mathrel{=\!\!\!\wedge} *_k$. As $\mathbf{place}(*_k) = \{k\}$ and as $*_k[\mathfrak{s}] \mathrel{=\!\!\!\wedge} \mathtt{s}_k$, the stated inequality holds trivially (we even have equality of both sides).

   Otherwise, $\mathtt{t}$ is a standard term, and we have $\mathbf{place}(\mathtt{t}) = \emptyset$. Therefore, the union on the left side equals $\emptyset$, which is a subset of any set.

2. $\mathtt{t} \mathrel{=\!\!\!\wedge} f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* Recall that $\mathtt{t}[\mathfrak{s}] \mathrel{=\!\!\!\wedge} f(\mathtt{t}_0[\mathfrak{s}], \ldots \mathtt{t}_n[\mathfrak{s}])$. Applying n'-many times induction hypothesis, we obtain for all $l \in n'$:

   $$\bigcup_{k \in \mathbf{place}(\mathtt{t}_l)} \mathbf{Sub}'(\mathtt{s}_k) \ \subseteq\ \mathbf{Sub}'(\mathtt{t}_l[\mathfrak{s}])$$

   The following both equations hold:

   (a) $\mathbf{Sub}'(\mathtt{t}[\mathfrak{s}]) = \{\mathtt{t}[\mathfrak{s}]\} \cup \bigcup_{l \in n'} \mathbf{Sub}'(\mathtt{t}_l[\mathfrak{s}])$

   (b) $\mathbf{place}(\mathtt{t}) = \bigcup_{l \in n'} \mathbf{place}(\mathtt{t}_l)$

   Therefore, we may calculate as follows:

   $$\begin{aligned} \bigcup_{k \in \mathbf{place}(\mathtt{t})} \mathbf{Sub}'(\mathtt{s}_k) \ &= \ \bigcup_{l \in n'} \bigcup_{k \in \mathbf{place}(\mathtt{t}_l)} \mathbf{Sub}'(\mathtt{s}_k) \\ &\subseteq \ \bigcup_{l \in n'} \mathbf{Sub}'(\mathtt{t}_l[\mathfrak{s}]) \ \subseteq\ \mathbf{Sub}'(\mathtt{t}[\mathfrak{s}]) \end{aligned}$$

Q.E.D.

If we focus on the free places (which means on the atomic nominal subterms), then we have a stronger result: the set of free places of the result of an application of the general substitution function is exactly the union of the free places of the entries determined by the free places of the nominal term.

5.11 Proposition (Free Places under Substitution): Let $\mathtt{t} \in \mathtt{T}$ be a nominal term, $\mathfrak{s} \in \mathtt{T}^{\omega}$ a sequence of nominal terms. The following equation holds:

$$\mathbf{place}(\mathtt{t}[\mathfrak{s}]) \;=\; \bigcup_{k \in \mathbf{place}(\mathtt{t})} \mathbf{place}(\mathtt{s}_k)$$

Proof. By induction over the structure of $\mathtt{t}$.

1. $\mathtt{t}$ *atomic:* If $\mathtt{t} \doteq *_k$, then $\mathtt{t}[\mathfrak{s}] \doteq \mathtt{s}_k$, and the stated equation holds trivially. Otherwise, $\mathtt{t}$ is a standard term with $\mathbf{place}(\mathtt{t}) = \emptyset$. As $\mathtt{t}[\mathfrak{t}] \doteq \mathtt{t}$, the left side of the equation equals to $\emptyset$. The same is true for the right side, as an empty union is empty.

2. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* Recall that $\mathtt{t}[\mathfrak{s}] \doteq f(\mathtt{t}_0[\mathfrak{s}], \ldots \mathtt{t}_n[\mathfrak{s}])$. Applying n'-many times induction hypothesis, we obtain for all $l \in n'$:

$$\mathbf{place}(\mathtt{t}_l[\mathfrak{s}]) \;=\; \bigcup_{k \in \mathbf{place}(\mathtt{t}_l)} \mathbf{place}(\mathtt{s}_k)$$

   The following both equations hold:

   (a) $\mathbf{place}(\mathtt{t}[\mathfrak{s}]) = \bigcup_{l \in n'} \mathbf{place}(\mathtt{t}_l)$
   (b) $\mathbf{place}(\mathtt{t}) = \bigcup_{k \in n'} \mathbf{place}(\mathtt{t}_k)$

   Therefore, we may calculate as follows:

$$\begin{aligned} \mathbf{place}(\mathtt{t}[\mathfrak{s}]) \;&=\; \bigcup_{l \in n'} \mathbf{place}(\mathtt{t}_l[\mathfrak{s}]) \\ &=\; \bigcup_{l \in n'} \bigcup_{k \in \mathbf{place}(\mathtt{t}_l)} \mathbf{place}(\mathtt{s}_k) \;=\; \bigcup_{k \in \mathbf{place}(\mathtt{t})} \mathbf{place}(\mathtt{s}_k) \end{aligned}$$

Q.E.D.

6 Homomorphisms on Nominal Terms

Homomorphisms are structure preserving functions; in the case of nominal terms, this structure is given by the standard symbols (the non-logical symbols, variables and auxiliary symbols) of the underlying formal language. We introduce these functions and discuss their close relationship to the general substitution function.

Furthermore, we provide criteria for special homomorphisms, with focus on isomorphisms, based on their restrictions to the set of nominal symbols. The discussion is complemented by a brief survey of the underlying algebraic structures discussed in these investigations. Additionally, the *expansions* and *contractions* of sequences of nominal terns according to so called *simple* homomorphisms are investigated. Similarly to essential equality, they provide a criterion for the equality of the result of applications of the general substitution function.

6.1 Introduction of Homomorphisms

We provide the formal definition of homomorphisms on nominal terms.

6.1 DEF (Homomorphisms):

1. *homomorphisms:* A function $F : \mathtt{T} \to \mathtt{T}$ is called a *homomorphisms* (on nominal terms), if the following conditions are satisfied:

   (a) $\mathtt{t}$ *constant symbol or variable:* $F(\mathtt{t}) \doteq \mathtt{t}$

   (b) $\mathtt{t} \doteq f(\mathtt{t}_0, \dots \mathtt{t}_n)$ *complex:* $F(f(\mathtt{t}_0, \dots \mathtt{t}_n)) \doteq f(F(\mathtt{t}_0), \dots F(\mathtt{t}_n))$

   $\mathbf{Hom}(\mathtt{T})$ is the set of *all homomorphisms* on $\mathtt{T}$.

2. *special homomorphisms:* Let $F \in \mathbf{Hom}(\mathtt{T})$ be a homomorphism.

   (a) *simple homomorphism:* F is *simple*, if $F(*_k) \in \mathbf{V}_*$ for all $k \in \omega$. $\mathbf{Hom}_s(\mathtt{T})$ denotes the set of *all simple homomorphisms* on $\mathtt{T}$.

   (b) *isomorphism:* F is called an *isomorphism*, if F is bijective. $\mathbf{Hom}^\circ(\mathtt{T})$ denotes the set of *all isomorphisms* on $\mathtt{T}$.

   (c) *constant homomorphism:* F is called a *constant homomorphism*, if there is a nominal term $\mathtt{t} \in \mathtt{T}$ such that $F(*_k) \doteq \mathtt{t}$.

Remarks (Homomorphisms):

1. *simple homomorphism:* A simple homomorphism maps nominal symbols to nominal symbols (which are, in particular, simple nominal terms), but not to arbitrary simple nominal terms. The latter means that the simplicity of homomorphisms is independent of the simplicity of nominal terms.

2. *constant homomorphisms:* A constant homomorphism is not a constant function (as no structure preserving function can be constant), but its restriction to the set of nominal symbols is constant.

3. *function spaces:* The composition of simple, bijective and constant homomorphisms are simple, bijective and constant, respectively.

   As $\mathbf{id}_{\mathtt{T}}$ is not a constant homomorphism, the constant homomorphisms do not form a function space, but the simple homomorphisms and the isomorphisms do as well as the set of all homomorphisms.

Universal Homomorphism: Homomorphisms are closely related with the general substitution function; we provide some details.

1. *restrictions to sequences:* Immediately by definition, every restriction $F_{\mathfrak{s}} : \mathtt{T} \to \mathtt{T} : \mathtt{t} \mapsto \mathtt{t}[\mathfrak{s}]$ of the general substitution function to a previously chosen sequence $\mathfrak{s}$ is a homomorphism.

2. *homomorphisms:* Vice versa, every homomorphism F on nominal terms equals to the restriction $F_{\mathfrak{s}}$ of the general substitution function determined by the sequence $\mathfrak{s} = \langle F(*_k);\ k \in \omega \rangle$.

This means that there is a bijective correspondence between homomorphisms and the restrictions of the general substitution function to sequences (and therefore also a correspondence to sequences of nominal terms). In particular, we can relate special homomorphisms with special sequences as follows:

1. *simple homomorphisms:* A sequence $\mathfrak{s}$ is a sequence of nominal symbols, if and only if the corresponding homomorphism $F_{\mathfrak{s}}$ is simple.

2. *constant homomorphisms:* A sequence $\mathfrak{s}$ is a constant sequence, if and only if the corresponding homomorphism $F_{\mathfrak{s}}$ is a constant homomorphism.

Under this perspective, the general substitution function may be understood as a *universal homomorphism* codifying (in its second argument) all homomorphisms on nominal terms.

Conceptual Remark (Homomorphisms): To deal with homomorphisms (instead of the general substitution function) means to focus on the first argument of the general substitution function, namely on nominal terms. It seems convenient to argue in the realm of homomorphisms, when we are not interested in the concrete choice of the sequences.

Due to the bijective correspondence between homomorphisms and restrictions of the general substitution function, we are able to carry over results belonging to one realm to the other. We mention some important properties of homomorphisms:

Properties of Homomorphisms:

1. *unique definability:* Every function $F_0 : \mathbf{V}_* \to \mathtt{T}$ uniquely determines a homomorphism F satisfying the condition $F_{|\mathbf{V}_*} = F_0$.
2. *essential equality:* $F(\mathtt{t}) = G(\mathtt{t})$, if and only if $F_{|\mathbf{V}_*(\mathtt{t})} = G_{|\mathbf{V}_*(\mathtt{t})}$.
3. *preimage:* The preimage $\mathtt{t}$ of the result $F(\mathtt{t})$ of an application of a homomorphisms F is given by the categorisation of the preimage with respect to the general substitution function.

6.2 Criteria for Special Homomorphisms

We provide criteria for the special homomorphisms (with focus on isomorphisms) based on their restrictions to the set of nominal terms.

6.2 Proposition (Criterion - Isomorphism): A homomorphism is an isomorphism, if and only if its restriction to nominal symbols is a permutation on that set. More formally, for all $F \in \mathbf{Hom}(\mathtt{T})$: $F \in \mathbf{Hom}^{\circ}(\mathtt{T})$, if and only if $F_{|\mathbf{V}_*} \in \mathbf{Sym}(\mathbf{V}_*)$.

Proof. We have to show two directions.

1. *"⇒":* We assume that $F \in \mathbf{Hom}^{\circ}(\mathtt{T})$ is an isomorphism. We first show that $F_0 = F_{|\mathbf{V}_*}$ is a function into $\mathbf{V}_*$. Assume that not. This means that there is a nominal symbol $*_k \in \mathbf{V}_*$ such that $F_0(*_k) \eqsim \mathtt{s} \notin \mathbf{V}_*$. We distinguish as follows:

(a) $\mathtt{s}$ *constant symbol of variable:* If $\mathtt{s}$ is a constant symbol or a variable, then we have that $F(\mathtt{s}) \eqsim \mathtt{s}$ according to the definition of homomorphisms. As F_0 is a restriction of F we may calculate as follows:

$$F(*_k) \;\eqsim\; F_0(*_k) \;\eqsim\; \mathtt{s} \;\eqsim\; F(\mathtt{s})$$

As $*_k \not\eqsim \mathtt{s}$, we obtain that F is not injective. This is a contradiction.

(b) $\mathtt{s} \eqsim f(\mathtt{s}_0, \ldots \mathtt{s}_n)$ *complex:* As F_0 is a restriction of F, we obtain $F(*_k) \eqsim f(\mathtt{s}_0, \ldots \mathtt{s}_n)$. As F is surjective, there are nominal terms $\mathtt{t}_k$ (with $k \in n'$) such that $F(\mathtt{t}_k) \eqsim \mathtt{s}_k$ for all $k \in n'$. As F is a homomorphism, we may calculate as follows:

$$\begin{aligned} F(*_k) \;&\eqsim\; f(\mathtt{s}_0, \ldots \mathtt{s}_m) \\ &\eqsim\; f(F(\mathtt{t}_0), \ldots F(\mathtt{t}_n)) \;\eqsim\; F(f(\mathtt{t}_0, \ldots, \mathtt{t}_n)) \end{aligned}$$

As $*_k \not\eqsim f(\mathtt{t}_0, \ldots \mathtt{t}_n)$, we obtain that F is not injective. This is again a contradiction.

The assumption that $F_0(*_k) \notin \mathbf{V}_*$ leads in all cases to a contradiction. Therefore, $F_0 : \mathbf{V}_* \to \mathbf{V}_*$ is, in deed, a function into the set of nominal symbols. Additionally, F_0 is injective, F_0 is the restriction of an injective function, namely of F. We still have to show that F_0 is surjective: let $*_k \in \mathbf{V}_*$ arbitrary. As F is surjective, there is a nominal term $\mathtt{t}$ such that $F(\mathtt{t}) \eqsim *_k$. According to the categorisation of the preimage, $\mathtt{t} \in \mathbf{V}_*$. The latter means that $F_0(\mathtt{t}) \eqsim *_k$.

2. *"$\Leftarrow$":* Let $F \in \mathbf{Hom}(\mathtt{T})$ be a homomorphism such that its restriction $F_0 = F_{|\mathbf{V}_*} \in \mathbf{Sym}(\mathbf{V}_*)$ is a permutation. Observe that F is the uniquely determined homomorphism extending F_0. We show by induction over the structure of nominal terms that every nominal $\mathtt{s}$ is hit by F and isolated with respect to F.

(a) $\mathtt{s}$ *nominal symbol:* Let $\mathtt{s} \in \mathbf{V}_*$ be a nominal symbol. As the restriction $F_0 \in \mathbf{Sym}(\mathbf{V}_*)$ of F is a permutation, and therefore surjective, there is a nominal symbol $*_k \in \mathbf{V}_*$ satisfying the condition $F(*_k) \eqsim F_0(*_k) \eqsim \mathtt{s}$. Therefore, $\mathtt{s}$ is hit by F.

Assume that $F(\mathtt{t}) \eqsim \mathtt{s}$ for a nominal term $\mathtt{t} \in \mathtt{T}$. According to the categorisation of the preimage, we obtain that $\mathtt{t} \in \mathbf{V}_*$. Therefore, $F_0(\mathtt{t}) \eqsim F(\mathtt{t}) \eqsim \mathtt{s} \eqsim F_0(*_k)$. As F_0 is injective, $\mathtt{t} \eqsim *_k$, and $\mathtt{s}$ is, indeed, isolated by F.

(b) $\mathtt{s}$ *constant symbol or variable:* Let $\mathtt{s}$ be a constant symbol or a variable. As F is a homomorphism, we have that $F(\mathtt{s}) \eqsim \mathtt{s}$. Therefore, $\mathtt{s}$ is hit by F.

Assume that $F(\mathtt{t}) \eqsim \mathtt{s}$ for a nominal term $\mathtt{t} \in \mathtt{T}$. According to the categorisation of the preimage, $\mathtt{t} \in \mathbf{V}_* \cup \{\mathtt{s}\}$. We can exclude that $\mathtt{t} \in \mathbf{V}_*$, as the restriction F_0 of F to $\mathbf{V}_*$ is a function into $\mathbf{V}_*$. Therefore, we obtain that $\mathtt{t} \eqsim \mathtt{s}$. The latter means that $\mathtt{s}$ is isolated by F.

(c) $\mathtt{s} \eqsim f(\mathtt{s}_0, \ldots \mathtt{s}_n)$ *complex:* Applying n'-many times induction hypothesis, we obtain that $\mathtt{s}_k$ is hit and isolated by F for all $k \in n'$. As $\mathtt{s}_k$ is hit by F, there are nominal terms $\mathtt{t}_k \in \mathtt{T}$ such that $F(\mathtt{t}_k) \eqsim \mathtt{s}_k$ for all $k \in n'$. Therefore:

$$F(f(\mathtt{t}_0, \ldots \mathtt{t}_n)) \eqsim f(F(\mathtt{t}_0), \ldots F(\mathtt{t}_n)) \eqsim f(\mathtt{s}_0, \ldots \mathtt{s}_n) \eqsim \mathtt{s}$$

This means that $\mathtt{s}$ is hit by F.

Assume that $F(\mathtt{t}) \eqsim \mathtt{s}$ for a nominal term $\mathtt{t} \in \mathtt{T}$. According to the categorisation of the preimage, we obtain that $\mathtt{t} \in \mathbf{V}_*$ or $\mathtt{t} \sim \mathtt{s}$. The former is excluded as argued before. Therefore, $\mathtt{t} \eqsim f(\mathtt{t}'_0, \ldots \mathtt{t}'_n)$ for suitable nominal terms $\mathtt{t}'_k \in \mathtt{T}$ (with $k \in n'$). We calculate as follows:

$$f(\mathtt{s}_0, \ldots \mathtt{s}_n) \eqsim \mathtt{s} \eqsim F(\mathtt{t}) \eqsim f(F(\mathtt{t}'_0), \ldots F(\mathtt{t}'_n))$$

As a consequence, we have that $F(\mathtt{t}'_k) \eqsim \mathtt{s}_k$ for all $k \in n'$. As $\mathtt{s}_k$ is isolated by F, we also obtain that $\mathtt{t}'_k \eqsim \mathtt{t}_k$ for all $k \in n'$. The latter means that $\mathtt{t} \eqsim f(\mathtt{t}_0, \ldots \mathtt{t}_n)$, and therefore, $\mathtt{s}$ is also isolated by F.

As all nominal terms $\mathtt{s} \in \mathtt{T}$ are hit and isolated by F, the homomorphism F is surjective and injective. The latter means that F is, indeed, bijective and, therefore, an isomorphism. Q.E.D.

Remarks (Criterion for Isomorphism):

1. *simple homomorphisms:* As the restriction of an isomorphism to the set $\mathbf{V}_*$ of nominal symbols is a permutation, we immediately obtain that every isomorphism is a simple homomorphism. Nevertheless, the converse does not hold: there are simple homomorphisms, which are no isomorphisms. Investigate, for example F induced by the function F_0 satisfying that $F_0(*_k) \eqsim *_0$ for all $k \in \omega$. Therefore:

$$\mathbf{Hom}^{\circ}(\mathtt{T}) \subsetneq \mathbf{Hom}_s(\mathtt{T}) \subsetneq \mathbf{Hom}(\mathtt{T})$$

2. *"bijective" sequences:* Isomorphisms correspond with sequences $\mathfrak{s}$ of nominal symbols, in which each nominal symbol $*_k$ is exactly once an entry of $\mathfrak{s}$. In comparison to the set $\mathtt{T}^\omega$ of sequences of nominal terms and to the set ${\mathbf{V}_*}^\omega$ of sequences of nominal symbols, there is no natural set of such "*bijective*" sequences. For this reason, we abstained from introducing artificially this set.

More Criteria: We mention some more criteria for other kinds of special homomorphisms:

1. *simple homomorphism:* Immediately by definition: a homomorphism F is simple, if and only if its restriction $F_{|\mathbf{V}_*}$ is a function into the set of nominal symbols.

2. *constant homomorphism:* Immediately by definition: a homomorphism F is a constant homomorphism, if and only if its restriction $F_{|\mathbf{V}_*}$ is a constant function.

3. *surjective homomorphism:* A homomorphism F is surjective, if and only if the set of nominal symbols is contained in the image of its restriction $F_{|\mathbf{V}_*}$ to that set. More formally, if and only if $\mathbf{V}_* \subseteq F[\mathbf{V}_*]$.

   In order to prove the direction "$\Rightarrow$", it is sufficient to mention that all nominal symbols $*_k$ are hit by a surjective F and, due to the categorisation of the preimage, the preimage of $*_k$ is a nominal symbol $*_l$. As a consequence, $\mathbf{V}_* \subseteq \mathbf{img}(F_{|\mathbf{V}_*}) = F[\mathbf{V}_*]$.

   In order to prove the direction "$\Leftarrow$", we have to show by an induction over the structure of nominal terms that if all nominal symbols are hit by F, then already all nominal terms are hit by F.

4. *injective homomorphisms:* Simple homomorphisms are injective, if and only if their restriction to the set of nominal symbols is injective. (One direction is immediate, the other direction is proved, as in the case of surjective homomorphisms, by induction, but with respect to the property of being isolated by F.)

   The situation is more involved in the general case of arbitrary injective homomorphisms. In this case, the injectivity of the restriction is not sufficient. Investigate the homomorphism F induced by the following function F_0:

$$F_0(*_0) \;\doteq\; *_0 + *_0 \quad ; \quad F_0(*_{n'}) \;\doteq\; *_n$$

Obviously, F_0 is injective, but the induced homomorphism F is not, as the following equation holds:

$$F(*_1 + *_1) \;\hat{=}\; F(*_1) + F(*_1) \;\hat{=}\; *_0 + *_0 \;\hat{=}\; F(*_0)$$

Nevertheless, it is possible to provide a criterion based on the restriction of F to nominal symbols: we have to check, whether $F_{|\mathbf{V}_*}(*_k)$ is isolated by F (and not only by $F_{|\mathbf{V}_*}$) for all $k \in \omega$.[75]

6.3 Excursus: Sequence Spaces and Function Spaces

For completeness reasons, we provide a brief excursus on the relationship between some algebraic structures present in our investigations.[76]

We already mentioned that homomorphisms and sequences of nominal terms correspond bijectively via the restrictions of the general substitution functions to the respective sequences. Introducing a suitable composition of sequences and, based on this notion, of sequence spaces, we can extend this correspondence to an isomorphism between the respective algebraic structures.

6.3 DEF (Composition of Sequence): The *composition* $\circ : \mathtt{T}^\omega \times \mathtt{T}^\omega \to \mathtt{T}^\omega$ of sequences of nominal terms is defined as follows:

$$\circ : \langle \mathfrak{t}, \mathfrak{s} \rangle \mapsto \langle \mathfrak{t}_k[\mathfrak{s}];\ k \in \omega \rangle$$

Remarks (Composition of Sequences):

1. *non-commutative:* The composition of sequences is not commutative:

$$\begin{aligned} \mathfrak{c}_{*_3} \circ \mathfrak{c}_{*_5} &= \langle *_3[\mathfrak{c}_{*_5}] \rangle = \mathfrak{c}_{*_5} \\ \mathfrak{c}_{*_5} \circ \mathfrak{c}_{*_3} &= \langle *_5[\mathfrak{c}_{*_3}] \rangle = \mathfrak{c}_{*_3} \end{aligned}$$

   Recall that $\mathfrak{c}_\mathtt{t}$ is the infinite constant sequence with the entry $\mathtt{t}$ at each position.

2. *neutral sequence:* It is easily checked that the neutral sequence $\mathfrak{e}$ is left-neutral and right-neutral with respect to the composition $\circ$ of sequences. Observe that the reason for neutrality is different in both

[75] We abstain here from a detailed proof, as injective, but not surjective homomorphisms are beyond the needs of our subsequent investigations.

[76] We abstain here from a detailed discussion, as the correspondences discussed below are beyond the needs of our investigations.

cases: if $\mathfrak{e}$ is the left argument, then $\mathfrak{e}$ "choses" the arguments from the right sequence in the order as given in the right sequence; if $\mathfrak{e}$ is the right argument, then the nominal symbols in the entries of the left sequence are replaced by themselves.

We introduce the *sequence spaces* formally.

Sequence Spaces: Let $\mathfrak{X} \subseteq \mathtt{T}^\omega$ be a set of sequences of nominal terms. The triple $\langle \mathfrak{X}, \circ, \mathfrak{e} \rangle$ is called a *sequence space*, if the following conditions are satisfied:

1. *closed under composition:* The set $\mathfrak{X}$ is closed under the composition of sequences. Formally, $\mathfrak{x} \circ \mathfrak{y} \in \mathfrak{X}$ for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}$.
2. *neutral element:* The neutral sequence $\mathfrak{e}$ is contained in $\mathfrak{X}$.

We extend the bijective correspondence between sequences and homomorphisms to the respective spaces.

Correspondence: Sequences // Homomorphisms: There is a bijective correspondence between the sequence space of all sequences of nominal terms and the function spaces of all homomorphisms via the following isomorphism:

$$\Phi : \mathtt{T}^\omega \to \mathbf{Hom}(\mathtt{T}) : \mathfrak{t} \mapsto F_\mathfrak{t}$$

This means that the respective spaces are isomorphic:

$$\langle \mathtt{T}^\omega, \circ, \mathfrak{e} \rangle \;\cong\; \langle \mathbf{Hom}(\mathtt{T}), \circ, \mathbf{id}_\mathtt{T} \rangle$$

In particular, the proper subspace of all sequences of nominal symbols is isomorphic to the subspace of all simple homomorphisms.

$$\langle \mathbf{V}_*{}^\omega, \circ, \mathfrak{e} \rangle \;\cong\; \langle \mathbf{Hom}_s(\mathtt{T}), \circ, \mathbf{id}_\mathtt{T} \rangle$$

Relating the nominal symbols with their indices, we may observe an additional isomorphism between function spaces:

Correspondence: Homomorphisms // Natural Functions: There is a bijective correspondence between the simple homomorphisms and the natural functions via the following isomorphism:

$$\Psi : \mathbf{Hom}_s(\mathtt{T}) \to \mathbf{Func}(\omega) : F \mapsto \{\langle k, !l : F(*_k) \doteq *_l \rangle;\ k \in \omega\}$$

This means that the respective function spaces are isomorphic:

$$\langle \mathbf{Hom}_s(\mathtt{T}), \circ, \mathbf{id}_{\mathtt{T}} \rangle \;\cong\; \langle \mathbf{Func}(\omega), \circ, \mathbf{id}_{\omega} \rangle$$

In particular, the proper subspace of isomorphisms on nominal terms is isomorphic to the symmetric group of the natural numbers:

$$\langle \mathbf{Hom}^{\circ}(\mathtt{T}), \circ, \mathbf{id}_{\mathtt{T}} \rangle \;\cong\; \langle \mathbf{Sym}(\omega), \circ, \mathbf{id}_{\omega} \rangle$$

Overview: Corresponding Spaces: The following overview illustrates the correspondences between the different algebraic spaces as sketched above. We do not display all existing subspaces, but only some canonical; we may easily find more (corresponding) subspaces.

General Substitution Function

$$\cdot[\cdot] : \mathtt{T} \times \mathtt{T}^{\omega} \to \mathtt{T}$$

$$\downarrow\downarrow\downarrow$$

Sequences		Homomorphisms		Natural Functions
$\langle \mathtt{T}^{\omega}, \ldots \rangle$	$\cong$	$\langle \mathbf{Hom}(\mathtt{T}), \ldots \rangle$		
$\bigvee$		$\bigvee$		
$\langle \mathbf{V}_*{}^{\omega}, \ldots \rangle$	$\cong$	$\langle \mathbf{Hom}_s(\mathtt{T}), \ldots \rangle$	$\cong$	$\langle \mathbf{Func}(\omega), \ldots \rangle$
		$\bigvee$		$\bigvee$
		$\langle \mathbf{Hom}^{\circ}(\mathtt{T}), \ldots \rangle$	$\cong$	$\langle \mathbf{Sym}(\omega), \ldots \rangle$

It is worth to mention that the concept of homomorphism seems to be the most general concept. In order to find a sequence space isomorphic to isomorphisms on nominal terms, we would have to introduce the (more or less) artificial notion of bijective sequences, and in order to find a function space in the context of natural numbers isomorphic to homomorphisms on nominal terms, we would have to discuss the natural numbers as a suitable term algebra.

6.4 Expansions and Contractions of Sequences

We provide a criterion for two nominal terms, namely a nominal term and its image under an application of a simple homomorphism, deciding, whether an application of the general substitution function on these nominal terms has the same result. Similarly to essential equality, the second arguments have to be related. This relationship is captured by the notions of *expansions* and *contractions* of sequences according to a simple homomorphism.

6.4 DEF (Expansion and Contraction): Let $\mathtt{t} \in \mathtt{T}$ be a nominal term, $F \in \mathbf{Hom}_s(\mathtt{T})$ a simple homomorphism and $\mathfrak{s}, \mathfrak{r} \in \mathtt{T}^\omega$ two infinite sequences of nominal terms.

1. *expansion:* $\mathfrak{s}$ is called an *F-expansion* (or an *expansion according to F*) of $\mathfrak{r}$ with respect to $\mathtt{t}$, if the following condition is satisfied for all $k \in \mathbf{place}(\mathtt{t})$:[77]

   $$\mathtt{s}_k \;\hat{=}\; \mathtt{r}_{F(k)}$$

   In this case, $\mathfrak{r}$ is called an *F-contraction* (or a *contraction according to F*) of $\mathfrak{s}$ with respect to $\mathtt{t}$.

2. *contractible:* The sequence $\mathfrak{s}$ is called *F-contractible* with respect to $\mathtt{t}$, if the following condition is satisfied for all $k, l \in \mathbf{place}(\mathtt{t})$:

   $$F(*_k) \;\hat{=}\; F(*_l) \quad \Rightarrow \quad \mathtt{s}_k \hat{=} \mathtt{s}_l$$

3. *relevant entries:* If $\mathfrak{s}$ is an expansion of $\mathfrak{r}$ according to F, then we use the following terminology:

   (a) *expansion:* An entry $\mathtt{s}_k$ of $\mathfrak{s}$ is called *relevant*, if $k \in \mathbf{place}(\mathtt{t})$; the respective position k of $\mathfrak{s}$ is also called *relevant*.

   (b) *contraction:* An entry $\mathtt{r}_k$ of $\mathfrak{s}$ is called *relevant*, if $k \in \mathbf{place}(F(\mathtt{t}))$; the respective position k of $\mathfrak{r}$ is also called *relevant*.

The definition applies analogously to finite sequences $\mathfrak{s}$ and $\mathfrak{r}$ provided that they are long enough. The latter means:

$$\mathbf{rank}(\mathtt{t}) \;\leq\; \mathbf{lng}(\mathfrak{s}) \quad ; \quad \mathbf{rank}(F(\mathtt{t})) \;\leq\; \mathbf{lng}(\mathfrak{r})$$

[77] Observe that we do not distinguish in the notation between the simple homomorphism F and the natural function induced by F. As long as the corresponding natural function is only applied on labels of nominal symbols or on numbers explicitly representing free places of a nominal term (as done in the definition), this ambiguity of notation is unproblematic.

Remarks (Expansion and Contraction):

1. *terminology:* Let $\mathfrak{s}$ be an expansion of $\mathfrak{r}$ according to a simple homomorphism F (with respect to a nominal term $\mathtt{t}$). If a relevant position l of $\mathfrak{r}$ is hit by two different relevant positions k, k' of $\mathfrak{s}$, then the entry $\mathtt{r}_l$ is "expanded" into the entries $\mathtt{s}_k$ and $\mathtt{s}_{k'}$. Changing the perspective, the entries $\mathtt{s}_k$ and $\mathtt{s}_{k'}$ of $\mathfrak{s}$ are "contracted" into the entry $\mathtt{r}_l$ of $\mathfrak{r}$.

   The length of the sequences does not, in the general case, behave as expected: if both sequences are infinite, then expansion and contraction are of the same length; even in the finite case, a contraction can be much longer than the corresponding expansion, namely if the relevant positions of the contraction are greater than those of the expansion.

   Nevertheless, the relevant positions (of both sequences) are finite, and the number of the relevant positions of the contraction is smaller than that of the expansion.

2. *contractibility:* Every sequence $\mathfrak{r}$ can be expanded (according to arbitrary simple homomorphisms and with respect to arbitrary nominal terms) without any problem. But the contraction of a sequence $\mathfrak{s}$ according to a simple homomorphism F and with respect to a nominal term $\mathtt{t}$ can be problematic, namely if we would have to "contract" two different positions with differing entries into a single entry of a contraction. Exactly this problem is excluded by the property of contractibility.

Example (Expansion and Contraction): We provide a detailed example illustrating our terminology.

1. *simple homomorphism:* Let F be the simple homomorphism induced by:

$$*_0 \mapsto *_1 \quad ; \quad *_{k'} \mapsto *_k$$

2. *nominal term:* Let $\mathtt{t} \doteq *_0 + (*_1 + *_2)$; therefore, $F(\mathtt{t}) \doteq *_1 + (*_0 + *_1)$.

3. *expansion:* Investigate $\mathfrak{r} = \langle \mathtt{r}_0, \mathtt{r}_1, \ldots \rangle$.

   If we intend to expand the sequence $\mathfrak{r}$ according to F with respect to $\mathtt{t}$, then $\{0, 1\}$ is the set of relevant positions of $\mathfrak{r}$. The relevant positions in an expansion $\mathfrak{s}$ of $\mathfrak{r}$ are given by the set $\{0, 1, 2\}$. We provide the relevant entries of such an expansion:

$$\mathfrak{s} \;=\; \langle \mathtt{s}_0, \mathtt{s}_1, \mathtt{s}_2, \ldots \rangle \;=\; \langle \mathtt{r}_1, \mathtt{r}_0, \mathtt{r}_1, \ldots \rangle$$

Observe that the entry $\mathtt{r}_1$ of $\mathfrak{r}$ is expanded into the entries at the positions 0 and 2 of $\mathfrak{s}$.

4. *contractibility:* Let $\mathfrak{s} = \langle \mathtt{s}_0, \mathtt{s}_1, \mathtt{s}_2, \ldots \rangle$ be a sequence, which we intend to contract according to F with respect to $\mathtt{t}$.

   Contractibility demands that $\mathtt{s}_0 \eqsim \mathtt{s}_2$, as $F(*_0) \eqsim *_1 \eqsim F(*_2)$.

5. *general substitution function:* We already can observe the relevance of our terminology with respect to applications of the general substitution function:

$$\begin{aligned} \mathtt{t}[\mathfrak{s}] &\eqsim (*_0 + (*_1 + *_2))[\mathtt{r}_1, \mathtt{r}_0, \mathtt{r}_1] \\ &\eqsim \mathtt{r}_1 + (\mathtt{r}_0 + \mathtt{r}_1) \\ &\eqsim (*_1 + (*_0 + *_1))[\mathtt{r}_0, \mathtt{r}_1] \eqsim F(\mathtt{t})[\mathfrak{r}] \end{aligned}$$

   Subsequently, we provide a precise formulation of this observation and prove that statement.

Simple Observations (Expansion and Contraction): We discuss some observations about expansions and contractions. For this purpose, let $\mathtt{t} \in \mathtt{T}$ and $F \in \mathbf{Hom}_s(\mathtt{T})$, $\mathfrak{s}$ and $\mathfrak{r} \in \mathtt{T}^\omega$.

1. *converse criterion:* Using a little bit sloppy notation, we can provide a converse criterion for expansions and contractions:

   Let $\mathfrak{s}$ be F-contractible with respect to $\mathtt{t}$. The sequence $\mathfrak{r}$ is a contraction of the sequence $\mathfrak{s}$, if the following condition is satisfied for all relevant positions $l \in \mathbf{place}(F(\mathtt{t}))$ of $\mathfrak{r}$:

$$\mathtt{r}_l \eqsim \mathtt{s}_{F^{-1}(l)}$$

   Here, $F^{-1}(l)$ can be made precise as $\min(k \in \mathbf{place}(\mathtt{t});\ F(*_k) \eqsim *_l)$.[78]

2. *standard terms:* Let $\mathtt{t} \in \mathtt{T}_0$ be standard. As there are no relevant positions, all sequences $\mathfrak{s}$ are trivially F-expansions (and F-contractions) of all sequences $\mathfrak{r}$ with respect to $\mathtt{t}$. (Recall that both $\mathbf{place}(\mathtt{t}) = \emptyset$ and $F(\mathtt{t}) \eqsim \mathtt{t}$ for standard terms $\mathtt{t}$.)

[78] Observe that F is, in general not invertible, not even if we restrict F to the nominal terms occurring in $F(\mathtt{t})$. Using here the notation of the inverse function makes insofar sense, as we are not interested in the preimage of F, but in the entries of the sequence $\mathfrak{s}$ determined by each of the preimages of F. Due to contractibility, these entries are syntactically equal.

3. *essential equality:* Essential equality is compatible with the concepts of expansions and contractions. More precisely:

   Let $\mathfrak{s}$ be an F-expansion of the sequence $\mathfrak{r}$ with respect to $\mathtt{t}$. A sequence $\mathfrak{s}'$ is an F-expansion of a sequence $\mathfrak{r}'$ with respect to $\mathtt{t}$, if and only if the following both conditions are satisfied:

   (a) *expansion:* $\mathfrak{s} \equiv_{\mathtt{t}} \mathfrak{s}'$

   (b) *contraction:* $\mathfrak{r} \equiv_{F(\mathtt{t})} \mathfrak{r}'$

   The stated equivalence holds immediately by definition.

4. *isomorphism:* Let $F \in \mathbf{Hom}^{\circ}(\mathtt{T})$ be an isomorphism. The following both statements are equivalent:

   (a) $\mathfrak{s}$ is an F-expansion of $\mathfrak{r}$ with respect to $\mathtt{t}$.

   (b) $\mathfrak{r}$ is an F^{-1}-expansion of $\mathfrak{s}$ with respect to $F(\mathtt{t})$.

   Assuming (a) we obtain (b), as $\mathtt{r}_l \doteq \mathtt{r}_{F(F^{-1}(l))} \doteq \mathtt{s}_{F^{-1}(l)}$ for all relevant positions $l \in \mathbf{place}(F(\mathtt{t})$ of $\mathfrak{r}$; the second equation holds, as the position $F^{-1}(l) \in \mathbf{place}(\mathtt{t})$ is relevant in $\mathfrak{s}$. The other direction holds, as F is the inverse of the isomorphism F^{-1}.

   In particular, the relevant entries of an F-expansion are a rearrangement of the relevant entries of the respective F-contraction (at shifted positions). We can improve this observation by formulating more restrictions on F:

   (a) If $F_{|\mathbf{V}_*(\mathtt{t})}$ is a permutation of $\mathbf{V}_*(\mathtt{t})$, then the rearrangement takes place at the same positions.

   (b) If that restriction is the identity function, then the relevant positions are pointwise equal. The latter means that F-expansion and F-contraction are essentially equal sequences with respect to $\mathtt{t}$.

In the next proposition, we provide the criterion for the equality of applications of the general substitution function on a nominal terms and on their images under a simple homomorphism.

6.5 Proposition (Expansions and Contractions): Let $\mathtt{t} \in \mathtt{T}$ a nominal term, $F \in \mathbf{Hom}_s(\mathtt{T})$ a simple homomorphism and $\mathfrak{s}, \mathfrak{r} \in \mathtt{T}^{\omega}$ two sequences of nominal terms. Then the following two statements are equivalent.

1. *equality:* $\mathtt{t}[\mathfrak{s}] \doteq F(\mathtt{t})[\mathfrak{r}]$

2. *expansion:* $\mathfrak{s}$ is an F-expansion of $\mathfrak{r}$ with respect to $\mathtt{t}$.

Proof. By induction over the structure of $\mathtt{t}$.

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* Let $\mathtt{t} \doteq *_k \in \mathbf{V}_*$. As F is simple, we have that $F(\mathtt{t}) \doteq *_l$ for a number $l \in \omega$. Furthermore, $\mathtt{t}[\mathfrak{s}] \doteq \mathtt{s}_k$ and $F(\mathtt{t})[\mathfrak{r}] \doteq \mathtt{r}_l$.

   As $\mathbf{place}(\mathtt{t}) = \{k\}$, we have that $\mathfrak{s}$ is an F-expansion of $\mathfrak{r}$, if and only if $\mathtt{s}_k \doteq \mathtt{r}_{F(k)} \doteq \mathtt{r}_l$. The latter means that $\mathtt{t}[\mathfrak{s}] \doteq F(\mathtt{t})[\mathfrak{r}]$. Therefore, both statements are equivalent.

2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* As $\mathtt{t} \notin \mathbf{V}_*$, but atomic, we have that $\mathtt{t}$ is a standard term. Therefore, both statements (1) and (2) hold trivially, which means that they are equivalent.

3. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_{n-1})$ *complex:* First, observe that the following statements hold:

   (a) $\mathtt{t}[\mathfrak{s}] \doteq F(\mathtt{t})[\mathfrak{r}]$, if and only if $\mathtt{t}_k[\mathfrak{s}] \doteq F(\mathtt{t}_k)[\mathfrak{r}]$ for all $k \in n'$.

   (b) $\mathfrak{s}$ is an F-expansion of $\mathfrak{r}$ with respect to $\mathtt{t}$, if and only if $\mathfrak{s}$ is an F-expansion of $\mathfrak{r}$ with respect to $\mathtt{t}_k$ for all $k \in n'$.

   Applying n' many times induction hypothesis, we also obtain that the following statement holds for all $k \in n'$:

   (a) $\mathtt{t}_k[\mathfrak{s}] \doteq F(\mathtt{t}_k)[\mathfrak{r}]$, if and only if $\mathfrak{s}$ is an F-expansion of $\mathfrak{r}$ with respect to $\mathtt{t}_k$.

   Putting the pieces together, we obtain that statement (1) with respect to $\mathtt{t}$ is equivalent to statement (2) with respect to $\mathtt{t}$, as demanded. Q.E.D.

Remark (Generalisation): We mention that the concepts and results in this section can be generalised. As expansion and contraction depend only on the behaviour of the simple homomorphism F on the nominal symbols occurring in $\mathtt{t}$, we can carry over the definition and our observations to arbitrary homomorphisms locally agreeing with F; the latter means to all homomorphisms essentially equal to F with respect to $\mathtt{t}$.

7 Relations Based on Homomorphisms

We discuss two relations both based on homomorphisms, namely the *isomorphism of nominal terms* and the *less-structured relation*.

7.1 The Isomorphism of Nominal Terms

The isomorphisms induce canonically an equivalence relation on the set of nominal terms: two nominal terms are called isomorphic, if there is an isomorphism mapping one to the other.[79]

7.1.1 Introduction of the Isomorphism of Nominal Terms

We provide the formal definition of the isomorphism of nominal terms.

7.1 DEF (Isomorphism of Nominal Terms): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ arbitrary. The nominal terms $\mathtt{t}$ and $\mathtt{s}$ are *isomorphic* (formally, $\mathtt{t} \cong \mathtt{s}$), if there is an isomorphism $F \in \mathbf{Hom}^{\circ}(\mathtt{T})$ such that $F(\mathtt{t}) \doteq \mathtt{s}$.

Examples (Isomorphic Nominal Terms): We provide some examples illustrating the concept of isomorphic nominal terms in the language $\mathfrak{L}_{\mathrm{PA}}$ of arithmetics. For this purpose, let $\mathtt{t} \doteq *_0 + (*_1 + *_0)$.

1. We investigate a nominal term $\mathtt{s} \doteq *_k + (*_l + *_m)$. We have $\mathtt{t} \cong \mathtt{s}$, if and only if there is a permutation $\pi \in \mathbf{Sym}(\omega)$ such that $\pi(0) = k$, $\pi(0) = m$ and $\pi(1) = l$. The latter is the case, if and only if $k = m$ and $l \neq k$ (if $l = k$ or $l = m$, then π is not injective).[80]

2. Let $\mathtt{s} \doteq (*_k + *_l) + *_m$. We have that $\mathtt{t} \ncong \mathtt{s}$. (One reason is that the right direct subterm of $\mathtt{t}$ is complex; applying a homomorphism on $\mathtt{t}$ maps the right direct subterm to a complex nominal term – but the right direct subterm of $\mathtt{s}$ is atomic. As a consequence, there is not even a homomorphism F such that $F(\mathtt{t}) \doteq \mathtt{s}$.)

[79] The less-structured relation is introduced analogously, but with respect to the more general notion of arbitrary homomorphisms.

[80] Due to the bijective correspondence between simple homomorphisms determined by their restriction on the set of nominal symbols and the natural functions, it is sufficient to discuss here permutation on ω.

Remarks (Isomorphic Nominal Terms):

1. *relabelling nominal terms:* Intuitively, it is clear that isomorphic nominal terms are equal modulo the relabelling of their nominal symbols according to a permutation of the set natural numbers (and not necessarily of the free places of that nominal term).

2. *equivalence relation:* As the function space $\langle \mathbf{Hom}^{\circ}(\mathtt{T}), \circ, \mathbf{id}_{\mathtt{T}} \rangle$ is, in particular, an algebraic group, the isomorphism $\cong$ of nominal terms is an equivalence relation on the set $\mathtt{T}$ of nominal terms:

   (a) *reflexive:* As $\mathbf{id}_{\mathtt{T}} \in \mathbf{Hom}^{\circ}(\mathtt{T})$, $\cong$ is reflexive.

   (b) *symmetric:* As $F^{-1} \in \mathbf{Hom}^{\circ}(\mathtt{T})$ for all $F \in \mathbf{Hom}^{\circ}(\mathtt{T})$, $\cong$ is symmetric.

   (c) *transitive:* As $F \circ G \in \mathbf{Hom}^{\circ}(\mathtt{T})$ for all $F, G \in \mathbf{Hom}^{\circ}(\mathtt{T})$, $\cong$ is transitive.

As a first result, we show that the restriction of the isomorphism of nominal terms to the set $\mathtt{T}^* = \mathtt{T}_0 \cup \mathtt{T}_1$ of all standard terms and unary nominal terms coincides with syntactic equality.

7.2 Proposition (Restriction to $\mathtt{T}^*$): The following statement holds for all nominal terms $\mathtt{t}, \mathtt{s} \in \mathtt{T}^*$:

$$\mathtt{t} \cong \mathtt{s} \quad \Leftrightarrow \quad \mathtt{t} \doteq \mathtt{s}$$

In particular, the equivalence classes with respect to $\cong_{|\mathtt{T}^*}$ are singletons.
Proof. Let $\mathtt{t} \cong \mathtt{s}$ for $\mathtt{t}, \mathtt{s} \in \mathtt{T}^*$. This means that there is an isomorphism $F \in \mathbf{Hom}^{\circ}(\mathtt{T})$ such that $F(\mathtt{t}) \doteq \mathtt{s}$. If $\mathtt{t} \in \mathtt{T}_0$ is a standard term, then we immediately obtain that $\mathtt{t} \doteq \mathtt{s}$ (invariance on standard terms). Otherwise, $\mathtt{t} \in \mathtt{T}_1$ is unary. Due to the proposition about the free places under substitution (in its version for homomorphisms) and that $F_{|\mathbf{V}_*} \in \mathbf{Sym}(\mathbf{V}_*)$, we obtain:

$$\begin{aligned} \mathbf{place}(\mathtt{s}) &= \mathbf{place}(F(\mathtt{t})) \\ &= \bigcup_{k \in \mathbf{place}(\mathtt{t})} \mathbf{place}(F(*_k)) \\ &= \mathbf{place}(F(*_0)) = \mathbf{place}(*_k) = \{k\} \end{aligned}$$

Therefore, $\mathtt{s}$ is not a standard term. As $\mathtt{s} \in \mathtt{T}^*$, we obtain that $\mathtt{s}$ is unary, and therefore that $k = 0$. As, $F(*_0) = \mathbf{id}_{\mathtt{T}}(*_0)$, we obtain via essential equality:

$$\mathtt{s} \;\hat{=}\; F(\mathtt{t}) \;\hat{=}\; \mathbf{id}_{\mathtt{T}}(\mathtt{t}) \;\hat{=}\; \mathtt{t}$$

This completes the direction "$\Rightarrow$"; the converse direction holds, as $\cong$ is, in particular, reflexive. The triviality of the equivalence classes is immediate.

Q.E.D.

7.1.2 Sequences of Indices

In order to simplify the discussion of isomorphic nominal terms, we introduce subsequently *sequences of indices* providing the indices of nominal symbols occurring in a nominal term. In order to define such sequence, we define first a special minimum function for nominal terms providing, essentially, the index of its leftmost nominal symbol.

7.3 DEF (Minimum Function): The *minimum function* $\min : \mathtt{T} \to \omega'$ for nominal terms is defined recursively as follows:

1. $\mathtt{t} \;\hat{=}\; *_k \in \mathbf{V}_*$: $\min(\mathtt{t}) = k$
2. $\mathtt{t}$ *standard atomic:* $\min(\mathtt{t}) = \omega$
3. $\mathtt{t} \;\hat{=}\; f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:*

$$\min(\mathtt{t}) \;=\; \begin{cases} \omega & \text{if } \min(\mathtt{t}_k) = \omega \text{ for all } k \in n' \\ \min(\mathtt{t}_k) & \text{otherwise, for suitable } k \end{cases}$$

In the second clause, k is suitable, if k is the smallest index such that the minimum of the respective direct subterm is finite. More formally, with the help of the minimum function for natural numbers:

$$k \;=\; \min(l \in n';\; \min(\mathtt{t}_l) \neq \omega)$$

Remarks (Minimum Function):

1. *maximum function:* An analogous maximum function $\max : \mathtt{T} \to \omega'$ mapping a nominal term to the index of its rightmost nominal symbol can be defined analogously to the definition of the minimum function.

Simple Observations (Minimum Function):

1. *standard terms:* A nominal term $\mathtt{t}$ is standard, if and only if its minimum is infinite (formally, if and only if $\min(\mathtt{t}) = \omega$).

2. *free places:* If $\mathtt{t} \notin \mathtt{T}_0$ is a proper nominal term, then $\min(\mathtt{t}) \in \mathbf{place}(\mathtt{t})$.

3. *minimum function under substitution:* Let $F \in \mathbf{Hom}(\mathtt{T})$ be an arbitrary homomorphism. If $\mathtt{t} \notin \mathtt{T}_0$ is a proper nominal term, then we can provide the minimum of the result of an application of F on $\mathtt{t}$ as follows:
$$\min(F(\mathtt{t})) \;=\; \min(F(*_k))$$
Here, $k = \min(\mathtt{t})$ is the index of the leftmost nominal symbol in $\mathtt{t}$. If F is a simple, then we can simplify the result:
$$\min(F(\mathtt{t})) \;= !l \in \omega :\; F(*_k) \mathrel{\widehat{=}} *_l$$
Observe that if $\mathtt{t}$ is a standard term, then $\min(F(\mathtt{t})) = \omega$ for all homomorphisms F, as $F(\mathtt{t}) \mathrel{\widehat{=}} \mathtt{t}$ is standard.

Determining all Nominal Symbols: Via the minimum function for nominal terms, we can identify all nominal symbols occurring in a nominal term in the order of their leftmost occurrences.

1. We presuppose that the first k-many indices $n_0, \ldots n_{k-1}$ of the nominal term $\mathtt{t}$ are already determined.

2. Let F_k be the uniquely determined homomorphism induced by the following function F on nominal symbols:
$$F : \mathbf{V}_* \to \mathtt{T} : *_l \;\mapsto\; \begin{cases} x_l & \text{if } l \in \{n_0, \ldots n_{k-1}\} \\ *_l & \text{otherwise} \end{cases}$$

3. Investigate the minimum of $F_k(\mathtt{t})$: If $\min(F_k(\mathtt{t})) = \omega$, then we have already determined all indices of nominal symbols in $\mathtt{t}$ in the order of their leftmost occurrences.

   Otherwise, $\min(F_k(\mathtt{t})) = l \in \omega$. As $n_0, \ldots n_{k-1}$ are the first k-many nominal symbols in $\mathtt{t}$ and as if these nominal symbols are replaced by standard terms in $F_k(\mathtt{t})$, $n_l = l$ is the next index of a nominal symbol according to their leftmost occurrences. We proceed with step (1).

As nominal terms are finite, the algorithm stops after finitely many steps.

The sequence of indices of a nominal term is generated according to the algorithm presented above and contains the indices of all nominal symbols occurring in that nominal term in the order of their leftmost occurrences. We provide the formal definition of the sequence of indices.

7.4 DEF (Sequence of Indices): The *sequence* $\sigma(\mathtt{t})$ *of the indices* of a nominal term $\mathtt{t} \in \mathtt{T}$ is defined with the help of intermediate sequences $\sigma(\mathtt{t}, k)$. The latter are defined recursively on $k \in \omega$:

1. $k = 0$*:* $\sigma(\mathtt{t}, 0)$ is the empty sequence (formally, $\sigma(\mathtt{t}, 0) = \epsilon$).

2. k'*:* We assume that $\sigma(\mathtt{t}, k) = \langle n_0, \dots n_{k-1} \rangle$ is already defined. Let F_k be the homomorphism determined by:

$$F : \mathbf{V}_* \to \mathtt{T} : *_l \ \mapsto \ \begin{cases} x_l & \text{if } l \in \{n_0, \dots n_{k-1}\} \\ *_l & \text{otherwise} \end{cases}$$

We define as follows:

$$\sigma(\mathtt{t}, k') \ = \ \begin{cases} \sigma(\mathtt{t}, k) & \text{if } \min(F_k(\mathtt{t})) = \omega \\ \langle \sigma(\mathtt{t}, k), \min(F_k(\mathtt{t})) \rangle & \text{otherwise} \end{cases}$$

Finally, we define: $\sigma(\mathtt{t}) = \bigcup_{k \in \omega} \sigma(\mathtt{t}, k)$.

Remarks (Sequence of Indices):

1. *finiteness:* As long as $k < |\mathbf{place}(\mathtt{t})|$, the intermediate sequences are proper initial segments of the next intermediate sequence. This changes for $l > k = |\mathbf{place}(\mathtt{t})|$: all sequences are equal (formally, $\sigma(\mathtt{t}, k) = \sigma(\mathtt{t}, l)$). As a consequence, $\sigma(\mathtt{t})$ is a finite sequence of natural numbers.

2. *entries:* The entries of $\sigma(\mathtt{t})$ are the free places of $\mathtt{t}$ in the order of their leftmost occurrence in $\mathtt{t}$ (which means that the entries are not necessarily sorted in their natural order).

3. n'*-ary:* If $\mathtt{t}$ is n'-ary (for a suitable $n \in \omega$), then $F_{n'}$ (as defined in the definition above) is determined by a the following function F on the set of all nominal symbols:

$$*_k \ \mapsto \ x_k \quad \text{for } k \in n' \quad ; \quad *_k \ \mapsto \ *_l \quad \text{for } k \notin n'$$

This means that $F_{n'}$ is the restriction of the general substitution function to the sequence $\langle x_0, \dots x_n \rangle$.

4. *rightmost occurrences:* If we would use a maximum function, as mentioned above, then we could determine the indices of the nominal symbols according to the order of their rightmost occurrences (from the right to the left). Observe that the sequence of the indices with respect to both functions are not necessarily converse to each other. Investigate the following nominal term in the language of arithmetics:

$$(*_0 + *_1) + (*_0 + *_0)$$

We obtain in both cases the sequence $\sigma(\mathtt{t}) = \langle 0, 1\rangle$.

In the next proposition, we discuss how the sequence of indices is influenced by isomorphisms.

7.5 Proposition (Sequence of Indices): Let $\mathtt{t} \in \mathtt{T}$ an arbitrary nominal term with sequence $\sigma(\mathtt{t}) = \langle n_0, \ldots n_{k-1}\rangle$ for $(k \in \omega)$ and $F \in \mathbf{Hom}^{\circ}(\mathtt{T})$ an isomorphism. The sequence $\sigma(F(\mathtt{t}))$ of indices of the nominal term $F(\mathtt{t})$ can be calculated as follows:

$$\sigma(F(\mathtt{t})) \;=\; \langle F(n_0), \ldots F(n_{k-1})\rangle$$

Proof. By induction over the length k of the sequences $\sigma(\mathtt{t})$ of indices:

1. *0:* If $\mathbf{lng}(\sigma(\mathtt{t})) = 0$, then $\mathtt{t} \in \mathtt{T}_0$ is standard. This means that $F(\mathtt{t}) \doteq \mathtt{t}$ and, therefore, $\sigma(F(\mathtt{t})) = \epsilon$. The stated equality holds trivially.

2. *k′:* Let $\mathtt{t} \in \mathtt{T}$ such that $\sigma(\mathtt{t}) = \langle n_0, \ldots n_k\rangle$ (the latter means that $\mathbf{lng}(\sigma(\mathtt{t})) = k'$). In particular, we have that $\mathtt{t}$ is a proper nominal term. Let $\mathtt{s} \doteq F(\mathtt{t})$. We first observe that $\min(\mathtt{s}) = \min(F(\mathtt{t})) = F(n_0)$.

   Let $G = F_1$ be the homomorphism as used in the construction of $\sigma(\mathtt{t})$; this means that F_1 is induced by the following function on the set of nominal symbols:

$$*_k \mapsto \begin{cases} x_k & \text{if } k = n_0 \\ *_k & \text{otherwise} \end{cases}$$

   It is easily checked that $F \circ G = G \circ F$. We define as follows: $\mathtt{t}' \doteq G(\mathtt{t})$ and $\mathtt{s}' \doteq G(\mathtt{s})$. We calculate as follows:

$$\mathtt{s}' \;\doteq\; G(\mathtt{s}) \;\doteq\; G(F(\mathtt{t})) \;\doteq\; F(G(\mathtt{t})) \;\doteq\; F(\mathtt{t}')$$

   The latter means that $\mathtt{t}' \cong \mathtt{s}'$ via the isomorphism F. As we replaced the leftmost nominal symbol of $\mathtt{t}$ in the nominal term $\mathtt{t}'$ by a suitable

variable, we have $\sigma(\mathtt{t}') = \langle n_1, \dots n_k \rangle$. This means that $\sigma(\mathtt{t}')$ is a sequence of length k; therefore, we may apply induction hypothesis and obtain $\sigma(\mathtt{s}') = \langle F(n_1), \dots F(n_k) \rangle$. As $\min(\mathtt{s}) = F(n_0)$ and as we replaced in $\mathtt{s}'$ this nominal symbol by a variable, we obtain, as demanded, that $\sigma(\mathtt{s}) = \langle F(n_0), \dots F(n_k) \rangle$. Q.E.D.

With the help of the proposition above, we obtain immediately as a corollary the following criterion for the identity of isomorphic nominal terms.

7.6 Corollary (Sequences of Indices): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$. The following statement holds:

$$\mathtt{t} \doteq \mathtt{s} \quad \Leftrightarrow \quad \mathtt{t} \cong \mathtt{s} \text{ and } \sigma(\mathtt{t}) = \sigma(\mathtt{s})$$

Proof. The direction "$\Rightarrow$" is trivial; we prove the other direction. Let $\mathtt{t} \cong \mathtt{s}$ and $\sigma(\mathtt{t}) = \sigma(\mathtt{s}) = \langle n_0, \dots n_{k-1} \rangle$ for suitable $k, n_0, \dots n_{k-1} \in \omega$. According to the proposition above, we obtain:

$$\langle n_0, \dots n_{k-1} \rangle \;=\; \sigma(\mathtt{s}) \;=\; \sigma(F(\mathtt{t})) \;=\; \langle F(n_0), \dots F(n_{k-1}) \rangle$$

This means that $F_{|\mathbf{V}_*(\mathtt{t})} = \mathbf{id}_{\mathbf{V}_*(\mathtt{t})}$. Therefore, F is essentially equal to $\mathbf{id}_{\mathtt{T}}$ with respect to $\mathtt{t}$. The latter means $\mathtt{s} \doteq F(\mathtt{t}) \doteq \mathbf{id}_{\mathtt{T}}(\mathtt{t}) \doteq \mathtt{t}$. Q.E.D.

7.1.3 Normal Nominal Terms

Via the sequences of indices, we are able to define normal nominal terms and to show that these nominal terms are the canonical representatives of the equivalence class with respect to the isomorphism of nominal terms.

7.7 DEF (Normal Nominal Term): A nominal term $\mathtt{t} \in \mathtt{T}$ is called *normal* (with respect to the isomorphism of nominal terms), if its sequence of indices is an initial segment of the sequence $\langle k;\ k \in \omega \rangle$ of natural numbers. More formally, if $\sigma(\mathtt{t}) = \langle k;\ k \in \mathbf{lng}(\sigma(\mathtt{t})) \rangle$.

Remarks (Normal Nominal Terms):

1. *characterisation:* A nominal term $\mathtt{t}$ is normal, if and only if $\mathtt{t}$ is n-ary (for $n \in \omega$) and if the labels of the leftmost occurrences of its nominal symbols are sorted according to the natural order.

2. *special cases:* All standard terms t are normal; all unary nominal terms are normal. This observation is as expected, as the equivalence classes of nominal term $\mathtt{t} \in \mathtt{T}^*$ are, as we have seen already, trivial.

3. *alternative account:* We based the definition of the sequences of indices on the minimum function for nominal terms; alternatively, we could have used the analogous maximum function for nominal terms. In this scenario, we would collect in the sequence of indices the labels of the rightmost occurrences of nominal symbols (from the right to the left). Using the same definition of normal nominal forms, but with respect to the alternative sequence of indices, would result in alternative normal nominal terms.

   There are cases, where a nominal term is normal with respect to both versions of normality, as the nominal term $*_0 + (*_1 + *_0)$. But in the general case, the normal forms are different: $*_0 + *_1$ is normal with respect to our notion of normality, the nominal term $*_1 + *_0$ with respect the alternative notion. (Observe that both nominal terms are isomorphic and, therefore, representing the same equivalence class).

Normal nominal terms are the canonical representatives of equivalence classes with respect to the isomorphisms of nominal terms.

7.8 Proposition (Normal Nominal Terms): Let $\mathtt{t} \in \mathtt{T}$ be an arbitrary nominal term. There is a uniquely determined normal nominal term $\mathtt{s} \in \mathtt{T}$ isomorphic to $\mathtt{t}$.

Proof. First, we show the existence of such a nominal term. We investigate the sequence of indices of $\mathtt{t}$: $\sigma(\mathtt{t}) = \langle n_0, \dots n_{k-1} \rangle$ for suitable natural numbers $k, n_0, \dots n_{k-1} \in \omega$. By construction, the indices in $\sigma(\mathtt{t})$ are all different. We extend $\sigma(\mathtt{t})$ to an infinite sequence σ of natural numbers as follows:

1. The first k entries s_l are equal to the the entries of $\sigma(\mathtt{t})$ (if $\sigma(\mathtt{t})$ is not the empty sequence).

2. The entry s_l of σ is the smallest number not yet occurring in the initial segment $\langle s_0, \dots s_{l-1} \rangle$ of σ for all $l \geq k$.

The sequence σ is a natural function. By construction, σ is injective (we extend by new entries) and surjective (investigate an arbitrary $l \in \omega$ such that l is not already an entry of $\sigma(\mathtt{t})$ and, therefore, trivially contained in σ: after finitely many steps of constructing σ, all previously unused numbers below l are used for extending $\sigma(\mathtt{t})$ and we have to use l in the next extension

step). σ corresponds to a permutation G_0 on the set of nominal symbols; let F_0 be the inverse function to G_0, and F the homomorphism induced by F_0. Obviously, F is an isomorphism. By construction, we have $F(*_{n_l}) = *_l$ for all $l \in k$ (for all n_l contained in $\sigma(\mathtt{t})$). As a consequence, $\sigma(\mathtt{s}) = \langle 0, 1, \ldots k-1 \rangle$ for $\mathtt{s} \doteq F(\mathtt{t})$. The latter means that $\mathtt{s}$ is normal. By construction, we also have that $\mathtt{t} \cong \mathtt{s}$, and the existence of the canonical representative is proved. We obtain uniqueness from the corollary to the proposition about the sequence of indices: if $\mathtt{r}$ is normal and isomorphic to $\mathtt{t}$, it is also isomorphic to $\mathtt{s}$ and as $\mathtt{r}$ is normal, we also have $\sigma(\mathtt{s}) = \sigma(\mathtt{r})$. Therefore, $\mathtt{s} \doteq \mathtt{r}$. Q.E.D.

Alternative Approach (Normal Form): In the proof above, we have used the injective fragment $\sigma(\mathtt{t})$ of a natural function to generate an isomorphism and the inverse isomorphism was used to map $\mathtt{t}$ to its normal form. Alternatively, we could have proceeded more directly by using transpositions (or more precisely, homomorphisms induced by transpositions or the identity function) with the following algorithm:

1. If $\min(\mathtt{t}) = \omega$, then $\mathtt{t}$ is a standard term, and nothing more is to be done; $\mathtt{t}$ is its own normal form. Otherwise, let $\alpha = \min(\mathtt{t})$. If $n \neq 0$, then $T = (*_0 *_\alpha)$ is a transposition of nominal symbols mapping $*_0$ to $*_\alpha$ and vice versa. Let T_0 be the isomorphism induced by T and $\mathtt{s}_0 \doteq T(\mathtt{t})$. (If $n = 0$, then we use $T_0 = \mathbf{id}_{\mathtt{T}}$.) Observe that $\min(\mathtt{s}_0) = 0$, and the first nominal symbol is sorted in $\mathtt{s}_0$.

2. Having sorted this way the first k'-many nominal symbols, we proceed as follows: Let $\alpha = \min(\mathtt{s}_k[x_0, \ldots x_k])$. If $\alpha = \omega$ then nothing more is to be done; the nominal term $\mathtt{s}_k$ is the demanded normal form of $\mathtt{t}$. If $\alpha = k'$, then $T_{k'} = \mathbf{id}_{\mathtt{T}}$, otherwise let $T_{k'}$ be induced by the transposition $T = (*_{k'} *_\alpha)$. (Observe that $\alpha < k'$ is not possible.) Then the nominal symbol $*_{k'}$ is, additionally, sorted in $\mathtt{s}_{k'} \doteq T_{k'}(\mathtt{s}_k)$.

The normal form $\mathtt{s}$ of $\mathtt{t}$ is directly constructed by the algorithm. But the isomorphism F mapping $\mathtt{t}$ to $\mathtt{s}$ is only implicitly given: in order to obtain F, we have to compose all transpositions F_k used in the algorithm. Observe that, in general, this isomorphism is different from that used in the proof of the proposition above. Nevertheless, the normal forms are equal; both isomorphisms are essentially equal with respect to $\mathtt{t}$.

7.2 The Less-Structured Relation

We discuss the *less-structured* relation, which is defined analogously to the isomorphism of nominal terms, but on the base of arbitrary homomorphisms.

7.2.1 Introduction of the Less-Structured Relation

We provide the formal definition of the less-structured relation.

7.9 DEF (Less-Structured Relation): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$. The nominal term $\mathtt{t}$ is *less-structured* than the nominal term $\mathtt{s}$ (formally, $\mathtt{t} \leq \mathtt{s}$), if there is a homomorphism $F \in \mathbf{Hom}(\mathtt{T})$ such that $F(\mathtt{t}) \eqsim \mathtt{s}$.

Remarks (Less-Structured Relation):

1. *terminology:* The denomination of the less-structured relation is motivated by the idea that replacing a nominal symbol in a nominal term by a (more complex) nominal term results in a nominal term having more structure (in terms of the symbols of the underlying formal language).

2. *alternative characterisation:* We may equivalently characterise the less-structured relation as follows:

$$\mathtt{t} \leq \mathtt{s} \quad \Leftrightarrow \quad \exists \mathfrak{t} \in \mathtt{T}^{\omega} : \ \mathtt{t}[\mathfrak{t}] \eqsim \mathtt{s}$$

3. *isomorphic nominal terms:* By definition (and due to symmetry), we have immediately that isomorphic nominal terms are also related by the less-structured relation. More formally:

$$\mathtt{t} \cong \mathtt{s} \quad \Rightarrow \quad \mathtt{t} \leq \mathtt{s} \text{ and } \mathtt{s} \leq \mathtt{t}$$

   Subsequently, we prove that the other direction also holds.

Examples (Less-Structured Relation): We provide some examples of less-structured nominal terms in the language $\mathfrak{L}_{\mathrm{PA}}$ of arithmetics.

$$* \ \leq \ *[* + *] \eqsim * + * \ \leq \ * + *[0] \eqsim 0 + 0 \quad ; \quad 0 + 0 \ \not\leq \ * + * \ \not\leq \ *$$

The first inequality holds, as $F(0+0) \eqsim 0+0 \neq * + *$ for all homomorphisms F; the second, as the preimage of $*$ under a homomorphism is a nominal symbol and, therefore, not $* + *$.

7.2.2 Criterion for Isomorphic Nominal Terns

Before we discuss the properties of the less-structured relation, we provide a criterion for the isomorphism of nominal terms based on the less-structured relation.

7.10 Proposition (Criterion - Isomorphic Nominal Terms): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ be two nominal terms. The nominal terms $\mathtt{t}$ and $\mathtt{s}$ are isomorphic, if and only if $\mathtt{t}$ is less-structured than $\mathtt{s}$ and vice versa. Formally:

$$\mathtt{t} \cong \mathtt{s} \quad \Leftrightarrow \quad \mathtt{t} \leq \mathtt{s} \text{ and } \mathtt{s} \leq \mathtt{t}$$

Proof. As stated before, the direction "$\Rightarrow$" is immediate; therefore, it is sufficient to prove "$\Leftarrow$" by induction over the structure of $\mathtt{t}$.

1. $\mathtt{t} \notin \mathbf{V}_*$ *standard atomic:* The condition $\mathtt{t} \leq \mathtt{s}$ already implies that $\mathtt{s} \doteq \mathtt{t}$, as $\mathtt{t}$ is standard. In particular, $\mathtt{t} \cong \mathtt{s}$.

2. $\mathtt{t} \in \mathbf{V}_*$ *nominal symbol:* Let $\mathtt{t} \doteq *_k$. As $\mathtt{s} \leq \mathtt{t}$, there is a homomorphism $F \in \mathbf{Hom}(\mathtt{T})$ such that $F(\mathtt{s}) \doteq \mathtt{t}$. We obtain by the categorisation of the preimage that $\mathtt{s} \doteq *_l \in \mathbf{V}_*$ is also a nominal term. If $\mathtt{t} \doteq \mathtt{s}$ nothing more is to be done. Otherwise, the transposition $T_0 = (*_k\, *_l)$ induces an isomorphism $T \in \mathbf{Hom}^\circ(\mathtt{T})$ witnessing $\mathtt{t} \cong \mathtt{s}$.

3. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* As $\mathtt{t} \leq \mathtt{s}$ and $\mathtt{s} \leq \mathtt{t}$, there are two homomorphisms $F, G \in \mathbf{Hom}(\mathtt{T})$ such that $F(\mathtt{t}) \doteq \mathtt{s}$ and $G(\mathtt{s}) \doteq \mathtt{t}$. As homomorphisms are structure preserving, we obtain that $\mathtt{t}$ and $\mathtt{s}$ are similar. The latter means that there are nominal terms $\mathtt{s}_0, \ldots \mathtt{s}_n$ such that $\mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$. In particular, $F(\mathtt{t}_k) \doteq \mathtt{s}_k$ and $G(\mathtt{s}_k) \doteq \mathtt{t}_k$ for all $k \in n'$. The latter means that we have both that $\mathtt{t}_k \leq \mathtt{s}_k$ and $\mathtt{s}_k \leq \mathtt{t}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain isomorphisms $H_k \in \mathbf{Hom}^\circ(\mathtt{T})$ such that $H_k(\mathtt{t}_k) \doteq \mathtt{s}_k$ for all $k \in n'$.

   As $F(\mathtt{t}_k) \doteq H_k(\mathtt{t}_k)$ and $G(\mathtt{s}_k) \doteq H_k^{-1}(\mathtt{s}_k)$, we have that F and H_k are essentially equal with respect to $\mathtt{t}_k$ as well as G and H_k^{-1} with respect to $\mathtt{s}_k$ for all $k \in n'$.

   We investigate the restriction $F_{|\mathbf{V}_*(\mathtt{t})}$ of F to the set of nominal symbols occurring in $\mathtt{t}$:

   (a) *into* $\mathbf{V}_*(\mathtt{s})$*:* Let $l \in \mathbf{place}(\mathtt{t})$ be arbitrary. There is $k \in n'$ such that $l \in \mathbf{place}(\mathtt{t}_k)$. Due to essential equality, we obtain:

   $$F(*_l) \;\doteq\; H_k(*_l) \;\in\; \mathbf{V}_*(\mathtt{s}_k) \;\subseteq\; \mathbf{V}_*(\mathtt{s})$$

   This means that $F_{|\mathbf{V}_*(\mathtt{t})}$ is a function into the set $\mathbf{V}_*(\mathtt{s})$.

(b) *surjective:* Let $l \in \mathbf{place}(\mathtt{s})$ be arbitrary. There is $k \in n'$ such that $l \in \mathbf{place}(\mathtt{s}_k)$. As H_k is an isomorphism, there is an index $\hat{l} \in \mathbf{place}(\mathtt{t}_k) \subseteq \mathbf{place}(\mathtt{t})$ such that $H_k(*_{\hat{l}}) \doteqdot *_l$. Due to essential equality, we obtain:

$$F(*_{\hat{l}}) \;\doteqdot\; H_k(*_{\hat{l}}) \;\doteqdot\; *_l$$

As a consequence, every nominal symbol $*_l \in \mathbf{V}_*(\mathtt{s})$ is hit by the restriction $F_{|\mathbf{V}_*(\mathtt{t})}$ of F. Therefore, $F_{|\mathbf{V}_*(\mathtt{t})}$ is surjective.

(c) *injective:* Let $l, \hat{l} \in \mathbf{place}(\mathtt{t})$ such that $F(*_l) \doteqdot F(*_{\hat{l}})$. As before, there are $k, \hat{k} \in n'$ such that $l \in \mathbf{place}(\mathtt{t}_k)$ and $\hat{l} \in \mathbf{place}(\mathtt{t}_{\hat{k}})$. We calculate:

$$\begin{aligned} *_l \doteqdot H_k^{-1}(H_k(*_l)) &\doteqdot G(H_k(*_l)) \\ &\doteqdot G(F(*_l)) \\ &\doteqdot G(F(*_{\hat{l}})) \\ &\doteqdot G(H_{\hat{k}}(*_{\hat{l}})) \doteqdot H_{\hat{k}}^{-1}(H_{\hat{k}}(*_{\hat{l}})) \doteqdot *_{\hat{l}} \end{aligned}$$

As a consequence, the restriction $F_{|\mathbf{V}_*(\mathtt{t})}$ is injective.

Putting the pieces together, we have that the function

$$F_{|\mathbf{V}_*(\mathtt{t})} : \mathbf{V}_*(\mathtt{t}) \to \mathbf{V}_*(\mathtt{s})$$

is bijective. The function $F_{|\mathbf{V}_*(\mathtt{t})}$ is easily extended to a permutation $H_0 \in \mathbf{Sym}(\mathbf{V}_*)$ of the set of all nominal symbols. Finally, the permutation H_0 induces canonically an isomorphism $H \in \mathbf{Hom}^{\circ}(\mathtt{T})$. By construction, F and H are essentially equal with respect to $\mathtt{t}$. Therefore, $H(\mathtt{t}) \doteqdot F(\mathtt{t}) \doteqdot \mathtt{s}$. The latter means that $\mathtt{t} \cong \mathtt{s}$. Q.E.D.

Remark (Criterion - Isomorphic Nominal Terms): The proposition above can be seen as the less-structured analogy of Bernstein's Theorem in set theory stating that if $x \leq y$ and $y \leq x$, then $x \equiv y$, where "$\leq$" means "*to be smaller*" and where "$\equiv$" means "*to be of the same size*".[81] As in our case, these relations are defined via the existence of injective and bijective functions, respectively.

[81] The theorem is found, for example, in Jech [19, p. 28]; observe that the terminology there is slightly different and that the theorem is referenced as Cantor-Bernstein Theorem.

7.2.3 Partial Order Modulo Isomorphism

Having proved the criterion for isomorphic nominal terms, we can investigate the less-structured relation and prove, in particular, that the less-structured relation is a partial order on the set of nominal terms *modulo the isomorphism of nominal terms.*[82]

7.11 Proposition (Less-Structured Relation): The following statements hold:

1. *partial order:* The less-structured relation $\leq$ is a partial order modulo the isomorphism of nominal terms on the set of nominal terms.
2. *least elements:* The nominal symbols are the least elements with respect to the less-structured relation.
3. *maximal elements:* The standard terms are maximal with respect to the less-structured relation.

Proof. We check each property.

1. *reflexive:* We have to show that $\mathtt{t} \leq \mathtt{s}$ for all nominal terms $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ satisfying the condition $\mathtt{t} \cong \mathtt{s}$. But this is, as mentioned before, immediate.
2. *anti-symmetric:* We have to show that if $\mathtt{t} \leq \mathtt{s}$ and $\mathtt{s} \leq \mathtt{t}$, then $\mathtt{t} \cong \mathtt{s}$ for all nominal terms $\mathtt{t}, \mathtt{s} \in \mathtt{T}$. This is exactly, what was proved in the proposition above.
3. *transitive:* We have to show that if $\mathtt{t} \leq \mathtt{s}$ and $\mathtt{s} \leq \mathtt{r}$, then $\mathtt{t} \leq \mathtt{r}$ for all $\mathtt{t}, \mathtt{s}, \mathtt{r} \in \mathtt{T}$. This is trivial, as the set $\mathbf{Hom}(\mathtt{T})$ is closed under the composition of functions.
4. *least elements:* Let $\mathtt{t} \in \mathbf{V}_*$ be a nominal symbol. We have to show that $\mathtt{t} \leq \mathtt{s}$ for all nominal terms $\mathtt{s} \in \mathtt{T}$. Let $\mathtt{s} \in \mathtt{T}$ arbitrary and F be the homomorphism induced by the constant function $F_0 : *_k \mapsto \mathtt{s}$. By construction, $F(\mathtt{t}) \doteq \mathtt{s}$, and therefore $\mathtt{t} \leq \mathtt{s}$.
5. *maximal elements:* Let $t \in \mathtt{T}_0$ be a standard term. We have to show that if $t \leq \mathtt{s}$, then $t \cong \mathtt{s}$ for all nominal terms $\mathtt{s} \in \mathtt{T}$. This is trivial, as applying a homomorphism F on the standard term t always results in the standard term t. Q.E.D.

[82] The latter means that the isomorphism of nominal terms is the relevant underlying equality relation (instead of syntactic equality). Cf. the preliminaries, section § 3.1.2 about relations.

Remarks (Less-Structured Relation):

1. *standard and unary nominal terms:* If we restrict the less-structured relation to the set $\mathtt{T}^*$ of standard terms and unary nominal terms, then isomorphism of nominal terms and syntactic equality coincide. As a consequence, the restriction of the less-structured relation to this set is, indeed, a partial order.

2. *proper chains:* As a consequence of the proposition above, every pair of a nominal symbol $*_k$ and of a standard term t is a proper chain with respect to the less-structured relation. More formally:

   $$*_k \quad < \quad t$$

   There are longer proper chains, if and only if there are function symbols f available in the underlying formal language $\mathfrak{L}$. In this case, there are even infinite proper chains.

   For example, if f is a unary function symbol, then $\mathfrak{t} = \langle \mathtt{t}_k;\ k \in \omega \rangle$ is a proper chain, where the $\mathtt{t}_k$ are defined as follows:

   $$\mathtt{t}_0 \ \hat{=} \ * \quad ; \quad \mathtt{t}_{k'} \ \hat{=} \ f(\mathtt{t}_k)$$

   The sequence $\mathfrak{t}$ is a chain: let F be the homomorphism induced by the function $F_0 : \mathbf{V}_* \to \mathtt{T} : *_k \mapsto f(*_k)$. We obtain that $F(\mathtt{t}_k) \ \hat{=} \ \mathtt{t}_{k'}$ and, therefore, $\mathtt{t}_k \leq \mathtt{t}_{k'}$ for all $k \in \omega$.

   In order to prove that $\mathfrak{t}$ is a proper chain, we have to show that different entries are not isomorphic. Assume that $k \neq l$, but $\mathtt{t}_k \cong \mathtt{t}_l$ for positions $k, l \in \omega$. Observe that the sequences of indices of $\mathtt{t}_k$ and $\mathtt{t}_l$ are equal (we have both $\sigma(\mathtt{t}_k) = \langle 0 \rangle$ and $\sigma(\mathtt{t}_l) = \langle 0 \rangle$). Due to the corollary of the proposition about the sequences of indices, we obtain immediately that $\mathtt{t}_k \ \hat{=} \ \mathtt{t}_l$. But this is obviously a contradiction. Therefore, different entries in $\mathfrak{t}$ are not isomorphic and $\mathfrak{t}$ is, indeed, a proper chain.

8 Elimination Forms and Occurrences

We introduce *elimination forms* of standard terms capable of representing the position of occurrences and, based on this notion, the notion of (standard) *occurrences* of terms in terms.

8.1 Elimination Forms

The elimination forms of standard terms are nominal terms less-structured than the respective standard term. After the introduction of this notion, we discuss the complete elimination function illustrating the underlying concept of elimination and we provide the number of some specific elimination forms.

8.1.1 Introduction of Elimination Forms

We provide the formal definition of elimination forms.

8.1 DEF (Elimination Form): Let $t \in \mathtt{T}_0$ be a standard term, $\mathtt{t} \in \mathtt{T}$ a nominal term, $\alpha \in \omega'$ an ordinal and $\mathfrak{s} \in \mathtt{T}^\alpha$ a sequence of nominal terms of length α.

1. *elimination form:* The nominal term $\mathtt{t}$ is called an *elimination form* of t, if $\mathtt{t}$ is less-structured than t (formally, if $\mathtt{t} \leq t$).

2. *eliminated sequences and entries:* We say that the sequence $\mathfrak{s}$ is *eliminated* in $\mathtt{t}$ (with respect to t), if $\mathtt{t}[\mathfrak{s}] \doteq t$; in this case, we also say that the entries $\mathtt{s}_k$ are *eliminated* in $\mathtt{t}$ for all $k \in \alpha$.

   Furthermore: if $k \in \mathbf{place}(\mathtt{t})$, then the entry $\mathtt{s}_k$ is *actually* eliminated, and if all entries of $\mathfrak{s}$ are actually eliminated, then we say that the sequence $\mathfrak{s}$ is *actually* eliminated.

3. *sets of elimination forms:* Furthermore, we define the following sets of elimination forms:

   - $\mathcal{E}(t) = \{\mathtt{t} \in \mathtt{T};\ \mathtt{t} \leq t\}$ is the *set of all elimination forms* of t.
   - $\mathcal{E}(t, \mathfrak{s}) = \{\mathtt{t} \in \mathtt{T};\ \mathtt{t}[\mathfrak{s}] \doteq t\}$ is the *set of all elimination forms* of t, in which the sequence $\mathfrak{s}$ is eliminated.

Remarks (Elimination Forms):

1. *notation:* Recall our notational conventions allowing the restriction of sets of nominal terms.[83] The use of those labels is permitted with respect to sets of elimination forms as defined above.

2. *limit cases:* Every standard term t is an elimination form of itself, in which no term is actually eliminated (in which the empty sequence ϵ is actually eliminated). There are no other standard terms $s \neq t$ such that s is an elimination form of t. More formally:

$$\mathcal{E}_0(t) \;=\; \{t\}$$

   Additionally, every nominal symbol $*_k$ is a trivial elimination form of every standard term t. More formally, for all $k \in \omega$:

$$*_k \in \mathcal{E}(t)$$

3. *actually eliminated terms:* All subterms of a standard term are standard. As a consequence, we obtain:

   (a) *free places:* The free places of an elimination form must be contained in the length α of the eliminated sequence $\mathfrak{s}$ (formally, $\mathbf{place}(\mathtt{t}) \subseteq \alpha$).

   (b) *actually eliminated terms:* If a nominal term $\mathtt{s}_k$ is actually eliminated, then $\mathtt{s}_k$ is a standard term. (If we have $k \in \mathbf{place}(\mathtt{t})$, then also $\mathtt{s}_k \in \mathbf{Sub}'(\mathtt{t}[\mathfrak{s}])$.)

   (c) *actually eliminated sequence:* If a sequence $\mathfrak{s}$ is actually eliminated in $\mathtt{t}$, then $\mathbf{place}(\mathtt{t}) = \alpha$, and $\mathfrak{s}$ is a finite sequence of standard terms. As a consequence, the length of an actually eliminated sequence $\mathfrak{s}$ is bound by the number of subterms of the standard term t.

4. *alternative characterisation:* The following conditions are equivalent:

   (a) A nominal term $\mathtt{t}$ is an elimination form of a standard term t.

   (b) There is a homomorphism $F \in \mathbf{Hom}(\mathtt{T})$ such that $F(\mathtt{t}) \backsimeq t$.

   (c) There is a sequence $\mathfrak{s} \in \mathtt{T}^{\omega}$ of nominal terms such that $\mathtt{t}[\mathfrak{s}] \backsimeq t$.

   (d) There is a sequence $\mathfrak{s} \in \mathtt{T}_0{}^{\omega}$ of standard terms such that $\mathtt{t}[\mathfrak{s}] \backsimeq t$.

[83] Cf. section § 4.4 about the basic categorisation of nominal terms.

Conceptual Remarks: We discuss briefly some aspects of the concept of elimination forms.

1. *relative notion:* The notion of an elimination form is a relative notion; every nominal term is an elimination form of some standard terms, but, in the general case, not of all standard terms.

   Investigate, for example, the nominal term $\mathtt{t} \doteq * + *$ in the language $\mathfrak{L}_{\mathrm{PA}}$ of arithmetics. The nominal term $\mathtt{t}$ is an elimination form of a standard term t, if and only if there is a standard term s such that $t \doteq s + s$. In particular, $\mathtt{t}$ is neither an elimination form of atomic terms, nor of terms generated by another function symbol (as $0 \cdot 0$), nor of a sum, if the direct subterms are different (as in $0 + 1$).

2. *change of perspective:* The relativity of the notion of elimination forms corresponds to a change of perspective. We had, essentially, an upward view on nominal terms. The central question was: what can be generated out of a given nominal term via an application of the general substitution function. Discussing elimination forms means to take a downward perspective. We first chose the result (the standard term) and then we ask: what are the nominal terms such that we can generate this result by an application of the general substitution function.

3. *elimination of subterms:* The underlying idea of elimination forms is that such nominal terms are the result of the replacement of occurrences of subterms in a given standard term by suitable nominal symbols.

   This replacement cannot be given, in the general case, by a recursive function, as the intended elimination affects arbitrary occurrences. In particular, it may even happen that two different occurrences of the same shape are eliminated by different nominal symbols: investigate, for example, the nominal term $(*_0 + *_1)$ understood as an elimination form of the standard term $(0 + 0)$. There is no function mapping the standard term 0 to $*_0$ as well as to $*_1$.

   This means: in order to describe the elimination as a function, we need a function on occurrences of terms in standard terms.

4. *representing positions:* The underlying idea of these elimination forms is that they represent the positions of the actually eliminated subterms of the standard term. This way, they are able to define the position of occurrences and, therefore, occurrences themselves. Observe that it is not necessary to define a function actually eliminating the subterms, the existence of elimination forms is sufficient.

Special Elimination Forms: We briefly mention some special kinds of elimination forms.

1. *n-ary elimination forms:* Let $\mathtt{t}$ be an n-ary elimination form of a standard term t. In contrast to arbitrary elimination forms, there is a sequence $\mathfrak{s}$ of standard terms actually eliminated in $\mathtt{t}$; this sequence $\mathfrak{s}$ is uniquely determined, all entries of $\mathfrak{s}$ are subterms of t, and $\mathfrak{s}$ is of length n. Furthermore, $\mathfrak{s}$ is an initial segment of every sequence $\mathfrak{s}'$ eliminated in $\mathtt{t}$ (with respect to t).

2. *simple elimination forms:* In a simple elimination form $\mathtt{t}$ of a standard term t, each occurrence of an eliminated subterm is eliminated separately.

3. *unary elimination forms:* In a unary elimination form $\mathtt{t}$ of a standard term t, all the eliminated occurrences of a subterm are of the same shape.

8.1.2 The Complete Elimination Function

In order to illustrate the concept of the elimination of subterms, we introduce the *complete elimination function*. This recursively defined function maps a pair of standard terms to that elimination form of the first argument, in which *all* occurrences of the second argument are eliminated.[84]

8.2 DEF (Complete Elimination Function): The *complete elimination function* $\mathbf{elim} : \mathtt{T}_0 \times \mathtt{T}_0 \to \mathtt{T}^\star$ is defined recursively (in the first argument) as follows for arbitrary standard terms $s \in \mathtt{T}_0$:

1. *t atomic:* $\mathbf{elim}(t, s) = \begin{cases} * & \text{if } t \backsimeq s \\ t & \text{otherwise} \end{cases}$

2. *$t \backsimeq f(t_0, \ldots t_n)$:*

$$\mathbf{elim}(t, s) = \begin{cases} * & \text{if } t \backsimeq s \\ f(\mathbf{elim}(t_0, s), \ldots \mathbf{elim}(t_n, s)) & \text{otherwise} \end{cases}$$

An application of the complete elimination function eliminates all occurrences of the term s in the term t by replacing the standard term s in the standard term t by the nominal symbol $*$.

[84] As we eliminate *all* occurrences, this function is a simple function (according to our distinction given in the introduction) and is definable recursively.

In the next proposition we show that the complete elimination functions works as demanded.

8.3 Proposition (Complete Elimination Function): Let $t, s \in \mathtt{T}_0$ be two arbitrary standard terms and $\mathtt{t} \eqsim \mathbf{elim}(t, s)$. The following statements hold:

1. The nominal term $\mathtt{t}$ is an elimination form of t in which the term s is eliminated. Formally, $\mathtt{t}[s] \eqsim t$.
2. The nominal term is proper (and unary), if and only if s is a subterm of t. Formally, if $s \in \mathbf{Sub}'(t)$, then $\mathtt{t} \in \mathtt{T}_1$; otherwise, $\mathtt{t} \eqsim t \in \mathtt{T}_0$.
3. All occurrences of s are eliminated in $\mathtt{t}$. Formally, $s \notin \mathbf{Sub}'(\mathtt{t})$.

Proof. All statements are proved in parallel by induction over the structure of the term t:

1. *t atomic:* We distinguish two case:
   - *$t \eqsim s$:* If $t \eqsim s$, then $\mathbf{elim}(t, s) \eqsim *$. Therefore, $\mathtt{t}[s] \eqsim t$. Furthermore, $s \in \mathbf{Sub}'(t)$ and $\mathtt{t} \in \mathtt{T}_1$. Finally, $s \notin \mathbf{Sub}'(\mathtt{t})$.
   - *$t \not\eqsim s$:* If $t \not\eqsim s$, then $\mathbf{elim}(t, s) \eqsim t$. Therefore, $\mathtt{t}[s] \eqsim t$. Furthermore, $s \notin \mathbf{Sub}'(t)$ and $\mathtt{t} \eqsim t \in \mathtt{T}_0$. Finally, still $s \notin \mathbf{Sub}'(\mathtt{t})$.
2. *$t \eqsim f(t_0, \ldots t_n)$:* The case $t \eqsim s$ is treated as in the atomic case. Therefore, we may assume that $t \not\eqsim s$. Let $\mathtt{t}_k \eqsim \mathbf{elim}(t_k, s)$ for all $k \in n'$. Due to the definition of **elim**, we have:

   $$\mathbf{elim}(t, s) \;\eqsim\; f(\mathbf{elim}(t_0, s), \ldots \mathbf{elim}(t_n, s)) \;\eqsim\; f(\mathtt{t}_0, \ldots \mathtt{t}_n)$$

   Applying n'-many times induction hypothesis, we obtain that all three statements hold with respect to the $\mathtt{t}_k$. We show each statement with respect to $\mathtt{t} \eqsim f(\mathtt{t}_0, \ldots \mathtt{t}_n)$:

   (a) In order to show that $\mathtt{t}$ is an elimination form of t in which s is eliminated, we calculate as follows:

   $$\mathtt{t}[s] \;\eqsim\; f(\mathtt{t}_0[s], \ldots \mathtt{t}_n[s]) \;\eqsim\; f(t_0, \ldots t_n) \;\eqsim\; t$$

   The second equation holds, as $\mathtt{t}_k$ is an elimination forms of t_k in which s is eliminated for all $k \in n'$.

(b) We assume that $s \in \mathbf{Sub}'(t)$. As $s \not\eqsim t$, we have $s \in \mathbf{Sub}(t)$. The latter means that there is $k \in n'$ such that $s \in \mathbf{Sub}'(t_k)$. As a consequence, $\mathtt{t}_k \in \mathtt{T}_1$, and therefore $\mathtt{t} \in \mathtt{T}_1$ (recall that $\mathtt{t}_l \in \mathtt{T}_0 \cup \mathtt{T}_1$ for all $l \in n'$).

Otherwise, $s \notin \mathbf{Sub}'(t)$. Therefore, $s \notin \mathbf{Sub}'(t_k)$ for all $k \in n'$. Therefore, $\mathtt{t}_k \eqsim t_k \in \mathtt{T}_0$. The latter implies that $\mathtt{t} \eqsim t \in \mathtt{T}_0$.

(c) Due to induction hypothesis, we have that $s \notin \bigcup_{k \in n'} \mathbf{Sub}'(\mathtt{t}_k)$. As $\mathtt{t} \not\eqsim s$ (even in the case that $\mathtt{t} \eqsim t$), we obtain $s \notin \mathbf{Sub}'(\mathtt{t})$.

Q.E.D.

Generalisations (Complete Elimination Function): We discuss briefly some generalisations of the complete elimination function allowing the elimination of more than one term in a given standard term.

1. *towards a generalisation:* A natural idea to eliminate more than one term in a standard term is to apply successively the complete elimination function more than once. There are some technical problems to be solved: having applied the complete elimination function on a standard term results, in the intended case, in a unary nominal term, which is not in the domain of the complete elimination function; furthermore, we may not use in the next elimination step the nominal symbols already used for elimination.

   Even, if we solve these technical problems, there remains an undesired property: the result of the elimination of finitely many terms depends, in general, on the order of the eliminations. If $t(s_0, s_1)$ denotes the result of first eliminating s_0 in t and then eliminating s_1, then we obtain, for example:

$$\begin{aligned} 0 + 0(0, 0 + 0) \;\eqsim\; * + *(0 + 0) \;&\eqsim\; * + * \\ &\not\eqsim\; * \;\eqsim\; *(0) \;\eqsim\; 0 + 0(0 + 0, 0) \end{aligned}$$

   Observe that the resulting nominal terms are not only different, but different in an essential way: they have a different structure.[85] In particular, the results are not isomorphic.

There are some possibilities to deal with this phenomenon.

[85] In the section about relations beyond homomorphisms, we introduce the equivalence of nominal terms capturing the concept of "having the same structure".

1. *presupposed order of elimination:* We can define a complete elimination function $\mathbf{elim} : \mathtt{T}_0 \times \mathtt{T}_0^{<\omega}$ such that $\mathbf{elim}(t, \mathfrak{s})$ is the result of eliminating in t the arguments given in the sequence $\mathfrak{s}$ in a specified order. In the example above, this order was "from the left to the right".

2. *restricted arguments:* We can also avoid the problem by restricting the arguments by demanding that all arguments s and s' given in the sequence $\mathfrak{s}$ satisfy the following both conditions:

$$s \not\trianglelefteq s' \quad \text{and} \quad s' \not\trianglelefteq s$$

   In this case, the elimination of the entries of $\mathfrak{s}$ in the standard term t does not depend on the order of the separate elimination steps (up to the isomorphism of nominal terms).

3. *fixed arguments:* A variation of the solution by restricted arguments is to fix suitable arguments a priori. Actually, a complete elimination of the first n variables $v_0, \ldots v_{n-1}$ in a standard term t is of some practical use. Observe that the variables v_k satisfy the condition for restricted arguments formulated in the clause above.

8.1.3 Counting Elimination Forms

We are able to count (formally) the number of (specific) elimination forms of a given standard term.[86]

In a first proposition, we show that the number of simple unary elimination forms, in which exactly one occurrence of a given subterm is eliminated, equals to the multiplicity of that subterm in the term.

8.4 Proposition (Simple Unary Elimination Forms): Let $t, s \in \mathtt{T}_0$ be two arbitrary standard terms. The following equation holds:

$$|\mathcal{E}_{1,s}(t, s)| \;=\; \mathbf{mult}(s, t)$$

Proof. By induction over the structure of t; let $s \in \mathtt{T}_0$ be arbitrary.

1. *t atomic:* Due to the categorisation of the preimage, $*$ is the only simple unary elimination form of t. We distinguish two cases:

   - *$t \doteq s$:* If $t \doteq s$, then $*$ is the only simple unary elimination form of t in which s is eliminated. Correspondingly, $\mathbf{mult}(s, t) = 1$.

[86] As all nominal symbols are elimination forms of an arbitrary standard terms, the number of all elimination forms of a standard term is in any case infinite.

- $t \not\eqsim s$: If $t \not\eqsim s$, then there is no simple unary elimination form of t in which s is eliminated. Correspondingly, $\mathbf{mult}(s,t) = 0$.

As a consequence, we have in both cases $|\mathcal{E}_{1,s}(t,s)| = \mathbf{mult}(s,t)$.

2. $t \eqsim f(t_0, \ldots t_n)$ *complex:* If $t \eqsim s$, then again we have that $*$ is the only suitable elimination form and that $\mathbf{mult}(s,t) = 1$. Otherwise, $t \not\eqsim s$.

   In the latter case: immediately, $*$ is not an elimination form of t in which s is eliminated. This implies, due to the categorisation of the preimage, that an elimination form $\mathtt{t}$ of t (in which s is actually eliminated) satisfies $\mathtt{t} \eqsim f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ for some suitable nominal terms $\mathtt{t}_k$ (for $k \in n'$). The term s is actually eliminated in $\mathtt{t}$, if and only if there is $k \in n'$ such that s is eliminated in $\mathtt{t}_k$ (with respect to t_k). Furthermore, $\mathtt{t}$ is additionally simple, if and only if there is only one such $k \in n'$ and if the respective nominal term $\mathtt{t}_k$ is simple. Therefore:

$$|\mathcal{E}_{1,s}(t,s)| \;=\; \sum_{k \in n'} |\mathcal{E}_{1,s}(t_k,s)|$$

   Applying n'-many times induction hypothesis and recalling the definition of the multiplicity function, we continue calculating and obtain:

$$|\mathcal{E}_{1,s}(t,s)| \;=\; \sum_{k \in n'} \mathbf{mult}(s,t_k) \;=\; \mathbf{mult}(s,t)$$

Q.E.D.

All Simple Unary Elimination Forms: As an immediate consequence of the proposition above, the number of all simple unary elimination forms of a standard term t is given as follows:

$$|\mathcal{E}_{1,s}(t)| \;=\; \sum_{s \in \mathtt{T}_0} \mathbf{mult}(s,t) \;=\; \sum_{s \in \mathbf{Sub}'(t)} \mathbf{mult}(s,t)$$

We complement the result above and provide in the next proposition the number of all elimination forms of a standard term, in which a given subterm is eliminated.

8.5 Proposition (Unary Elimination Forms): Let $t, s \in \mathtt{T}_0$ be two arbitrary standard terms. The following equation holds:

$$|\mathcal{E}_1(t,s)| \;=\; 2^{\mathbf{mult}(s,t)} - 1$$

Proof. Recalling that t is the only standard elimination form of t in which the term s is (trivially) eliminated (more formally, $\mathcal{E}_0(t,s) = \{t\}$), it is sufficient to prove the following slightly modified equation:

$$|\mathcal{E}^*(t,s)| \;=\; 2^{\mathbf{mult}(s,t)}$$

We prove the latter equation by induction over the structure of t; let $s \in \mathtt{T}_0$.

1. *t atomic:* $\mathcal{E}^*(t) = \{*, t\}$. We distinguish two cases:
   - *$t \doteq s$:* If $t \doteq s$, then both $*$ and t are elimination forms of t in which s is eliminated. Furthermore, $\mathbf{mult}(s,t) = 1$, and therefore:
   $$|\mathcal{E}^*(t,s)| \;=\; 2^1 \;=\; 2^{\mathbf{mult}(s,t)}$$
   - *$t \not\doteq s$:* If $t \not\doteq s$, then only $*$ is an elimination form of t (contained in $\mathcal{E}^*(t)$) in which s is eliminated. Furthermore, $\mathbf{mult}(s,t) = 0$, and therefore:
   $$|\mathcal{E}^*(t,s)| \;=\; 2^0 \;=\; 2^{\mathbf{mult}(s,t)}$$
2. *$f \doteq f(t_0, \ldots t_n)$:* The case that $t \doteq s$ is treated as above. Therefore, we may assume without loss $t \not\doteq s$; the latter excludes $*$ as an elimination form of t in which the term s is eliminated. Due to the categorisation of the preimage, we obtain that every elimination form $\mathtt{t}$ of t equals to a nominal term $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ for some suitable nominal terms $\mathtt{t}_k$ (with $k \in n'$). The term s is eliminated in $\mathtt{t}$, if and only if s is eliminated in all direct subtermms $\mathtt{t}_k$ with respect to the respective subterm t_k of t (for all $k \in n'$). This means that the elements $\mathtt{t}$ of $\mathcal{E}(t,s)$ are determined by the combination of such elimination forms. Therefore:
   $$|\mathcal{E}^*(t,s)| \;=\; \prod_{k \in n'} |\mathcal{E}^*(t_k,s)|$$
   Applying n'-many times induction hypothesis and recalling the definition of the multiplicity function, we calculate as follows:
   $$|\mathcal{E}^*(t,s)| \;=\; \prod_{k \in n'} 2^{\mathbf{mult}(s,t_k)} \;=\; 2^{\sum_{k \in n'} \mathbf{mult}(s,t_k)} \;=\; 2^{\mathbf{mult}(s,t)}$$

Q.E.D.

All Unary Elimination Forms: As an immediate consequence of the proposition above, we provide the number of all unary elimination forms of a standard term t as follows:

$$|\mathcal{E}_1(t)| \;=\; \sum_{s \in \mathbf{Sub}'(t)} (2^{\mathbf{mult}(s,t)} - 1)$$

8.2 Standard Occurrences

With the help of the elimination forms, we are able to represent adequately the position of occurrences and, therefore, to introduce the formal notion of an occurrence (of terms in terms). We accompany the introduction of occurrences with a brief discussion of some applications of this notion.

8.2.1 Introduction of Occurrences

We provide the definition of occurrences of terms in terms.

8.6 DEF (Occurrences): Let $t, s \in \mathtt{T}_0$ be two standard terms, $\mathtt{t} \in \mathtt{T}$ be an arbitrary nominal term.

1. *occurrence:* The triple $\mathfrak{o} = \langle t, s, \mathtt{t} \rangle$ is called a *(standard) occurrence* of the term s in the term t at the position $\mathtt{t}$, if $\mathtt{t}$ is an elimination form of t in which the term s is actually eliminated (formally, if both $\mathtt{t} \notin \mathtt{T}_0$ and $t \doteqdot \mathtt{t}[s]$).

   If $\mathtt{t}$ is simple (and therefore single), then $\mathfrak{o}$ is called *single*; otherwise, $\mathfrak{o}$ is called *multiple*. Furthermore, if $\mathtt{t} \doteqdot *$ is a nominal symbol (i.e. $\mathfrak{o} = \langle t, t, * \rangle$), then $\mathfrak{o}$ is also called *trivial*.

2. *projections:* Let $\mathfrak{o} = \langle t, s, \mathtt{t} \rangle$ be an occurrence. We define as follows:

   (a) *context:* The standard term $t \doteqdot \mathbf{con}(\mathfrak{o})$ is the *context* of $\mathfrak{o}$.

   (b) *shape:* The standard term $s \doteqdot \mathbf{shape}(\mathfrak{o})$ is the *shape* of $\mathfrak{o}$.

   (c) *position:* The nominal term $\mathtt{t} \doteqdot \mathbf{pos}(\mathfrak{o})$ is the *position* of $\mathfrak{o}$.

3. *sets of occurrences:* We introduce the following sets of occurrences:

   (a) *all occurrences:* $\mathcal{O}_1 = \{\langle t, s, \mathtt{t} \rangle \in {\mathtt{T}_0}^2 \times \mathtt{T}_1;\ t \doteqdot \mathtt{t}[s]\}$ is the *set of all occurrences*.

   (b) *occurrences in t:* $\mathcal{O}_1(t) = \{\mathfrak{o} \in \mathcal{O}_1;\ \mathbf{con}(\mathfrak{o}) \doteqdot t\}$ is the *set of all occurrences* with context t.

   (c) *occurrences of s:* $\mathcal{O}_1(t, s) = \{\mathfrak{o} \in \mathcal{O}_1;\ \mathbf{con}(\mathfrak{o}) \doteqdot t,\ \mathbf{shape}(\mathfrak{o}) = s\}$ is the *set of all occurrences* with context t and shape s.

Remarks (Occurrences):

1. *terminology:* In order to distinguish the occurrences, as defined above, from their generalisations discussed in the course of these investigations, we call the former also *standard* occurrences; in order to simplify terminology, we usually omit the denomination "standard", as long as the reference is clear from the context.

2. *Notation:* Sets of occurrences are notated with the label "1"; in contrast to the labels used for sets of nominal terms, this label refers to the number of shapes and not to the arity of the position. This notational convention is motivated by our notation with respect to the generalisations of standard occurrences.

   As long as it is meaningful, we permit also the use of additional labels indicating further restrictions analogously to the restrictions of sets of nominal terms.

Examples (Occurrences): We illustrate the notion of occurrences by an example in the language $\mathfrak{L}_{\mathrm{PA}}$ of arithmetics: we provide all occurrences in the standard term $t \doteq (0+0)+0$. First, we provide all single occurrences; the formal occurrences are accompanied by their informal representation, in which the positions of the intended occurrences are underlined:

- $(\underline{0}+0)+0 \quad \rightsquigarrow \quad \langle t, 0, (*+0)+0\rangle$
- $(0+\underline{0})+0 \quad \rightsquigarrow \quad \langle t, 0, (0+*)+0\rangle$
- $(0+0)+\underline{0} \quad \rightsquigarrow \quad \langle t, 0, (0+0)+*\rangle$
- $\underline{(0+0)}+0 \quad \rightsquigarrow \quad \langle t, 0+0, *+0\rangle$
- $\underline{(0+0)+0} \quad \rightsquigarrow \quad \langle t, (0+0)+0, *\rangle$

Observe that the first three occurrences have the same shape 0. The other subterms of t determine uniquely a (single) occurrence, as the multiplicity of these subterms in t is equal to 1. Besides the single occurrences, there are the following multiple occurrences of 0 in t:

- $(\underline{0}+\underline{0})+0 \quad \rightsquigarrow \quad \langle t, 0, (*+*)+0\rangle$
- $(\underline{0}+0)+\underline{0} \quad \rightsquigarrow \quad \langle t, 0, (*+0)+*\rangle$
- $(0+\underline{0})+\underline{0} \quad \rightsquigarrow \quad \langle t, 0, (0+*)+*\rangle$
- $(\underline{0}+\underline{0})+\underline{0} \quad \rightsquigarrow \quad \langle t, 0, (*+*)+*\rangle$

Observations (Occurrences):

1. *subterm property:* Due to the proposition about subterms under substitution, it is immediate that the shape s of an occurrence $\mathfrak{o}$ in a term t is a subterm of the context t. More formally:

$$\mathbf{shape}(\mathfrak{o}) \in \mathbf{Sub}'(\mathbf{con}(\mathfrak{o}))$$

2. *redundant information:* Occurrences codify redundant informations:

   (a) Context and position determine the shape.
   (Due to essential equality with respect to the position.)

   (b) Shape and position determine the context.
   (Trivial, as the general substitution function is a function.)

   But:

   (c) Context and shape do not, in general, determine the position.
   (Only if the multiplicity of the shape in the context equals 1.)

   As discussed in the introduction, statement (c) is a necessary property of any meaningful concept of the notion of occurrences; by statements (a) and (b), our approach is distinguished from simpler approaches resulting in weak theories of occurrences.

3. *equivalent definitions:* As a consequence of the redundancies mentioned above, there are two equivalent alternatives of defining occurrences parsimoniously; we decided to encode all three aspects of an occurrences, in order to obtain an faithful formal representation of occurrences according to our informal intuitions.

4. *bijective correspondence:* Due to the mentioned redundancies, there is a bijective correspondence between occurrences in a standard term and the unary elimination forms of this term (representing the positions of the respective occurrences). As a consequence, a great number of the hard problems (not solvable on the base of the recursive structure of standard terms) are solvable with reference to the position of occurrences without an explicit reference to the full notion of occurrences.

5. *definable occurrences:* Some special occurrences are "recursively" definable. For example, using the (recursively defined) complete elimination function, we can explicitly provide for all standard terms t and

all subterms $s \in \mathbf{Sub}'(t)$ of t the occurrence $\mathfrak{o} = \langle t, s, \mathbf{elim}(t, s)\rangle$ in which *all* occurrences of the subterm s of t are intended. (Observe that if $s \notin \mathbf{Sub}'(t)$, the tuple given above is not an occurrence at all, as $\mathbf{elim}(t, s)$ is, in this case, not a proper nominal term.)

8.2.2 Application: Counting Occurrences

The main advantage of a formal notion of occurrences is the possibility of discussing intuitively clear results on formal grounds. As a first formal result, we provide the number of occurrences in a given standard term.

8.7 Proposition (Counting Occurrences): Let $t, s \in \mathtt{T}_0$ be tow standard terms. The following statements hold:

1. *single occurrences:* The number of single occurrences of the term s in the term t equals to the multiplicity $\mathbf{mult}(s, t)$ of s in t. More formally:
$$|\mathcal{O}_{1,s}(t, s)| \;=\; \mathbf{mult}(s, t)$$
2. *occurrences:* The number of arbitrary occurrences of the term s in the term t equals to the number of possibilities to combine some single occurrences in a (multiple) occurrence. More formally:
$$|\mathcal{O}_1(t, s)| \;=\; 2^{\mathbf{mult}(s,t)} - 1$$

Proof. Due to the bijective correspondence between unary elimination forms and occurrences, we can immediately carry over the analogous results proved with respect to unary elimination forms. Q.E.D.

8.2.3 Application: The Lies-Within Relation

The paradigmatic example of a hard problem was the question, whether an intended occurrence lies within another intended occurrence (in the same context). Having introduced the theory of occurrences so far, we are able to provide the necessary methods for a formal treatment of this problem: the informal concept of an occurrence lying within another is, essentially, captured by the less-structured relation on nominal terms. We provide the definition of the formal lies-within relation for occurrences.

8.8 DEF (Lies-Within Relation): An occurrence $\mathfrak{o} = \langle t, s, \mathtt{t}\rangle$ *lies-within* an occurrence $\mathfrak{o}' = \langle t', s', \mathtt{t}'\rangle$ (formally, $\mathfrak{o} \leq \mathfrak{o}'$), if the following both conditions are satisfied:

1. *same context:* Both occurrences share their contexts (formally, $t \doteq t'$).
2. *less-structured position:* The position $\mathfrak{t}'$ of $\mathfrak{o}'$ is less-structured than the position $\mathfrak{t}$ of $\mathfrak{o}$ (formally, $\mathfrak{t}' \leq \mathfrak{t}$).

Example (Lies-Within Relation): We illustrate the lies-within relation by providing all occurrences in the standard term $t \doteq (0+0)+0$ according to their order given by the lies-within relation.

$$\left.\begin{array}{l}\langle t,0,(*+0)+0\rangle\\ \langle t,0,(0+*)+0\rangle\\ \langle t,0,(*+*)+0\rangle\end{array}\right\} \leq \mathfrak{o} = \langle t,0+0,*+0\rangle \left.\begin{array}{r}\\ \\ \langle t,0,(0+0)+*\rangle\\ \langle t,0,(*+0)+*\rangle\\ \langle t,0,(0+*)+*\rangle\\ \langle t,0,(*+*)+*\rangle\end{array}\right\} \leq \langle t,t,*\rangle$$

Observe that the position $*$ of the trivial occurrence $\langle t,t,*\rangle$ is less-structured than every position of an occurrence in t and that the position $\mathfrak{t} \doteq *+0$ of the occurrence $\mathfrak{o}$ is less-structured than the positions of the occurrences in the leftmost column, as we may obtain these positions by replacing $*$ in $\mathfrak{t}$ by the nominal terms $*+0$, $0+*$ and $*+*$, respectively.

Additionally, we provide the occurrences discussed above in informal notation by underlining the intended positions.

$$\left.\begin{array}{l}(\underline{0}+0)+0\\ (0+\underline{0})+0\\ (\underline{0}+\underline{0})+0\end{array}\right\} \leq \mathfrak{o} = \underline{(0+0)}+0 \left.\begin{array}{r}\\ \\ (0+0)+\underline{0}\\ (\underline{0}+0)+\underline{0}\\ (0+\underline{0})+\underline{0}\\ (\underline{0}+\underline{0})+\underline{0}\end{array}\right\} \leq \underline{(0+0)+0}$$

Remarks (Lies-Within Relation):

1. *shared context:* We demand in the definition of the lies-within relation that related occurrences have the same context; correspondingly, occurrences in different terms are not related by this relation, even if their positions are comparable.
2. *converse relation:* Observe that in the lies-within relation the less-structured relation is inverted: the position of a greater occurrence

(with respect to the lies-within relation) is smaller (with respect to the less-structured relation) than the position of a smaller occurrence (again, with respect to the lies-within relation). Under this perspective, the lies-within relation is converse to the less-structured relation.

3. *respecting the subterm relation:* The lies-within relation respects the subterm relation: if an occurrence $\mathfrak{o}$ lies-within an occurrence $\mathfrak{o}'$, then the shape of $\mathfrak{o}$ is a subterm of the shape of $\mathfrak{o}'$. More formally:

$$\mathfrak{o} \leq \mathfrak{o}' \quad \Rightarrow \quad \mathbf{shape}(\mathfrak{o}) \in \mathbf{Sub}'(\mathbf{shape}(\mathfrak{o}'))$$

4. *partial order:* The lies-within relation is a partial order on the set $\mathcal{O}$ of all occurrences. The trivial occurrences $\langle t, t, * \rangle$ are maximal with respect to the lies-within relation for all standard terms t; an occurrence $\langle t, s, \mathtt{t} \rangle$ is minimal with respect to the lies-within relation, if and only if its shape s is atomic.

Hard Problem: The hard problem (as discussed in the introduction) of deciding, whether an intended occurrence lies within another intended occurrence, is solved: in order to provide a formally justified answer, we have to check, whether the formal occurrences representing the informally given occurrences are related by the lies-within relation or not.

8.2.4 Excursus: Independence of Occurrences

We conclude our introduction of occurrences with a brief consideration of the (yet) informal concept of the independence of occurrences.

Independence of Occurrences:

1. *intuition:* Roughly spoken, two occurrences are considered as independent, if their positions do not interfere. We provide an example (in informal notation) illustrating this intuition:

$$\mathfrak{o} \;=\; 0 + \underline{(0+0)} \quad ; \quad \mathfrak{o}' \;=\; \underline{0} + (0+0) \quad ; \quad \mathfrak{o}'' \;=\; 0 + (\underline{0} + 0)$$

   While the pairs $\mathfrak{o}$ and $\mathfrak{o}'$ as well as $\mathfrak{o}'$ and $\mathfrak{o}''$ are independent, the pair $\mathfrak{o}$ and $\mathfrak{o}''$ is not, as $\mathfrak{o}''$ lies within $\mathfrak{o}$.

2. *first approach:* An obvious idea to define the independence of two occurrences $\mathfrak{o}$ and $\mathfrak{o}'$ (having a shared context) is to demand that neither

$\mathfrak{o}$ lies within $\mathfrak{o}'$ nor vice versa. Unfortunately, this idea only works fine with respect to single occurrences.

Having multiple occurrences at hands, we easily find counterexamples of pairs of occurrences satisfying the formal condition, but which are not independent according to our intuitions. Investigate the following two examples (again given in informal notation):

$$\underline{0} + (\underline{0} + 0) \;;\; 0 + (\underline{0} + \underline{0}) \quad \text{and} \quad 0 + \underline{(0 + 0)} \;;\; \underline{0} + (\underline{0} + 0)$$

In the first example, a common intended occurrence of a subterm is marked by both positions (we would call such a pair of occurrences weakly independent). In the second example, the second occurrence is partially separated from and partially inside the first occurrence.

3. *towards a definition:* As a consequence of the existence of the counterexamples, we have to dismiss the first idea to define independence. Actually, we develop only later in the course of our investigations the methods necessary to discuss independence adequately.

9 Relabelling Nominal Symbols

We introduce new methods for the treatment of nominal terms, namely two (in a way dual) functions, both allowing to relabel the nominal symbols occurring in a nominal term (without changing the structure of these nominal terms): the *unification function*, mapping all nominal symbols to the distinguished nominal symbol $*$, and the *simplification function*, mapping each nominal symbol to different nominal symbols.

The formal treatment of the simplification function demands the introduction of some new methods: first, we introduce the *extended minimum function* mapping a set of natural numbers and a natural number k to the set containing containing the first k members of the set. Via this function, we introduce a *labelling function* relabelling the nominal symbols occurring in a nominal term according to a given set of labels. Additionally, we investigate *shift operations* increasing (or decreasing) the labels of all nominal symbols in a nominal term by a fixed number.

9.1 The Unification Function

We provide the formal definition of the unification function.

9.1 DEF (Unification Function): The *unification function* $\mathbf{uni} : \mathtt{T} \to \mathtt{T}$ is defined recursively as follows:

1. $\mathtt{t}$ *atomic:*
$$\mathbf{uni}(\mathtt{t}) \;\doteqdot\; \begin{cases} *_0 & \text{if } \mathtt{t} \in \mathbf{V}_* \\ \mathtt{t} & \text{otherwise} \end{cases}$$
2. $\mathtt{t} \doteqdot f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* $\mathbf{uni}(\mathtt{t}) \doteqdot f(\mathbf{uni}(\mathtt{t}_0), \ldots \mathbf{uni}(\mathtt{t}_n))$

Remarks (Unification Function):

1. *terminology:* The unification function is called so, as it unifies all nominal symbols in a nominal term turning this way proper nominal terms into unary nominal terms.

   The introduction of this function is not motivated by the concept of unification as discussed in the theory of substitution. There unification means to find a unifying substitution such that applying such a substitution on each member of a given set of expressions results always in the same expression.[87] In particular, we are not searching here for such a (most general) unifier.

[87] Cf., for example, Baader and Siekmann [3] for more details.

Nevertheless, we there is a relationship to the concept of unification: subsequently, we identify via an equivalence relation nominal terms, which are mapped by the unification function onto the same nominal term. In slightly different terminology: we identify nominal terms unified by the unification function.

2. *alternative definition:* The unification function is the restriction of the general substitution function to the constant sequence $\mathfrak{c}_* = \langle *;\ k \in \omega \rangle$ in which each entry equals to the nominal symbol $*$. Therefore:

$$\mathbf{uni}(\mathtt{t}) \ \hat{=} \ \mathtt{t}[\mathfrak{c}_*]$$

In particular, the unification function is a simple homomorphism, but not an isomorphism.

3. *elimination form:* In the general case: as different nominal symbols are unified by the unification function, applying this function on an elimination form $\mathtt{t}$ of a standard term t does not result in an elimination form of t. Investigate, for example, the following example:

$$*_0 + *_1 \ \leq \ 0 + 1 \quad \text{but} \quad \mathbf{uni}(*_1 + *_0) \ \hat{=} \ *_0 + *_0 \ \not\leq \ 0 + 1$$

Basic Properties (Unification Function): The unification function has the following properties:

1. *restriction to* $\mathtt{T}^*$*:* The restriction of the unification function to the set $\mathtt{T}^*$ of standard terms and unary nominal terms is the identity function on this set. More formally:

$$\mathbf{uni}_{|\mathtt{T}^*} \ = \ \mathbf{id}_{\mathtt{T}^*}$$

(It is sufficient to mention that the constant sequence $\mathfrak{c}_*$ is essentially equal to the neutral sequence $\mathfrak{e}$ with respect to each nominal term $\mathtt{t} \in \mathtt{T}^*$.)

2. *idempotence:* The unification function is idempotent. More formally, for all nominal terms $\mathtt{t}$:

$$\mathbf{uni}(\mathtt{t}) \ \hat{=} \ \mathbf{uni}(\mathbf{uni}(\mathtt{t}))$$

(Immediate, as $\mathbf{uni}(\mathtt{t}) \in \mathtt{T}^*$ for all nominal terms $\mathtt{t} \in \mathtt{T}$.)

3. *isomorphic nominal terms:* Isomorphic nominal terns have the same unification. More formally, for all $\mathtt{t}, \mathtt{s} \in \mathtt{T}$:

$$\mathtt{t} \cong \mathtt{s} \quad \Rightarrow \quad \mathbf{uni}(\mathtt{t}) \doteq \mathbf{uni}(\mathtt{s})$$

   (Induction over the structure of $\mathtt{t}$.)

4. *weight and dual weight:* The unification function does neither change the number of nominal symbols nor that of standard atomic terms in a nominal term. More formally:

$$\mathbf{weight}(\mathtt{t}) = \mathbf{weight}(\mathbf{uni}(\mathtt{t})) \quad ; \quad \overline{\mathbf{weight}}(\mathtt{t}) = \overline{\mathbf{weight}}(\mathbf{uni}(\mathtt{t}))$$

   (Induction over the structure of $\mathtt{t}$.)

9.2 The Extended Minimum Function

In order to define the simplification function, we have to introduce first the *extended minimum function*, a generalisation of the minimum function. This function maps, if it is possible, a set of natural numbers to the set of the first k elements of this set. We provide the formal definition of this function.

9.2 DEF (Extended Minimum Function): The extended minimum function $\min : \mathfrak{p}(\omega) \times \omega \to \mathfrak{p}(\omega)$ is defined recursively (in the second argument) as follows:

1. $k = 0$: $\min_0(X) = \emptyset$
2. $k + 1$: $\min_{k+1}(X) = \begin{cases} \{\min(X)\} \cup \min_k(X \backslash \{\min(X)\}) & \text{if } X \neq \emptyset \\ \emptyset \quad \text{otherwise} \end{cases}$

The set $\min_k(X)$ contains the first k members of X (according to the well-order of ω), if X contains at least k members; otherwise, $\min_k(X)$ is equal to the set X itself.

Examples (Extended Minimum Function): We illustrate the extended minimum function by discussing some examples:

1. Let $\mathbb{E} = \{2n;\ n \in \omega\}$ be the set of even numbers.

$$\begin{aligned} \min_2(\mathbb{E}) &= \{\min(\mathbb{E})\} \cup \min_1(\mathbb{E} \backslash \{\min(\mathbb{E})\}) \\ &= \{0\} \cup \min_1(\{2, 4, 6, \ldots\}) \\ &= \{0\} \cup \{2\} \cup \min_0(\{4, 6, \ldots\}) \\ &= \{0\} \cup \{2\} \cup \emptyset = \{0, 2\} \end{aligned}$$

2. Recall that $2 = \{0, 1\}$ and $3 = \{0, 1, 2\}$.

$$\begin{aligned} \min_3(2) &= \{\min(2)\} \cup \min_2(2 \backslash \{\min(2)\}) \\ &= \{0\} \cup \min_2(\{1\}) \\ &= \{0\} \cup \{1\} \cup \min_1(\emptyset) \\ &= \{0\} \cup \{1\} \cup \emptyset \;=\; \{0, 1\} \;=\; 2 \end{aligned}$$

   And:

$$\begin{aligned} \min_2(3) &= \{\min(3)\} \cup \min_1(3 \backslash \{\min(3)\}) \\ &= \{0\} \cup \min_1(\{1, 2\}) \\ &= \{0\} \cup \{1\} \cup \min_0(\{2\}) \\ &= \{0\} \cup \{1\} \cup \emptyset \;=\; \{0, 1\} \;=\; 2 \end{aligned}$$

   More generally: $\min_n(m) = \min(n, m)$ for all $n, m \in \omega$.

9.3 The Labelling Function

With the help of the extended minimum function, the labelling function is defined. This function relabels the nominal symbols in a nominal term according to a given set of labels. We provide the formal definition.

9.3 DEF (Labelling Function): The labelling function $\lambda : \mathtt{T} \times \mathfrak{p}(\omega) \to \mathtt{T}$ is defined recursively (in its first argument) as follows:

1. $\mathtt{t}$ *atomic:* $\lambda(\mathtt{t}, X) \;=\; \begin{cases} *_{\min(X)} & \text{if } \mathtt{t} \in \mathbf{V}_* \text{ and } X \neq \emptyset \\ \mathtt{t} & \text{otherwise} \end{cases}$

2. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* $\lambda(\mathtt{t}, X) = f(\lambda(\mathtt{t}_0, X_0), \ldots \lambda(\mathtt{t}_n, X_n))$

   Here, the sets X_k are defined recursively as follows for all $k \in n'$:

$$X_k \;=\; \min_{w_k}(X \backslash \bigcup_{l<k} X_l) \quad \text{where } w_k = \mathbf{weight}(\mathtt{t}_k)$$

Conception (Labelling Function): We investigate the labelling function in some details.

1. *weight:* Recall that the weight function provides the number of nominal symbols occurring in a nominal term.

2. *complex case:* In the complex case, the labelling function is applied to the direct subterms $\mathtt{t}_k$ and a sets X_k containing as many labels as the direct subterms have nominal symbols (provided that the set X is sufficiently large). Furthermore, if a direct subterm $\mathtt{t}_k$ is left of a direct subterm $\mathtt{t}_l$, then the labels provided in X_k are all strictly smaller than the labels contained in X_l.

   Observe that if the initial set X of labels contains at least as many members as the weight of the initial nominal term $\mathtt{t}$ on which the labelling function is applied, then the intermediate sets X_k have all sufficiently many members, namely exactly as many members as nominal symbols occur in the respective subterm.

3. *atomic case:* In the typical case, the nominal term $\mathtt{t}$ is a nominal symbol and the respective set X a singleton or the nominal term $\mathtt{t}$ is a standard term and the respective set X empty. In the first typical case, the label of the nominal symbol is replaced by the label contained in X, in the latter case nothing happens.

   We discuss the unintended cases: it is possible that the set of labels contains more than one member, if we apply the labelling function directly on an atomic nominal term. In this case, a nominal symbol becomes relabelled with the smallest label and the standard terms still remain unchanged.

   On the other hand, if the initial set X of labels is not sufficiently large, then it may also happen, that the labelling function is applied on an atomic nominal term and the empty set. In this case, the the nominal term also remains unchanged.

4. *homomorphisms:* In the typical case (if $\mathtt{t}$ is not simple and if X is sufficiently large), the same nominal symbol $*_k$ is mapped by the labelling function to different nominal symbols. As a consequence, an application of the labelling function cannot be replaced, in general, by an application of a homomorphism.

5. *standard terms:* The labelling function is invariant on standard terms.

Putting the pieces together: if $k = \min(|X|, \mathbf{weight}(\mathtt{t}))$, then $\lambda(\mathtt{t}, X)$ is the result of replacing the labels of the first k nominal symbols in $\mathtt{t}$ (from the left to the right) by the first k labels contained in X.

Examples (Labelling Function): We illustrate the labelling function by some examples.

1. Let $\mathtt{t} \mathrel{\widehat{=}} (* + *) + *$.

$$\begin{aligned}\lambda(\mathtt{t}, \omega) &\mathrel{\widehat{=}} \lambda(* + *, \min_2(\omega)) + \lambda(*, \min_1(\omega \backslash \min_2(\omega))) \\ &\mathrel{\widehat{=}} \lambda(* + *, \{0, 1\}) + \lambda(*, \{2\}) \\ &\mathrel{\widehat{=}} (\lambda(*, \{0\}) + \lambda(*, \{1\})) + \lambda(*, \{2\}) \\ &\mathrel{\widehat{=}} (*_0 + *_1) + *_2\end{aligned}$$

2. Let $\mathtt{t} \mathrel{\widehat{=}} (*_0 + *_5) + *_0$.

$$\begin{aligned}\lambda(\mathtt{t}, \{5\}) &\mathrel{\widehat{=}} \lambda(* + *, \{5\}) + \lambda(*, \emptyset) \\ &\mathrel{\widehat{=}} (\lambda(*, \{5\}) + \lambda(*, \emptyset)) + \lambda(*, \emptyset) \\ &\mathrel{\widehat{=}} (*_5 + *_5) + *_0\end{aligned}$$

3. Let $\mathtt{t} \mathrel{\widehat{=}} * + (* + *)$; recall that $\mathbb{E} = \{2n;\ n \in \omega\}$.

$$\begin{aligned}\lambda(\mathtt{t}, \mathbb{E}) &\mathrel{\widehat{=}} \lambda(*, \{0\}) + \lambda(* + *, \{2, 4\}) \\ &\mathrel{\widehat{=}} \lambda(*, \{0\}) + (\lambda(*, \{2\}) + \lambda(*, \{4\})) \\ &\mathrel{\widehat{=}} *_0 + (*_2 + *_4)\end{aligned}$$

9.4 The Shift Operations

We introduce the *shift operations* for nominal terms increasing and decreasing uniformly the labels of all nominal symbols in a nominal term.

9.4 DEF (Shift Operations): Let $n \in \omega$ be arbitrary.

1. *right shift:* The *n-th right-shift operation* $\cdot^{+n} : \mathtt{T} \to \mathtt{T}$ is defined recursively as follows:
   (a) $\mathtt{t}$ *atomic:* $\mathtt{t}^{+n} \mathrel{\widehat{=}} \begin{cases} *_{k+n} & \text{if } \mathtt{t} \mathrel{\widehat{=}} *_k \\ \mathtt{t} & \text{otherwise} \end{cases}$
   (b) $\mathtt{t} \mathrel{\widehat{=}} f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* $\mathtt{t}^{+n} \mathrel{\widehat{=}} f(\mathtt{t}_0{}^{+n}, \ldots \mathtt{t}_n{}^{+n})$
2. *left shift:* The *n-th left-shift operation* $\cdot^{-n} : \mathtt{T} \to \mathtt{T}$ is defined recursively as follows:
   (a) $\mathtt{t}$ *atomic:* $\mathtt{t}^{-n} \mathrel{\widehat{=}} \begin{cases} *_k & \text{if } \mathtt{t} \mathrel{\widehat{=}} *_{k+n} \\ *_0 & \text{if } \mathtt{t} \mathrel{\widehat{=}} *_k \text{ for } k \in n \\ \mathtt{t} & \text{otherwise} \end{cases}$
   (b) $\mathtt{t} \mathrel{\widehat{=}} f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* $\mathtt{t}^{-n} \mathrel{\widehat{=}} f(\mathtt{t}_0{}^{-n}, \ldots \mathtt{t}_n{}^{-n})$

Remarks (Shift Operations):

1. *simple homomorphism:* Immediately by definition, the shift operations are simple homomorphisms. Besides the trivial case $n = 0$, the right-shift operation is not surjective, the left-shift operation is not injective. Therefore, these shift-operation are no isomorphisms. Nevertheless, the left-shift operations are right-inverse to the respective right-shift operations. More formally, for arbitrary number n and all nominal terms $\mathtt{t}$:
$$(\mathtt{t}^{+n})^{-n} \;\doteqdot\; \mathtt{t}$$

2. *alternative definition:* The n-th right-shift operation is the restriction of the general substitution function to the following sequence of nominal symbols:
$$\mathfrak{e}^{+n} \;=\; \langle *_{k+n};\; k \in \omega \rangle$$
This means that $\mathtt{t}^{+n} \doteqdot \mathtt{t}[\mathfrak{e}^{+n}]$ for all nominal terms $\mathtt{t}$ and all natural numbers $n \in \omega$.

   The n-th left-shift operation is the restriction of the general substitution function to the following sequence of nominal symbols:
$$\mathfrak{e}^{-n} \;=\; \langle *_{k \dot{-} n};\; k \in \omega \rangle$$
   Here, $\dot{-}$ is the (recursively definable) total subtraction on ω, where results below 0 are replaced by 0.

3. *isomorphism:* Every application of a right-shift operation can be replaced by an application of a suitable isomorphism. In other words: there is an isomorphisms F for every nominal term $\mathtt{t}$ and every natural number n such that:
$$F(\mathtt{t}) \;\doteqdot\; \mathtt{t}^{+n}$$
   Investigate the following set for $m = \mathbf{rank}(\mathtt{t})$:
$$S \;=\; \{\langle *_k, *_{k+n} \rangle;\; k \in \mathbf{rank}(\mathtt{t})\} \cup \{\langle *_{k+m}, *_k \rangle;\; k \in n\}$$
   The function S is a permutation of the set $\{*_k;\; k \in \mathbf{rank}(\mathtt{t}) + n\}$. This function S is easily extended to a bijective function F_0 on $\mathbf{V}_*$. The latter function induces an isomorphism F essentially equal to the n-th right-shift operation with respect to the nominal term $\mathtt{t}$.

   It is not possible to provide uniformly (independently of the nominal term under discussion) an isomorphism for a right-shift operation (besides the trivial case $n = 0$).

4. *notation:* Occasionally, we omit in our notation the natural number n specifying the concrete shift operation. We may read $\mathtt{t}^+$ as "the nominal term $\mathtt{t}$ in which the labels of its nominal symbols are suitably increased"; analogously, with respect to the left-shift operations. This is done in cases, where n is clear from the context or when we are not interested in the exact specification.

9.5 The Simplification Function

The simplification function is defined as an application of the labelling function on a sufficiently large set of labels. We provide the formal definition.

9.5 DEF (Simplification Function): We define the *simplification function* $\mathbf{simp} : \mathtt{T} \to \mathtt{T}$ as follows for all nominal terms $\mathtt{t} \in \mathtt{T}$:

$$\mathbf{simp}(\mathtt{t}) \;\doteq\; \lambda(\mathtt{t}, \mathbf{weight}(\mathtt{t}))$$

Remarks (Simplification Function): Our observations about the labelling function are easily carried over to the special case of the simplification function.

1. *result:* As we supply sufficiently many labels, an application of the simplification function results in a simple nominal term.

   As the nominal symbols are sorted in $\mathbf{simp}(\mathtt{t}) \doteq \lambda(\mathtt{t}, \mathbf{weight}(\mathtt{t}))$, the simplification function results in a normal nominal term.

2. *alternative definition:* We may use any superset of the set $\mathbf{weight}(\mathtt{t})$ in the definition of the simplification function. In particular, we have

   $$\mathbf{simp}(\mathtt{t}) \;\doteq\; \lambda(\mathtt{t}, \omega)$$

3. *homomorphisms:* The simplification function is not a homomorphism.

In the next proposition, we relate the simplification of a complex nominal term with the simplifications of its direct subterms.

9.6 Proposition (Complex Simplified Nominal Terms): The following equation holds for all complex nominal terms $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n) \in \mathtt{T}$:

$$\mathbf{simp}(\mathtt{t}) \;\doteq\; f(\mathbf{simp}(\mathtt{t}_0)^{+n_0}, \ldots \mathbf{simp}(\mathtt{t}_n)^{+n_n})$$

Here, the natural numbers n_k specifying the shift operations are defined recursively as follows (for all $k \in n'$):

$$n_k \;=\; \sum_{l<k} \mathbf{weight}(\mathtt{t}_l)$$

Proof. It is sufficient to mention that the simplification $\mathbf{simp}(\mathtt{t}_k)$ of the direct subterms $\mathtt{t}_k$ are all normal and simple; this means that each of their nominal symbols is sorted from the left to the right according to the natural numbers. In order to simplify the complex nominal term $\mathtt{t}$, we can simplify all direct subterms and then shift their labels as many steps as there are nominal symbols left of them in the complex nominal term. Q.E.D.

Remarks (Complex Simplified Nominal Terms):

1. *similarity preserving:* The proposition above illustrates that the simplification function is not structure preserving (we have to shift nominal terms in the direct subterms), but still well-behaved, as the main function symbol is preserved; this property could be called *similarity preserving*.

2. *relevance:* From a technical point of view, the proposition above is relevant and useful, as it allows to carry over the recursive structure of nominal terms to their simplifications. The latter means: via the proposition above, we are able to prove statements about simplified nominal terms by induction over simplified nominal terms.

We provide the basic properties of the simplification function.

Basic Properties: The simplification function has the following properties:

1. *identity:* If $\mathtt{t}$ is a standard term, then $\mathtt{t} \mathrel{\hat{=}} \mathbf{simp}(\mathtt{t})$. Otherwise, we have that $\mathtt{t} \mathrel{\hat{=}} \mathbf{simp}(\mathtt{t})$, if and only if $\mathtt{t}$ is simple and normal.

   (Induction over the structure of $\mathtt{t}$.)

2. *idempotence:* The simplification function is idempotent. More formally:

   $$\mathbf{simp}(\mathbf{simp}(\mathtt{t})) \;\mathrel{\hat{=}}\; \mathbf{simp}(\mathtt{t})$$

   (It is sufficient to mention that $\mathbf{simp}(\mathtt{t})$ is simple and normal.)

3. *absorption:* The simplification function absorbs the unification function and vice versa. More formally:

$$\mathbf{simp}(\mathbf{uni}(\mathtt{t})) \;\hat{=}\; \mathbf{simp}(\mathtt{t}) \quad ; \quad \mathbf{uni}(\mathbf{simp}(\mathtt{t})) \;\hat{=}\; \mathbf{uni}(\mathtt{t})$$

(Induction over the structure of $\mathtt{t}$.)

4. *isomorphic nominal terms:* If $\mathtt{t}$ and $\mathtt{s}$ are isomorphic nominal terms, then their simplifications are equal. More formally:

$$\mathtt{t} \cong \mathtt{s} \quad \Rightarrow \quad \mathbf{simp}(\mathtt{t}) \;\hat{=}\; \mathbf{simp}(\mathtt{s})$$

(Isomorphism of nominal terms implies that the respective unifications are equal. Absorption implies that the simplifications of the respective unifications are equal.)

5. *weight and dual weight:* The simplification function does neither change the number of nominal symbols nor that of standard atomic terms in a nominal term. More formally:

$$\mathbf{weight}(\mathtt{t}) \;=\; \mathbf{weight}(\mathbf{simp}(\mathtt{t})) \quad ; \quad \overline{\mathbf{weight}}(\mathtt{t}) \;=\; \overline{\mathbf{weight}}(\mathbf{simp}(\mathtt{t}))$$

(Induction over the structure of $\mathtt{t}$.)

In contrast to the unification function, the simplification function preserves the property of being an elimination form.

9.7 Proposition (Simplification of Elimination Forms): Let $\mathtt{t} \in \mathtt{T}$ and $t \in \mathtt{T}_0$. If $\mathtt{t}$ is an elimination form of t, then also $\mathbf{simp}(\mathtt{t})$. More formally, if $\mathtt{t} \leq t$, then $\mathbf{simp}(\mathtt{t}) \leq t$.
Proof. By induction over the structure of $\mathtt{t}$.

1. $\mathtt{t}$ *atomic:* If $\mathtt{t}$ is an atomic standard term, then $\mathtt{t} \leq t$ implies $\mathtt{t} \;\hat{=}\; t$. Furthermore, $\mathbf{simp}(\mathtt{t}) \;\hat{=}\; \mathtt{t}$. Therefore, $\mathbf{simp}(\mathtt{t}) \leq t$.

   Otherwise, $\mathtt{t} \in \mathbf{V}_*$ is a nominal symbol. In this case, $\mathtt{t} \leq t$ for all standard terms t. The same holds for $\mathbf{simp}(\mathtt{t}) \;\hat{=}\; *$.

2. $\mathtt{t} \;\hat{=}\; f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* $\mathtt{t} \leq t$ implies that $t \;\hat{=}\; f(t_0, \ldots t_n)$ is similar to $\mathtt{t}$. In particular, we have $\mathtt{t}_k \leq t_k$ for all $k \in n'$. Applying induction hypothesis n'-many times, we obtain $\mathbf{simp}(\mathtt{t}_k) \leq t_k$ for all $k \in n'$. The latter means that there are n'-many sequences $\mathfrak{s}_k$ of length $\mathbf{place}(\mathbf{simp}(\mathtt{t}_k))$ actually eliminated in $\mathbf{simp}(\mathtt{t}_k)$. Recall:

$$\mathbf{simp}(\mathtt{t}) \;\hat{=}\; f(\mathbf{simp}(\mathtt{t}_0)^+, \ldots \mathbf{simp}(\mathtt{t}_n)^+)$$

Here, $\mathbf{simp}(\mathtt{t}_k)^+$ means that the nominal symbols of $\mathbf{simp}(\mathtt{t}_k)$ are suitably shifted. As a consequence:

$$\mathbf{simp}(\mathtt{t})[\mathfrak{s}_0 \circ \ldots \mathfrak{s}_n] \mathrel{\widehat{=}} t$$

Here, $\circ$ denotes the concatenation of finite sequences. The latter means that $\mathbf{simp}(\mathtt{t})$ is, indeed, an elimination form of t. Q.E.D.

Remarks (Simplification of Elimination Forms):

1. *generalisation:* The proposition above is the special case of a more general observation:

$$\mathtt{t} \leq \mathtt{s} \quad \Rightarrow \quad \mathbf{simp}(\mathtt{t}) \leq \mathtt{s} \text{ and } \mathtt{t} \leq \mathbf{uni}(\mathtt{s})$$

In general, the other inequalities do not hold. More formally:

$$\mathtt{t} \leq \mathtt{s} \quad \not\Rightarrow \quad \mathtt{t} \leq \mathbf{simp}(\mathtt{s}) \text{ or } \mathbf{uni}(\mathtt{t}) \leq \mathtt{s}$$

Investigate the following both examples:

$$* + * \;\leq\; * + * \quad \text{but} \quad * + * \;\not\leq\; *_0 + *_1 \;\mathrel{\widehat{=}}\; \mathbf{simp}(* + *)$$

$$*_0 + *_1 \;\leq\; *_0 + *_1 \quad \text{but} \quad \mathbf{uni}(*_0 + *_1) \;\mathrel{\widehat{=}}\; * + * \;\not\leq\; *_0 + *_1$$

9.6 Inverting a Simplification

As mentioned before, the simplification function is not a homomorphism.[88] Nevertheless, the situation is different, if we investigate the inverse direction: there is a simple homomorphism mapping the simplification of a nominal term to the nominal term itself.

9.8 Proposition (Preimage of the Simplification): Let $\mathtt{t} \in \mathtt{T}$ arbitrary. There is a simple homomorphism F such that $F(\mathbf{simp}(\mathtt{t})) \mathrel{\widehat{=}} \mathtt{t}$.
Proof. By induction over the structure of $\mathtt{t}$.

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* Let $\mathtt{t} \mathrel{\widehat{=}} *_k$. The constant homomorphism F induced by the function $F_0 : *_l \mapsto *_k$ is simple and satisfies the demanded condition $F(\mathbf{simp}(\mathtt{t})) \mathrel{\widehat{=}} F(*) \mathrel{\widehat{=}} *_k$.

[88] Even applications of the simplification function cannot be replaced, in general, by a homomorphism.

2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* As $\mathtt{t}$ is a standard term, every simple homomorphism, in particular the identity function, satisfies the demanded condition.

3. $\mathtt{t} \doteq f(\mathtt{t}_0, \dots \mathtt{t}_n)$ *complex:* Applying the induction hypothesis n-many times, we obtain simple homomorphisms F_k satisfying the condition $F_k(\mathbf{simp}(\mathtt{t}_k)) \doteq \mathtt{t}_k$ for all $k \in n'$. Furthermore, we have:

$$\mathbf{simp}(\mathtt{t}) \;\doteq\; f(\mathbf{simp}(\mathtt{t}_0)^+, \dots \mathbf{simp}(\mathtt{t}_n)^+)$$

Recall that $\cdot^+$ denotes a suitable shift of the labels of nominal symbols.

By definition of the simplification function, the free places of $\mathbf{simp}(\mathtt{t})$ are the disjoint union of the free places of its direct subterms. More formally:

$$\mathbf{place}(\mathbf{simp}(\mathtt{t})) \;=\; \dot{\bigcup_{k \in n'}} \mathbf{place}(\mathbf{simp}(\mathtt{t}_k)^+)$$

We define a homomorphism F_k^- as the composition of a suitable left-shift and the respective homomorphism F_k for every $k \in n'$ such that the following condition is satisfied for all $k \in n'$:

$$F_k^-(\mathbf{simp}(\mathtt{t}_k)^+) \;\doteq\; F(\mathbf{simp}(\mathtt{t}_k))$$

Having these modified functions F_k^- at hands, we define a function F_0 on the set of all nominal symbols as follows:

$$F_0(*_l) \;\doteq\; \begin{cases} F_k^-(*_l) & \text{for } k \text{ satisfying } l \in \mathbf{place}(\mathbf{simp}(\mathtt{t}_k)^+) \\ *_0 & \text{otherwise, if there is no such } k \end{cases}$$

The function F_0 is well-defined, as the respective sets of free places are disjoint. Furthermore, F_0 induces canonically a homomorphism F. By construction, the homomorphism F is simple and essentially equal to F_k^- with respect to $\mathbf{simp}(\mathtt{t}_k)^+$ for all $k \in n'$. Therefore, we may calculate as follows:

$$\begin{aligned} F(\mathbf{simp}(\mathtt{t})) &\doteq f(F(\mathbf{simp}(\mathtt{t}_0)^+), \dots F(\mathbf{simp}(\mathtt{t}_n)^+)) \\ &\doteq f(F_0^-(\mathbf{simp}(\mathtt{t})^+), \dots F_n^-(\mathbf{simp}(\mathtt{t}_n)^+)) \\ &\doteq f(F_0(\mathbf{simp}(\mathtt{t})), \dots F_n(\mathbf{simp}(\mathtt{t}_n))) \\ &\doteq f(\mathtt{t}_0, \dots \mathtt{t}_n) \;\doteq\; \mathtt{t} \end{aligned}$$

Q.E.D.

10 Relations Beyond Homomorphisms

On the base of the methods discussed in the last section, we introduce two new binary relations on the set of nominal terms, namely the *equivalence* of nominal terms and the *covered-by* relation (in a weak and in a strong version).

10.1 The Equivalence of Nominal Terms

The equivalence of nominal terms identifies nominal terms, which differ only with respect to the labels of their nominal symbols. We provide the formal definition of the equivalence of nominal terms.

10.1 DEF (Equivalence of Nominal Terms): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$. The nominal terms $\mathtt{t}$ and $\mathtt{s}$ are *equivalent* (formally, $\mathtt{t} \equiv \mathtt{s}$), if their unifications are equal (formally, if $\mathbf{uni}(\mathtt{t}) \doteq \mathbf{uni}(\mathtt{s})$).

Remarks (Equivalence of Nominal Terms):

1. *alternative characterisation:* Equivalently, we could have defined the equivalence of nominal terms with respect to the simplification function. More formally:

   $$\mathtt{t} \equiv \mathtt{s} \quad \Leftrightarrow \quad \mathbf{simp}(\mathtt{t}) \doteq \mathbf{simp}(\mathtt{s})$$

   (Almost immediate, due to absorption.)

2. *direct subterms:* Two complex nominal terms $\mathtt{t}$ and $\mathtt{s}$ are equivalent, if and only if the following both conditions are satisfied:

   (a) *similarity:* $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n) \sim f(\mathtt{s}_0, \ldots \mathtt{s}_n) \doteq \mathtt{s}$

   (b) *subterms:* $\mathtt{t}_k \equiv \mathtt{s}_k$ for all $k \in n'$

   (Essentially, as **uni** is a homomorphism.)

3. *equivalence relation:* It is easily checked that the equivalence of nominal terms is, indeed, an equivalence relation.

   (Essentially, as syntactic equality is an equivalence relation.)

4. *normal form:* There are two distinguished representatives in each equivalence class with respect to the equivalence of nominal terms, namely the unification and the simplification of any element in the respective class. More formally, for all nominal terms $\mathtt{t}$:

   $$[\mathtt{t}]_{\equiv} \;=\; [\mathbf{uni}(\mathtt{t})]_{\equiv} \;=\; [\mathbf{simp}(\mathtt{t})]_{\equiv}$$

Observe that both representatives of the equivalence of nominal terms are also normal with respect to the isomorphism of nominal terms. The converse does not hold: there are normal nominal terms (with respect to the isomorphism of nominal terms) such that they are no canonical representatives of the equivalence of nominal terms. Investigate, as an example, the normal nominal term $\mathtt{t} \mathrel{\widehat{=}} *_0 + (*_1 + *_0)$:

$$\mathbf{uni}(\mathtt{t}) \;\mathrel{\widehat{=}}\; *_0 + (*_0 + *_0) \;\neq\; \mathtt{t} \;\neq\; *_0 + (*_1 + *_2) \;\mathrel{\widehat{=}}\; \mathbf{simp}(\mathtt{t})$$

This means that the isomorphism of nominal terms is a finer equivalence relation on the set of nominal terms than the equivalence of nominal terms.

5. *restriction to* $\mathtt{T}^*$*:* The restriction of the equivalence of nominal terms to the set $\mathtt{T}^*$ of standard terms and unary nominal terms is the identity relation on nominal terms. More formally, for all $\mathtt{t}, \mathtt{s} \in \mathtt{T}^*$:

$$\mathtt{t} \equiv \mathtt{s} \quad \Leftrightarrow \quad \mathtt{t} \mathrel{\widehat{=}} \mathtt{s}$$

(Immediate, as $\mathbf{uni}_{|\mathtt{T}^*} = \mathbf{id}_{\mathtt{T}^*}$.)

Observe that if we use the simplification function for identifying equivalent nominal terms, then we have to deal with n-ary nominal terms, where n is equal to the multiplicity of the nominal symbol $*$ in the respective nominal term. Nevertheless, in the restriction to $\mathtt{T}^*$ only unary nominal terms are related (even if this is decided outside of $\mathtt{T}^*$).

6. *recursive characterisation:* Our observation with respect to the direct subterms of equivalent complex nominal terms motivates the following recursive characterisation of equivalent nominal terms: two nominal terms $\mathtt{t}$ and $\mathtt{s}$ are equivalent, if and only if one of the following conditions is satisfied:

   (a) $\mathtt{t} \in \mathbf{V}_*$ *atomic:* The nominal tern $\mathtt{s}$ is a nominal symbol ($\mathtt{s} \in \mathbf{V}_*$).

   (b) $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* The nominal term $\mathtt{s}$ is equal to $\mathtt{t}$ ($\mathtt{s} \mathrel{\widehat{=}} \mathtt{t}$).

   (c) $\mathtt{t} \mathrel{\widehat{=}} f(\mathtt{t}_0, \ldots \mathtt{t}_n)$*:* The nominal term $\mathtt{s}$ i s similar to $\mathtt{t}$ and the respective direct subterms are already equivalent. Formally: there are nominal terms $\mathtt{s}_k \in \mathtt{T}$ (for $k \in n'$) satisfying the following both conditions:

$$\mathtt{s} \mathrel{\widehat{=}} f(\mathtt{s}_0, \ldots \mathtt{s}_n) \quad \text{and} \quad \mathtt{t}_k \equiv \mathtt{s}_k \text{ for all } k \in n'$$

(Straightforward induction over the structure of $\mathtt{t}$.)

Conceptual Remark (Equivalence of Nominal Terms): We discuss the advantages of the different characterisations of the equivalence of nominal terms:

1. *unification:* The unification function is a (very simple) homomorphism; using the unification function, we can easily check, whether two nominal terms are equivalent. (To calculate the simplification function is more involved.)

2. *simplification function:* The advantage of the simplification function is that, under a certain perspective, no information is lost under an application of this function.[89] This property of the simplification function is of use in theoretical discussions.

3. *recursive characterisation:* The recursive characterisation of the equivalence of nominal terms corresponds, essentially, to the characterisation in terms of the unification function. But sometimes it is convenient to have a recursive characterisation at hands allowing inductions over the structure of equivalent nominal terms.

   Furthermore, the recursive characterisation is interesting from a theoretical point of view: we introduce subsequently a number of relations having a definition similar to this characterisation. The different properties of these relations correspond to variations of the atomic clauses.

Observations (Equivalence of Nominal Terms): We provide some observations about the equivalence of nominal terms.

1. *weight functions:* The weight functions are compatible with the equivalence of nominal terms. More formally, for all nominal terms $\mathtt{t}$ and $\mathtt{s}$ satisfying $\mathtt{t} \equiv \mathtt{s}$:

   $$\mathbf{weight}(\mathtt{t}) \;=\; \mathbf{weight}(\mathtt{s}) \quad ; \quad \overline{\mathbf{weight}}(\mathtt{t}) \;=\; \overline{\mathbf{weight}}(\mathtt{s})$$

   (Straightforward induction over the structure of $\mathtt{t}$ using the recursive characterisation of the equivalence of nominal terms.)

[89] The concept of an information loss is understood here informally and is associated with the idea of inverting an application of the respective function: to obtain the preimage of an application of the unification function, we have to use methods beyond homomorphisms, whereas the preimage of an application of the simplification function can be given via a simple homomorphism. This phenomenon corresponds to the observation that, for example, the property of being an elimination form of a standard term is preserved under the application of the simplification function, whereas it may be lost under the application of the unification function.

2. *simple homomorphism:* If there is a simple homomorphism F such that $F(\mathtt{t}) \doteq \mathtt{s}$, then $\mathtt{t}$ and $\mathtt{s}$ are equivalent. More formally:

$$\exists F \in \mathbf{Hom}_s(\mathtt{T}) : \; F(\mathtt{t}) \doteq \mathtt{s} \quad \Rightarrow \quad \mathtt{t} \equiv \mathtt{s}$$

(Straightforward induction over $\mathtt{t}$.)

Observe that the other direction does not hold in general. Investigate, for example, the nominal terms $\mathtt{t} \doteq *_0 + (*_0 + *_1)$ and $\mathtt{s} \doteq *_0 + (*_1 + *_1)$. There is no homomorphism F at all such that $F(\mathtt{t}) \doteq \mathtt{s}$. But we have $\mathtt{t} \equiv \mathtt{s}$, as $\mathbf{uni}(\mathtt{t}) \doteq \mathbf{uni}(\mathtt{s})$.

In the next proposition, we show that if the nominal term $\mathtt{t}$ is simple, then the other direction of our observation holds.

10.2 Proposition (Criterion: Simple Homomorphism): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$. If $\mathtt{t}$ is simple, then the following statement holds: if $\mathtt{t} \equiv \mathtt{s}$, then there is a simple homomorphism $F \in \mathbf{Hom}_s(\mathtt{T})$ such that $F(\mathtt{t}) \doteq \mathtt{s}$.
Proof. By induction over the structure of the simple homomorphism $\mathtt{t}$:

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* $\mathtt{t} \equiv \mathtt{s}$ implies that $\mathtt{s} \in \mathbf{V}_*$. Obviously, there is $F \in \mathbf{Hom}_s(\mathtt{T})$ such that $F(\mathtt{t}) \doteq \mathtt{s}$.

2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* $\mathtt{t} \equiv \mathtt{s}$ means that $\mathtt{t} \doteq \mathtt{s}$. Obviously, there is $F \in \mathbf{Hom}_s(\mathtt{T})$ such that $F(\mathtt{t}) \doteq \mathtt{s}$.

3. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$*:* $\mathtt{t} \equiv \mathtt{s}$ implies that $\mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$ and $\mathtt{t}_k \equiv \mathtt{s}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain simple homomorphism F_k such that $F(\mathtt{t}_k) \doteq \mathtt{s}_k$. Observe that there is exactly one $l \in n'$ such that $*_k \in \mathbf{place}(\mathtt{t}_l)$ for all $k \in \mathbf{place}(\mathtt{t})$, as $\mathtt{t}$ is simple. Therefore, the following definition of a function $\hat{F}$ on the set of all variables is well-defined:

$$\hat{F} : \; *_k \; \mapsto \; \begin{cases} F_l(*_k) & \text{for } l \text{ with } k \in \mathbf{place}(\mathtt{t}_l), \text{ if } k \in \mathbf{place}(\mathtt{t}) \\ *_k & \text{otherwise} \end{cases}$$

$\hat{F}$ induces canonically a homomorphism F. By construction, F is simple and F is essentially equal to F_k with respect to $\mathtt{t}_k$ for all $k \in n'$. Therefore:

$$\begin{aligned} F(\mathtt{t}) \; &\doteq \; f(F(\mathtt{t}_0), \ldots F(\mathtt{t}_n)) \\ &\doteq \; f(F_0(\mathtt{t}_0), \ldots F_n(\mathtt{t}_n)) \; \doteq \; f(\mathtt{s}_0, \ldots \mathtt{s}_n) \; \doteq \; \mathtt{s} \end{aligned}$$

Q.E.D.

10.2 The Covered-By Relation

A nominal term is *covered by* another nominal term, if the latter is the result of replacing (locally) some nominal symbols in the former nominal term by some standard terms. As this replacement is independently of the concrete labels of the respective nominal symbols, the covered-by relation corresponds to the equivalence of nominal terms. We discuss first a weak version of this relation, where the nominal symbols, which are not covered by a standard term, may be different in both nominal terms.

10.2.1 Introduction of the Covered-By Relation

We provide the definition of the covered-by relation.

10.3 DEF (Covered-By Relation): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$. The nominal term $\mathtt{t}$ is *covered* by the nominal term $\mathtt{s}$ (formally, $\mathtt{t} \ll \mathtt{s}$), if one of the following conditions is satisfied:

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* $\mathtt{s} \in \mathbf{V}_*$ or $\mathtt{s} \in \mathtt{T}_0$
2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* $\mathtt{s} \backsimeq \mathtt{t}$
3. $\mathtt{t} \backsimeq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* If the following both conditions are satisfied:
   (a) $\mathtt{t}$ and $\mathtt{s}$ are similar. Formally: there are nominal terms $\mathtt{s}_k \in \mathtt{T}$ (for $k \in n'$) such that $\mathtt{s} \backsimeq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$.
   (b) The direct subterms of $\mathtt{t}$ are covered by the respective direct subterms of $\mathtt{s}$. Formally, for all $k \in n'$: $\mathtt{t}_k \ll \mathtt{s}_k$ for all $k \in n'$.

Remarks (Covered-By Relation):

1. *local replacement:* Due to clause (1) of the definition of the covered-by relation, nominal symbols in the covered nominal term can be covered by an arbitrary nominal symbol or by a standard term. The latter corresponds to our intuition that some nominal symbols of the covered nominal term are replaced *locally* by some standard terms.

   This local replacement can be understood as a substitution as given by the general substitution function, but not applied on the full nominal term, but locally (directly to the respective nominal symbol). This "locality" corresponds to the fact that we do not consider the labels of the nominal symbols. Subsequently, we use the informal expression "local" to refer to similar phenomena.

2. *less-structured relation:* Besides that the covered-by relation is a relation beyond homomorphisms (as the labels of the covered nominal symbols are neglected), the covered-by relation reminds on the less-structured relation.

   Nevertheless, there is a conceptional difference to the less-structured relation: if we would define the covered-by relation in full analogy to the less-structured relation, we would have to allow arbitrary nominal terms to cover a nominal symbol. By restricting these nominal terms to standard terms (and nominal symbols), the covered-by relation becomes a hybrid between the less-structured relation and the notion of elimination forms (which are nominal terms less-structured than standard terms).

3. *elimination forms:* If a nominal term $\mathtt{t}$ is an elimination form of a standard term t, then $\mathtt{t}$ is covered by t. More formally:

   $$\mathtt{t} \leq t \quad \Rightarrow \quad \mathtt{t} \ll t$$

   The converse direction does not hold in the general case. Investigate, for example:

   $$* + * \ \ll \ 0 + 1 \quad \text{but} \quad * + * \ \not\leq \ 0 + 1$$

   This clash of nominal symbols can be resolved by the use of the simplification function: if $\mathtt{t}$ is covered by $\mathtt{s}$, then the simplification of $\mathtt{t}$ is less structured than $\mathtt{s}$ (and even less structured than the unification of $\mathtt{s}$). More formally:

   $$\mathtt{t} \ \ll \ \mathtt{s} \quad \Rightarrow \quad \mathbf{simp}(\mathtt{t}) \ \leq \ \mathbf{uni}(\mathtt{s})$$

10.2.2 Criterion for the Covered-By Relation

We provide two equivalent conditions for the the covered-by relation based on the unification function and on the simplification function.

10.4 Proposition (Criterion - Covered-By Relation): The following three statements are equivalent for all nominal terms $\mathtt{t}, \mathtt{s} \in \mathtt{T}$.

1. *covered-by:* $\mathtt{t} \ll \mathtt{s}$
2. *unification:* $\mathbf{uni}(\mathtt{t}) \ll \mathbf{uni}(\mathtt{s})$
3. *simplification:* $\mathbf{simp}(\mathtt{t}) \ll \mathbf{simp}(\mathtt{s})$

Proof. We first prove the equivalence of the statements (1) and (2) in parallel, by induction over the structure of $\mathtt{t}$.

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* $\mathbf{uni}(\mathtt{t}) \doteq * \in \mathbf{V}_*$.

   Let $\mathtt{t} \ll \mathtt{s}$. By definition, $\mathtt{s} \in \mathtt{T}_0$ or $\mathtt{s} \in \mathbf{V}_*$. We have $\mathbf{uni}(\mathtt{s}) \doteq \mathtt{s} \in \mathtt{T}_0$ or $\mathbf{uni}(\mathtt{s}) \doteq * \in \mathbf{V}_*$. As a consequence, $\mathbf{uni}(\mathtt{t}) \ll \mathbf{uni}(\mathtt{s})$.

   Let $\mathtt{t} \not\ll \mathtt{s}$. As $\mathtt{t} \in \mathbf{V}_*$, $\mathtt{s} \notin \mathtt{T}_0 \cup \mathbf{V}_*$. Therefore, $\mathtt{s}$ is a complex proper nominal term and, therefore, $\mathbf{uni}(\mathtt{s}) \notin \mathtt{T}_0 \cup \mathbf{V}_*$. As a consequence, $\mathbf{uni}(\mathtt{t}) \not\ll \mathbf{uni}(\mathtt{s})$.

2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* $\mathbf{uni}(\mathtt{t}) \doteq \mathtt{t} \notin \mathbf{V}_*$, but atomic.

   If $\mathtt{t} \ll \mathtt{s}$, then $\mathtt{t} \doteq \mathtt{s}$ by definition of the covered-by relation. Therefore, $\mathbf{uni}(\mathtt{s}) \doteq \mathbf{uni}(\mathtt{t})$. As a consequence, $\mathbf{uni}(\mathtt{t}) \ll \mathbf{uni}(\mathtt{s})$.

   If $\mathtt{t} \not\ll \mathtt{s}$, then $\mathtt{t} \neq \mathtt{s}$. Assume $\mathbf{uni}(\mathtt{s}) \doteq \mathbf{uni}(\mathtt{t}) \doteq \mathtt{t}$. This would imply that $\mathtt{s} \doteq \mathtt{t}$, which is already excluded. Therefore $\mathbf{uni}(\mathtt{s}) \neq \mathbf{uni}(\mathtt{t})$. As a consequence, $\mathbf{uni}(\mathtt{t}) \not\ll \mathbf{uni}(\mathtt{s})$.

3. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:*

   Let $\mathtt{t} \ll \mathtt{s}$. First, we obtain that $\mathtt{t} \sim \mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$. As the unification function is structure preserving, also $\mathbf{uni}(\mathtt{t}) \sim \mathbf{uni}(\mathtt{s})$.

   Second, $\mathtt{t}_k \ll \mathtt{s}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain $\mathbf{uni}(\mathtt{t}_k) \ll \mathbf{uni}(\mathtt{s}_k)$ for all $k \in n'$. As the homomorphism $\mathbf{uni}$ is structure preserving, these are the direct subterms of $\mathbf{uni}(\mathtt{t})$ and of $\mathbf{uni}(\mathtt{s})$, respectively. Therefore, $\mathbf{uni}(\mathtt{t}) \ll \mathbf{uni}(\mathtt{s})$.

   Let $\mathbf{uni}(\mathtt{t}) \ll \mathbf{uni}(\mathtt{s})$. As $\mathtt{t}$ is complex, $\mathbf{uni}(\mathtt{t})$ is also complex. Therefore, $\mathbf{uni}(\mathtt{t}) \sim \mathbf{uni}(\mathtt{s})$. As $\mathbf{uni}$ is structure preserving, we also obtain that $\mathtt{t} \sim \mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$.

   Furthermore, the direct subterms of $\mathbf{uni}(\mathtt{t})$ are covered by the respective direct subterms of $\mathbf{uni}(\mathtt{s})$. As $\mathbf{uni}$ is structure preserving, these direct subterms are $\mathbf{uni}(\mathtt{t}_k)$ and $\mathbf{uni}(\mathtt{s}_k)$. The latter means that $\mathbf{uni}(\mathtt{t}_k) \ll \mathbf{uni}(\mathtt{s}_k)$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain $\mathtt{t}_k \ll \mathtt{s}_k$ for all $k \in n'$. As a consequence, $\mathtt{t} \ll \mathtt{s}$.

We show the equivalence between (1) and (3):

"⇒" Let $\mathtt{t} \ll \mathtt{s}$. As $\mathbf{uni}$ absorbs $\mathbf{simp}$ and with the equivalence shown above, we obtain:

$$\mathbf{uni}(\mathbf{simp}(\mathtt{t})) \doteq \mathbf{uni}(\mathtt{t}) \ll \mathbf{uni}(\mathtt{s}) \doteq \mathbf{uni}(\mathbf{simp}(\mathtt{s}))$$

Applying a second time the equivalence proved above, we obtain that $\mathbf{simp}(\mathtt{t}) \ll \mathbf{simp}(\mathtt{s})$.

"$\Leftarrow$" Let $\mathbf{simp}(\mathtt{t}) \ll \mathbf{simp}(\mathtt{s})$. Similarly as above, we obtain:

$$\mathbf{uni}(\mathtt{t}) \;\hat{=}\; \mathbf{uni}(\mathbf{simp}(\mathtt{t})) \;\ll\; \mathbf{uni}(\mathbf{simp}(\mathtt{s})) \;\hat{=}\; \mathbf{uni}(\mathtt{s})$$

As a consequence, $\mathtt{t} \ll \mathtt{s}$.

Q.E.D.

10.2.3 Partial Order modulo Equivalence

In the next proposition, we show that the covered-by relation is a partial order modulo the equivalence of nominal terms.

10.5 Proposition (Partial Order): The covered-by relation is a partial order modulo the equivalence of nominal terms on the set of nominal terms. A nominal term is maximal with respect to the covered-by relation (modulo the equivalence of nominal terms), if and only if it is a standard term, and minimal, if and only if its dual weight equals to zero.
Proof. We check the relevant statements.

1. *reflexive:* We have to show that if $\mathtt{t} \equiv \mathtt{s}$, then $\mathtt{t} \ll \mathtt{s}$.

   First, we observe $\mathtt{t} \ll \mathtt{t}$ for all nominal terms $\mathtt{t} \in \mathtt{T}$ $(\star)$ (trivial induction).

   Let $\mathtt{t} \equiv \mathtt{s}$, which means $\mathbf{uni}(\mathtt{t}) \;\hat{=}\; \mathbf{uni}(\mathtt{s})$. By $(\star)$, $\mathbf{uni}(\mathtt{t}) \ll \mathbf{uni}(\mathtt{s})$. By the criterion for the covered-by relation (unification), we also have $\mathtt{t} \ll \mathtt{s}$.

2. *maximal terms:* First, we show that standard terms are maximal. The latter means that if $t \ll \mathtt{s}$, then $t \equiv \mathtt{s}$ for all standard terms t and nominal terms $\mathtt{s}$. In order to show this statement, we prove (by induction over the structure of the standard term t) the stronger statement that $t \ll \mathtt{s}$ implies $t \;\hat{=}\; \mathtt{s}$.

   (a) t *atomic:* $t \ll \mathtt{s}$ implies by definition $t \;\hat{=}\; \mathtt{s}$.

   (b) $t \;\hat{=}\; f(t_0, \ldots t_m)$ *complex:* $t \ll \mathtt{s}$ implies first that $t \sim \mathtt{s}$, which means that $\mathtt{s} \;\hat{=}\; f(\mathtt{s}_0, \ldots \mathtt{s}_n)$ for some nominal terms $\mathtt{s}_k$. Furthermore, we have that $t_k \ll \mathtt{s}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain that $t_k \;\hat{=}\; \mathtt{s}_k$ for all $k \in n'$. As $t \sim \mathtt{s}$, the latter implies $t \;\hat{=}\; \mathtt{s}$.

We can conclude immediately $t \equiv \mathtt{s}$, as $t \doteq \mathtt{s}$ implies $\mathbf{uni}(t) \doteq \mathbf{uni}(\mathtt{s})$.

Second, we show by induction that proper nominal terms are not maximal. This means that if $\mathtt{t} \notin \mathtt{T}_0$, then there is a nominal term $\mathtt{s}$ such that $\mathtt{t} \ll \mathtt{s}$ and $\mathtt{t} \not\equiv \mathtt{s}$.

(a) $\mathtt{t}$ *atomic:* $\mathtt{t} \notin \mathtt{T}_0$ implies that $\mathtt{t} \in \mathbf{V}_*$. Investigate, for example, the nominal term $\mathtt{s} \doteq v_0 \in \mathtt{T}_0$. By definition of the covered-by relation, $\mathtt{t} \ll \mathtt{s}$. As $\mathbf{uni}(\mathtt{t}) \doteq * \not\equiv v_0 \doteq \mathbf{uni}(\mathtt{s})$, we also have that $\mathtt{t} \not\equiv \mathtt{s}$.

(b) $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$*:* Let $\mathtt{t} \notin \mathtt{T}_0$. Therefore, there is $k \in n'$ such that $\mathtt{t}_k \notin \mathtt{T}_0$. By induction hypothesis, we obtain that there is a nominal term $\mathtt{s}_k$ such that both $\mathtt{t}_k \ll \mathtt{s}_k$ and $\mathtt{t}_k \not\equiv \mathtt{s}_k$.

Let $\mathtt{s}_l \doteq \mathtt{t}_l$ for all $k \neq l \in n'$ and $\mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$.

First, we observe that $\mathtt{t} \sim \mathtt{s}$. Furthermore, we obtain by reflexivity (statement (1)) that $\mathtt{t}_l \ll \mathtt{s}_l$ for all $k \neq l \in n'$. As we also have $\mathtt{t}_k \ll \mathtt{s}_k$, we can conclude that $\mathtt{t} \ll \mathtt{s}$. Finally, we have $\mathtt{t} \not\equiv \mathtt{s}$, as $\mathtt{t}_k \not\equiv \mathtt{s}_k$.

By both inductions, we have shown that a nominal term $\mathtt{t}$ is maximal with respect to the covered-by relation (modulo the equivalence of nominal terms), if and only if $\mathtt{t}$ is a standard term.

3. *anti-symmetric:* We have to show that if $\mathtt{t} \ll \mathtt{s}$ and $\mathtt{s} \ll \mathtt{t}$, then $\mathtt{t} \equiv \mathtt{s}$. This is proved by an induction over the structure of $\mathtt{t}$.

(a) $\mathtt{t} \in \mathbf{V}_*$ *atomic:* $\mathtt{t} \ll \mathtt{s}$ implies that $\mathtt{s} \in \mathbf{V}_*$ or $\mathtt{s} \in \mathtt{T}_0$. The second case is excluded, as $\mathtt{s} \ll \mathtt{t}$ would imply, by statement (2), the contradiction that $\mathtt{t} \doteq \mathtt{s} \in \mathtt{T}_0$. Therefore, $\mathtt{s} \in \mathbf{V}_*$. The latter implies $\mathbf{uni}(\mathtt{t}) \doteq * \doteq \mathbf{uni}(\mathtt{s})$, which means that $\mathtt{t} \equiv \mathtt{s}$.

(b) $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* $\mathtt{t} \ll \mathtt{s}$ implies $\mathtt{t} \doteq \mathtt{s}$. We obtain immediately that $\mathbf{uni}(\mathtt{t}) \doteq \mathbf{uni}(\mathtt{s})$, which means that $\mathtt{t} \equiv \mathtt{s}$.

(c) $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* Let $\mathtt{t} \ll \mathtt{s}$ and $\mathtt{s} \ll \mathtt{t}$. By $\mathtt{t} \ll \mathtt{s}$, we obtain $\mathtt{t} \sim \mathtt{s}$, which means that $\mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$ for some suitable nominal terms $\mathtt{s}_k$. Furthermore, by $\mathtt{t} \ll \mathtt{s}$ and $\mathtt{s} \ll \mathtt{t}$ we obtain both $\mathtt{t}_k \ll \mathtt{s}_k$ and $\mathtt{s}_k \ll \mathtt{t}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain $\mathtt{t}_k \equiv \mathtt{s}_k$ for all $k \in n'$. The latter implies $\mathtt{t} \equiv \mathtt{s}$, as $\mathtt{t} \sim \mathtt{s}$.

4. *transitive:* We show by induction over the structure of $\mathtt{t}$ that if $\mathtt{t} \ll \mathtt{s}$ and $\mathtt{s} \ll \mathtt{r}$, then $\mathtt{t} \ll \mathtt{r}$.

   (a) $\mathtt{t} \in \mathbf{V}_*$ *atomic:* Let $\mathtt{t} \ll \mathtt{s}$ and $\mathtt{s} \ll \mathtt{r}$. As $\mathtt{t} \in \mathbf{V}_*$, we have that $\mathtt{s} \in \mathbf{V}_*$ or $\mathtt{s} \in \mathtt{T}_0$.

   - If $\mathtt{s} \in \mathbf{V}_*$: $\mathtt{s} \ll \mathtt{r}$ implies that $\mathtt{r} \in \mathbf{V}_*$ or $\mathtt{r} \in \mathtt{T}_0$. In both subcases, $\mathtt{t} \ll \mathtt{r}$.
   - If $\mathtt{s} \in \mathtt{T}_0$: by maximality of standard terms (statement (2)), $\mathtt{s} \ll \mathtt{r}$ implies that $\mathtt{s} \doteq \mathtt{r}$. Therefore, $\mathtt{t} \ll \mathtt{r}$.

   In all cases, we have shown that $\mathtt{t} \ll \mathtt{r}$.

   (b) $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* Let $\mathtt{t} \ll \mathtt{s}$ and $\mathtt{s} \ll \mathtt{r}$. As $\mathtt{t} \notin \mathbf{V}_*$, we have that $\mathtt{t} \in \mathtt{T}_0$. Due to the maximality of standard terms (statement (2)), $\mathtt{t} \ll \mathtt{s}$ implies that $\mathtt{t} \doteq \mathtt{s}$. Therefore, $\mathtt{t} \doteq \mathtt{s} \ll \mathtt{r}$.

   (c) $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* Let $\mathtt{t} \ll \mathtt{s}$ and $\mathtt{s} \ll \mathtt{r}$. First, we obtain that $\mathtt{t} \sim \mathtt{s}$ and $\mathtt{s} \sim \mathtt{r}$. This means that there are suitable nominal terns $\mathtt{s}_k$ and $\mathtt{r}_k$ such that:

   $$\mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n) \quad ; \quad \mathtt{r} \doteq f(\mathtt{r}_0, \ldots \mathtt{r}_n)$$

   In particular, $\mathtt{t} \sim \mathtt{r}$.

   Furthermore, we have $\mathtt{t}_k \ll \mathtt{s}_k$ and $\mathtt{s}_k \ll \mathtt{r}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we also have $\mathtt{t}_k \ll \mathtt{s}_k$ for all $k \in n'$. Therefore, $\mathtt{t} \ll \mathtt{r}$.

5. *minimal terms:* First, we show by induction that nominal terms with dual weight zero are minimal. The latter means that if both $\mathtt{t} \ll \mathtt{s}$ and $\overline{\mathbf{weight}(\mathtt{s})} = 0$, then $\mathtt{t} \equiv \mathtt{s}$ for all nominal terms $\mathtt{t}$ and $\mathtt{s}$.

   (a) $\mathtt{s}$ *atomic:* $\overline{\mathbf{weight}(\mathtt{s})} = 0$ implies that $\mathtt{s} \in \mathbf{V}_*$. As $\mathtt{t} \ll \mathtt{s}$, we obtain by case distinction $\mathtt{t} \in \mathbf{V}_*$. Therefore, we have that $\mathbf{uni}(\mathtt{t}) \doteq * \doteq \mathbf{uni}(\mathtt{s})$, which means that $\mathtt{t} \equiv \mathtt{s}$.

   (b) $\mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$ *complex:* Let $\mathtt{t} \ll \mathtt{s}$. We can exclude the case that $\mathtt{t} \in \mathbf{V}_*$, as $\mathtt{s} \notin \mathtt{T}_0$ (as $\overline{\mathbf{weight}}(\mathtt{s}) = 0$). We can also exclude the case that $\mathtt{t} \notin \mathbf{V}_*$, but atomic (as in this case $\mathtt{t} \doteq \mathtt{s}$, which is a contradiction). Therefore, $\mathtt{t}$ is complex. $\mathtt{t} \ll \mathtt{s}$ implies that $\mathtt{t} \sim \mathtt{s}$, which means that $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ for suitable nominal terms $\mathtt{t}_k$. Furthermore, we have that $\mathtt{t}_k \ll \mathtt{s}_k$ for all $k \in n'$.
   As $\overline{\mathbf{weight}(\mathtt{s})} = 0$, we have that $\overline{\mathbf{weight}(\mathtt{s}_k)} = 0$ for all $k \in n'$. Therefore, we may apply n'-many times induction hypothesis and obtain that $\mathtt{t}_k \equiv \mathtt{s}_k$ for all $k \in n'$. Therefore, $\mathtt{t} \equiv \mathtt{s}$.

Next, we show, again by induction, that nominal terms with non-zero dual weight are not minimal. This means that if $\overline{\mathbf{weight}}(\mathtt{s}) \neq 0$, then there is a nominal term $\mathtt{t}$ such that $\mathtt{t} \not\equiv \mathtt{s}$ and $\mathtt{t} \ll \mathtt{s}$ for all nominal terms $\mathtt{s}$.

(a) $\mathtt{s}$ *atomic:* $\overline{\mathbf{weight}}(\mathtt{s}) \neq 0$ means that $\mathtt{s} \notin \mathbf{V}_*$. Therefore, $\mathtt{s} \in \mathtt{T}_0$ is a standard term. The nominal term $\mathtt{t} \doteq *$ satisfies both that $\mathtt{t} \ll \mathtt{s}$ and $\mathtt{t} \not\equiv \mathtt{s}$ (the latter, as their unifications are different).

(b) $\mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$ *complex:* If $\mathtt{s} \in \mathtt{T}_0$, then $\mathtt{t} \doteq *$ satisfies the demanded conditions.

Otherwise, $\mathtt{s} \notin \mathtt{T}_0$ is a proper nominal term. As $\overline{\mathbf{weight}}(\mathtt{s}) \neq 0$ there is $k \in n'$ such that $\overline{\mathbf{weight}}(\mathtt{s}_k) \neq 0$. By induction hypothesis, there is a nominal term $\mathtt{t}_k$ such that both $\mathtt{t}_k \ll \mathtt{s}_k$ and $\mathtt{t}_k \not\equiv \mathtt{s}$.

Let $\mathtt{t}_l \doteq \mathtt{s}_l$ for all $k \neq l \in n'$ and $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$. By construction, we have $\mathtt{t} \sim \mathtt{s}$. As $\mathtt{t}_k \not\equiv \mathtt{s}_k$, we have $\mathtt{t} \not\equiv \mathtt{s}$. By reflexivity of the covered-by relation, we have $\mathtt{t}_l \ll \mathtt{s}_l$ for all $k \neq l \in n'$. As we also have that $\mathtt{t}_k \ll \mathtt{s}_k$, we can conclude $\mathtt{t} \ll \mathtt{s}$.

By both inductions, we have shown that a nominal term $\mathtt{s}$ is minimal with respect to the covered-by relation (modulo the equivalence of nominal terms), if and only if $\overline{\mathbf{weight}}(\mathtt{s}) = 0$. Q.E.D.

10.2.4 Chains with Respect to the Covered-By Relation

In the next proposition, we show that the strict version of the covered-by relation is compatible with the dual weight function and conversely compatible with the weight function. As an immediate consequence, the weight functions turn out to be a useful tool for the discussion of chains with respect to the covered-by relation.

10.6 Proposition (Weight of Covered Nominal Terms): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ be two nominal terms such that both $\mathtt{t} \ll \mathtt{s}$ and $\mathtt{t} \not\equiv \mathtt{s}$. The following both statements hold:

1. *decreasing weight:* $\mathbf{weight}(\mathtt{t}) > \mathbf{weight}(\mathtt{s})$.
2. *increasing dual weight:* $\overline{\mathbf{weight}}(\mathtt{t}) < \overline{\mathbf{weight}}(\mathtt{s})$.

Proof. We prove both statements in parallel by induction over the structure of the nominal term $\mathtt{t}$.

1. $\mathtt{t}$ *atomic:* $\mathtt{t} \ll \mathtt{s}$ and $\mathtt{t} \not\equiv \mathtt{s}$ means that $\mathtt{t} \in \mathbf{V}_*$ and $\mathtt{s} \in \mathtt{T}_0$. Therefore:

   $$\mathbf{weight}(\mathtt{t}) = 1 > 0 = \mathbf{weight}(\mathtt{s}) \quad ; \quad \overline{\mathbf{weight}}(\mathtt{t}) = 0 < \overline{\mathbf{weight}}(\mathtt{s})$$

   Observe that we only know that $\overline{\mathbf{weight}}(\mathtt{s}) \neq 0$, as the nominal term $\mathtt{s}$ can be complex.

2. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* Let $\mathtt{t} \ll \mathtt{s}$ and $\mathtt{t} \not\equiv \mathtt{s}$. As $\mathtt{t} \ll \mathtt{s}$, we have both that $\mathtt{t} \sim \mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$ for some suitable nominal terms $\mathtt{s}_k$ and $\mathtt{t}_k \ll \mathtt{s}_k$ for all $k \in n'$.

   $\mathtt{t} \not\equiv \mathtt{s}$ means that the respective unifications are not equal, and therefore, there is $k \in n'$ such that $\mathtt{t}_k \not\equiv \mathtt{s}_k$. By induction hypothesis, we obtain:

   $$\mathbf{weight}(\mathtt{t}_k) > \mathbf{weight}(\mathtt{s}_k) \quad ; \quad \overline{\mathbf{weight}}(\mathtt{t}_k) < \overline{\mathbf{weight}}(\mathtt{s}_k)$$

   Furthermore, for all $k \neq l \in n'$:

   $$\mathbf{weight}(\mathtt{t}_k) \geq \mathbf{weight}(\mathtt{s}_k) \quad ; \quad \overline{\mathbf{weight}}(\mathtt{t}_k) \leq \overline{\mathbf{weight}}(\mathtt{s}_k)$$

   (Equality holds, if the respective pairs of nominal terms are equivalent, otherwise we obtain strict inequality by induction hypothesis.)

   Altogether:

   $$\mathbf{weight}(\mathtt{t}) \;=\; \sum_{k \in n'} \mathbf{weight}(\mathtt{t}_k) \;>\; \sum_{k \in n'} \mathbf{weight}(s_k) \;=\; \mathbf{weight}(\mathtt{s})$$

   $$\overline{\mathbf{weight}}(\mathtt{t}) \;=\; \sum_{k \in n'} \overline{\mathbf{weight}}(\mathtt{t}_k) \;<\; \sum_{k \in n'} \overline{\mathbf{weight}}(s_k) \;=\; \overline{\mathbf{weight}}(\mathtt{s})$$

Q.E.D.

Remarks (Weight of Covered Terms):

1. *minimal and maximal terms:* As a corollary of the proposition above, we obtain that:

   - Standard terms t are maximal, as their weight equals to zero.
   - Nominal terms $\mathtt{t}$ having no standard terms as atomic subterms are minimal, as their dual weight equals to zero.

   Observe that we proved already a stronger statement in the proposition about the covered-by relation.

2. *chains:* Another immediate consequence of the proposition above is that every chain $\mathfrak{t} = \langle \mathtt{t}_k;\ k \in \omega \rangle$ with respect to the covered-by relation becomes stationary (modulo the equivalence of nominal terns). More precisely:

   (a) *ascending chains:* If $\mathfrak{t}$ is an ascending chain (this means $\mathtt{t}_k \ll \mathtt{t}_{k'}$ for all $k \in \omega$), then there is $k \in \omega$ such that $\mathtt{t}_k \equiv \mathtt{t}_l$ for all $l \in \omega$ such that $k < l$.

   (b) *descending chains:* If $\mathfrak{t}$ is a descending chain (this means $\mathtt{t}_{k'} \ll \mathtt{t}_k$ for all $k \in \omega$), then there is $k \in \omega$ such that $\mathtt{t}_k \equiv \mathtt{t}_l$ for all $l \in \omega$ such that $k < l$.

   Furthermore, if all nominal terms $\mathtt{t}_k$ in the chain are standard terms or unary terms (formally, if $\mathtt{t}_k \in \mathtt{T}^*$ for all $k \in \omega$), then both statements hold even with respect to syntactic equality instead of the equivalence of nominal terms.

   Finally, we observe that there are proper chains of length greater than 2, if and only if there is a proper function symbol available in the underlying formal language $\mathfrak{L}$.

10.2.5 Intermediate Nominal Terms

We conclude our introduction of the covered-by relation by providing the number of intermediate nominal terms (modulo the equivalence of nominal terms) modulo the equivalence of nominal terms.

10.7 Proposition (Intermediate Nominal Terms): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ such that $\mathtt{t}$ is minimal and $\mathtt{s}$ is maximal with respect to the covered-by relation. If $\mathtt{t} \ll \mathtt{s}$, then number of intermediate nominal terms $\mathtt{r} \in \mathtt{T}^*$ (intermediate means that $\mathtt{r}$ satisfies the condition $\mathtt{t} \ll \mathtt{r} \ll \mathtt{s}$) equals to $2^{\mathbf{weight}(\mathtt{t})}$.
Proof. By induction over the structure of $\mathtt{t}$.

1. $\mathtt{t}$ *atomic:* As $\mathtt{t}$ is minimal, we have that $\mathtt{t} \in \mathbf{V}_*$ and $\mathbf{weight}(\mathtt{t}) = 1$. As $\mathtt{s}$ is maximal, we have that $\mathtt{s} \in \mathtt{T}_0$ is a standard term. Trivially, $\mathtt{t} \ll \mathtt{s}$. Furthermore, $\mathtt{r} \in \mathtt{T}^*$ is intermediate, if and only if $\mathtt{r} \in \{*, \mathtt{s}\}$. Therefore, the postulated statement holds with respect to atomic nominal terms $\mathtt{t}$.

2. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* $\mathtt{t} \ll \mathtt{s}$ implies that $\mathtt{t} \sim \mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$ for suitable nominal terms $\mathtt{s}_k$ and $\mathtt{t}_k \ll \mathtt{s}_k$ for all $k \in n'$. As $\mathtt{t}$ is minimal, $\overline{\mathbf{weight}}(\mathtt{t}) = 0$. Therefore, $\overline{\mathbf{weight}}(\mathtt{t}_k) = 0$ for all $k \in n'$. This means that the direct subterms of $\mathtt{t}$ are also minimal. Furthermore, as $\mathtt{s}$ is

maximal, $\mathtt{s}$ is a standard term. Therefore, the direct subterms of $\mathtt{s}$ are also standard terms, which means that they are also maximal.

Every intermediate nominal term $\mathtt{r} \in \mathtt{T}^*$ satisfies the following conditions:

- $\mathtt{r} \doteq f(\mathtt{r}_0, \ldots \mathtt{r}_n)$ for nominal terms $\mathtt{r}_k$ (as $\mathtt{t} \ll \mathtt{r}$ implies $\mathtt{t} \sim \mathtt{r}$).
- $\mathtt{t}_k \ll \mathtt{r}_k \ll \mathtt{s}_k$ for all $k \in n'$ (as $\mathtt{t} \ll \mathtt{r} \ll \mathtt{s}$).
- $\mathtt{r}_k \in \mathtt{T}^*$ for all $k \in n'$ (as $\mathtt{r} \in \mathtt{T}^*$).

Observe that every combination of nominal terms $\mathtt{r}_k$ (for $k \in n'$) satisfying the conditions given above results in exactly one intermediate nominal term $\mathtt{r}$. Therefore, applying n'-many times induction hypothesis, we can calculate the number m of intermediate nominal terms as follows:

$$m \;=\; \prod_{k \in n'} 2^{\mathbf{weight}(\mathtt{t}_k)} \;=\; 2^{\sum_{k \in n'} \mathbf{weight}(\mathtt{t}_k)} \;=\; 2^{\mathbf{weight}(\mathtt{t})}$$

Q.E.D.

Remarks (Intermediate Nominal Terms):

1. *infinite intermediate terms:* Observe that if we drop the restriction in to the set $\mathtt{T}^*$ the proposition above, then the number of intermediate nominal terms is trivially infinite, as the equivalence classes of the minimal nominal terms are infinite.

   Slightly changing the perspective: the number of strictly intermediate nominal terms is infinite or zero, as strictly intermediate nominal terms are proper nominal terms with infinite equivalence classes.

2. *modulo equivalence:* The proposition above is formulated for terms contained in $\mathtt{T}^*$. Recalling that statements modulo an equivalence relation are to be understood as the respective statement with respect to the respective equivalence classes, we can read the result above as a statement about the number of intermediate terms modulo the equivalence of nominal terms. (Observe that there is exactly one canonical representative in the set $\mathtt{T}^*$ of every equivalence class with respect to the equivalence of nominal terms.)

10.3 The Strong Covered-By Relation

Besides the covered-by relation, as introduced above, we need in the course of our investigations the strong version covered-by relation. In the weak version, a nominal symbols of the covered nominal term was either covered by a standard term or by an arbitrary nominal symbol. In the strong version, we demand with respect to the second case that these nominal symbols are, additionally, equal. As a consequence of this restriction, the strong covered-by relation turns out to be a proper partial order (and not only modulo the equivalence of nominal terms).

10.3.1 Introduction of the Strong Covered-By Relation

We provide the definition of the strong covered-by relation.

10.8 DEF (Strong Covered-By Relation): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$. The nominal term $\mathtt{t}$ is *strongly covered* by the nominal term $\mathtt{s}$ (formally, $\mathtt{t} \ll_* \mathtt{s}$), if one of the following conditions is satisfied:

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* $\mathtt{s} \doteq \mathtt{t}$ or $\mathtt{s} \in \mathtt{T}_0$
2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* $\mathtt{s} \doteq \mathtt{t}$
3. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* If the following both conditions are satisfied:
   - $\mathtt{t}$ and $\mathtt{s}$ are similar. Formally: there are nominal terms $\mathtt{s}_k \in \mathtt{T}$ (for $k \in n'$) such that $\mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$.
   - The respective direct subterms are strongly covered by each other. Formally: $\mathtt{t}_k \ll_* \mathtt{s}_k$ for all $k \in n'$.

Remarks (Strong Covered-By Relation):

1. *covered-by relation:* Immediate by the definition, strongly covered nominal terms are also related by the covered-by relation. More formally, for all nominal terms $\mathtt{t}$ and $\mathtt{s}$:

$$\mathtt{t} \ll_* \mathtt{s} \quad \Rightarrow \quad \mathtt{t} \ll \mathtt{s}$$

2. *restriction:* The restriction of the strong covered-by relation to the set $\mathtt{T}^*$ of standard terms and unary nominal terms equals to the respective restriction of the covered-by relation. Formally, for all $\mathtt{t}, \mathtt{s} \in \mathtt{T}^*$:

$$\mathtt{t} \ll_* \mathtt{s} \quad \Leftrightarrow \quad \mathtt{t} \ll \mathtt{s}$$

10.3.2 Partial Order

In the next proposition, we show that the strong version of the covered-by relation is, indeed, a partial order on the set of nominal terms (and not only modulo the equivalence of nominal terms).

10.9 Proposition (Partial Order): The strong covered-by relation is a partial order on the set $\mathtt{T}$ of all nominal terms. Minimal elements are nominal terms having a dual weight equal to zero, standard terms are maximal.
Proof. We check the relevant properties.

1. *reflexive:* $*_k \ll_* *_k$ for all nominal symbols $*_k \in \mathbf{V}_*$ according to clause (1) of the definition; $\mathtt{t} \ll_* \mathtt{t}$ for standard atomic nominal terms $\mathtt{t}$ is given by clause (2); $\mathtt{t} \ll_* \mathtt{t}$ for complex nominal terms $\mathtt{t}$ by induction hypothesis and clause (3).

2. *maximal terms:* We have to show for all standard terms t: if $t \ll_* \mathtt{s}$, then $t \doteq \mathtt{s}$. This is immediate, if t is atomic, as $t \ll_* \mathtt{s}$ is given by clause (2) of the definition. If $t \doteq f(t_0, \dots t_n)$ is complex, then $t \ll_* \mathtt{s}$ implies that $t \sim \mathtt{s} \doteq f(\mathtt{s}_0, \dots \mathtt{s}_n)$. Furthermore, $t_k \ll_* \mathtt{s}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain $t_k \doteq \mathtt{s}_k$ for all $k \in n'$. The latter means that $t \doteq \mathtt{s}$.

3. *minimal terms:* We have to show for all nominal terms $\mathtt{t}$ having dual weight 0: if $\mathtt{s} \ll_* \mathtt{t}$, then $\mathtt{t} \doteq \mathtt{s}$. $\mathtt{s} \ll_* *_k$ is only possible according to clause (1) of the definition, which means that $\mathtt{s} \doteq *_k$. Standard atomic nominal terms have dual weight $1 \neq 0$. If $\mathtt{t} \doteq f(\mathtt{t}_0, \dots \mathtt{t}_n)$ has a dual weight of 0, then all direct subterms $\mathtt{t}_k$ have a dual weight of 0. As $\mathtt{t}$ cannot be a standard term (due to its dual weight), $\mathtt{s} \ll_* \mathtt{t}$ has to be given by clause (3) of the definition. Therefore, $\mathtt{t} \sim \mathtt{s}$, which means that $\mathtt{s} \doteq f(\mathtt{s}_0, \dots \mathtt{s}_n)$, and $\mathtt{s}_k \ll_* \mathtt{t}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain $\mathtt{t}_k \doteq \mathtt{s}_k$ for all $k \in n'$. The latter means that $\mathtt{t} \doteq \mathtt{s}$.

4. *transitive:* We have to show for all nominal terms $\mathtt{t}, \mathtt{s}, \mathtt{r}$: if $\mathtt{t} \ll_* \mathtt{s}$ and $\mathtt{s} \ll_* \mathtt{r}$, then $\mathtt{t} \ll_* \mathtt{r}$. If $\mathtt{t} \in \mathbf{V}_*$, then $\mathtt{s} \doteq \mathtt{t}$ or $\mathtt{s} \in \mathtt{T}_0$. In the first case, trivially $\mathtt{t} \ll_* \mathtt{r}$; in the second case: as standard terms are maximal, we obtain $\mathtt{s} \doteq \mathtt{r}$ and, therefore, $\mathtt{t} \ll_* \mathtt{r}$. If $\mathtt{t}$ is atomic and standard, then immediately $\mathtt{t} \doteq \mathtt{s}$ and, therefore, trivially, $\mathtt{t} \ll_* \mathtt{s}$. Finally, if $\mathtt{t} \doteq f(\mathtt{t}_0, \dots \mathtt{t}_n)$ is complex, then we obtain successively that $\mathtt{t} \sim \mathtt{s}$ and $\mathtt{s} \sim \mathtt{r}$. Furthermore, denoting the direct subterms as expected, for all $k \in n'$:

$$\mathtt{t}_k \ \ll_* \ \mathtt{s}_k \quad \text{and} \quad \mathtt{s}_k \ \ll_* \ \mathtt{r}_k$$

Applying n'-many times induction hypothesis, we obtain $\mathtt{t}_k \ll_* \mathtt{r}_k$ and, therefore, $\mathtt{t} \ll_* \mathtt{r}$.

5. *anti-symmetry:* We have to show for all nominal terms $\mathtt{t}$ and $\mathtt{s}$: if $\mathtt{t} \ll_* \mathtt{s}$ and $\mathtt{s} \ll_* \mathtt{t}$, then $\mathtt{t} \doteq \mathtt{s}$. If $\mathtt{t} \in \mathbf{V}_*$, then $\mathtt{t} \ll_* \mathtt{s}$ implies $\mathtt{s} \doteq \mathtt{t}$ (and nothing more is to be done) or $\mathtt{s} \in \mathtt{T}_0$. As standard terms are maximal, the latter is impossible, as $\mathtt{s} \ll_* \mathtt{t} \neq \mathtt{s}$. If $\mathtt{t}$ is standard atomic, then $\mathtt{t} \ll_* \mathtt{s}$ implies immediately by clause (2) of the definition $\mathtt{t} \doteq \mathtt{s}$. If $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$, then $\mathtt{t} \ll_* \mathtt{s}$ implies $\mathtt{t} \sim \mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$ and $\mathtt{t}_k \ll_* \mathtt{s}_k$ for all $k \in n'$. Furthermore, $\mathtt{s} \ll_* \mathtt{t}$ implies $\mathtt{s}_k \ll_* \mathtt{t}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain $\mathtt{t}_k \doteq \mathtt{s}_k$ for all $k \in n'$ and, therefore, $\mathtt{t} \doteq \mathtt{s}$. Q.E.D.

10.3.3 Chains and Intermediate Terms

We conclude our introduction of the strong covered-by relation by carrying over the results about chains and the number of intermediate nominal terms. In order to do so, we briefly discuss the weight and dual weight of strongly covered nominal terms

10.10 Proposition (Weight and Dual Weight): Let $\mathtt{t}, \mathtt{s} \in \mathbf{T}$. The following both statements hold:

1. *weight:* If $\mathtt{t} \neq \mathtt{s}$ and $\mathtt{t} \ll_* \mathtt{s}$, then $\mathbf{weight}(\mathtt{t}) > \mathbf{weight}(\mathtt{s})$
2. *dual weight:* If $\mathtt{t} \neq \mathtt{s}$ and $\mathtt{t} \ll_* \mathtt{s}$, then $\overline{\mathbf{weight}}(\mathtt{t}) < \overline{\mathbf{weight}}(\mathtt{s})$

Proof. Both statements are proved as in the analogous proposition with respect to the covered-by relation, but with reference to syntactic equality instead of the equivalence of nominal terms. Q.E.D.

Remarks (Weight and Dual Weight):

1. *covered-by relation:* We already mentioned that if $\mathtt{t} \ll_* \mathtt{s}$, then also $\mathtt{t} \ll \mathtt{s}$ (with respect to the non-strict version of both relations). Due to the proposition above, we can carry over this observation to the strict case. More formally: if $\mathtt{t} \ll_* \mathtt{s}$ and $\mathtt{t} \neq \mathtt{s}$, then $\mathtt{t} \ll \mathtt{s}$ and $\mathtt{t} \not\equiv \mathtt{s}$.

2. *chains:* As an immediate consequence, we can carry over our results concerning chains:

   (a) *ascending chains:* Ascending chains with respect to the strong covered-by relation are stationary.

   (b) *descending chains:* Descending chains with respect to the strong covered-by relation are stationary.

   (c) *intermediate terms:* If $\mathtt{t} \in \mathtt{T}$ is minimal and $\mathtt{s} \in \mathtt{T}$ is maximal with respect to the strong covered-by relation, then the number of intermediate nominal terms between $\mathtt{t}$ and $\mathtt{s}$ equals to $2^{\mathbf{weight}(\mathtt{t})}$. More formally:

$$|\{\mathtt{r} \in \mathtt{T};\ \mathtt{t} \ll_* \mathtt{r} \ll_* \mathtt{s}\}| \ = \ 2^{\mathbf{weight}(\mathtt{t})}$$

11 Multi-Shape Occurrences

We investigate the *multi-shape occurrences*, the generalisation of the standard occurrences allowing not only to represent a single shape (at possibly multiple positions), but a finite sequences of shapes.

11.1 Introduction of Multi-Shape Occurrences

We provide the formal definition of the multi-shape occurrences.

11.1 DEF (Multi-Shape Occurrences): Let $t \in \mathtt{T}_0$ be standard term, $\mathfrak{s} \in {\mathtt{T}_0}^n$ a finite sequence of standard terms (for $n \in \omega$) and $\mathtt{t} \in \mathtt{T}$ a nominal term.

1. *multi-shape occurrence:* The triple $\mathfrak{o} = \langle t, \mathfrak{s}, \mathtt{t} \rangle$ is called an n-place *multi-shape* occurrence, if $\mathtt{t}$ is an elimination form of t in which the sequence $\mathfrak{s}$ is eliminated (formally, if $t \doteq \mathtt{t}[\mathfrak{s}]$).

   In this case, the numbers $k \in n$ are also called *the places* of $\mathfrak{o}$.

2. *projections:* The standard term $t \doteq \mathbf{con}(\mathfrak{o})$ is called the *context*, the sequence $\mathfrak{s} = \mathbf{shape}(\mathfrak{o})$ the *sequence of shapes* and the nominal term $\mathtt{t} \doteq \mathbf{pos}(\mathfrak{o})$ the *position* of $\mathfrak{o}$.

3. *sets of occurrences:* We use $\mathcal{O}$, $\mathcal{O}(t)$ and $\mathcal{O}(t, \mathfrak{s})$ to denote the expected sets of multi-shape occurrences; $\mathcal{O}_n$, $\mathcal{O}_n(t)$ and $\mathcal{O}_n(t, \mathfrak{s})$ are the respective restrictions to occurrences having n places (for $n \in \omega$).

Remarks (Multi-Shape Occurrences):

1. *standard occurrences:* Identifying the shape of a standard occurrence with the sequence of length 1 containing the respective term as only entry, the standard occurrences become a special case of multi-shape occurrences.

2. *n-place occurrence:* The denomination as n-place occurrence does not depend on the position of a multi-shape occurrence, but on the finite length n of its sequence of shapes.

3. *standard properties:* If meaningful, then we attribute properties of the position $\mathbf{pos}(\mathfrak{o})$ also to the respective multi-shape occurrence $\mathfrak{o}$. In particular, we distinguish n-ary, simple, multiple and single multi-shape occurrences.

4. *restrictions on the position:* In contrast to the standard notion of occurrences, there is no a priori restriction on the position of a multi-shape occurrence. A posteriori, we observe that $\mathbf{rank}(\mathfrak{t}) \leq n = \mathbf{lng}(\mathfrak{s})$. (The position is an elimination form of a standard term and therefore, all nominal symbols have to be replaced by entries of the sequence of shapes.) In particular, the limit case that the position is a standard term and, therefore, syntactically equal to the context (formally, that $\mathbf{con}(\mathfrak{o}) \doteq \mathbf{pos}(\mathfrak{o})$) is not excluded.

5. *redundancies:* As the notion of standard occurrences, the notion of multi-shape occurrences is redundant. We provide the details:

   - Position and sequence of shapes determine the context.
   - Context and position partially determine the sequence of shapes.
   - Context and sequence of shapes do not determine the position.

   Partial determination means determination up to essential equality with respect to the position. (In particular, the length of the sequence of shapes is not determined, but has the rank of the position as lower bound.)

6. *empty occurrence:* The limit case of a 0-place occurrence $\mathfrak{o} = \langle t, \epsilon, t \rangle$, where ϵ is the empty sequence, is subsumed in the definition of multi-shape occurrences. We can use such empty occurrences as a mathematical objects representing the informal concept of "no occurrence".

11.2 Representation of Informal Occurrences

We presuppose that informally given occurrences in the same context and of the same shapes at the same informally given positions are equal, independently of the way how these positions are actually given. As a consequence and in contrast to the simple case of standard occurrence, different multi-shape occurrences can represent the same informally given occurrence. We provide some examples illustrating this phenomenon.

Example (Identical Occurrences): Let $\underline{0} + \underline{0}$ be an informally given occurrence in the context $t \mathrel{\widehat{=}} 0 + 0$ (where the intended positions are underlined). We provide some exemplary multi-shape occurrences representing the intended occurrence:

1. $\mathfrak{o}_1 = \langle t, \langle 0 \rangle, *_0 + *_0 \rangle$

   The representation is parsimonious, as we use the minimal possible numbers of nominal symbols for representing the intended positions, but for this reason the position is not a simple nominal term. Furthermore, the position of $\mathfrak{o}_1$ is normal and the sequence of shapes does not contain unnecessary entries (which is possible, as the position is, in particular, n-ary).

2. $\mathfrak{o}_2 = \langle t, \langle 0, 0 \rangle, *_0 + *_1 \rangle$

   The representation is not parsimonious, as we use different nominal symbols to represent the same shape (at different positions), but simple. As in the example above, the position is normal and the sequence of shapes does not contain unnecessary entries.

3. $\mathfrak{o}_2 = \langle t, \langle 0, 0 \rangle, *_1 + *_0 \rangle$

   In contrast to the second example, the labels of the position are permuted. Therefore, the position is simple, but not normal. The sequence of shapes does still not contain unnecessary entries.

4. $\mathfrak{o}_4 = \langle t, \langle 2, 0, 0, 1 \rangle, *_1 + *_2 \rangle$

   The sequence of shapes is longer than the rank of the position. Therefore, the last entry of the sequence of shapes can be omitted without any loss. As the position is not n-ary, the first entry of the sequence of shapes is unnecessary; we may replace this entry by arbitrary standard terms. But it is not possible to eliminate this entry without relabelling the nominal symbols in the position.

We additionally investigate the occurrence $\underline{0}+\underline{1}$ in the standard term $t \mathrel{\widehat{=}} 0+1$:

1. $\mathfrak{o} = \langle t, \langle 1, 0 \rangle, *_1 + *_0 \rangle$

   The representation is parsimonious and simple together. (This is possible as the multiplicity of each intended shape in t is equal to 1.)

2. There is no multi-shape occurrence $\mathfrak{o}' = \langle t, \mathfrak{s}, \mathfrak{t} \rangle$ representing the informally given occurrence satisfying the condition that the position $\mathfrak{t}$ is the unification $\mathbf{uni}(*_1 + *_0) \mathrel{\widehat{=}} *_0 + *_0$ of the position $\mathbf{pos}(\mathfrak{o})$ of $\mathfrak{o}$, as there are two different shapes which must be eliminated in $\mathfrak{t}$.

3. Observe that we may use the simplification $\mathbf{simp}(*_1 + *_0) \mathrel{\hat{=}} *_0 + *_1$ as position and obtain, for example, the following multi-shape occurrence:

$$\langle t, \langle 0, 1\rangle, *_0 + *_1\rangle$$

We could have used instead as shape any sequence of standard terms extending the sequence $\langle 0, 1\rangle$.

Analysing these examples, we observe:

1. *context:* The context of all multi-shape occurrences representing the informally given occurrence are equal and, in particular, equal to the context of the informally given occurrence.

2. *position:* The positions of the multi-shape occurrences representing the same intended occurrence are all equivalent. Nevertheless, there are, in the general case, equivalent nominal terms not capable of representing the intended position.

3. *shapes:* As every extension of a suitable sequence of shapes result again in a suitable sequence of shapes, the number of formal representatives of the same informally given occurrence is infinite.

11.3 Equivalence of Occurrences

Even though it is not possible to provide a formal criterion, whether a multi-shape occurrence represents, indeed, an informal occurrence, we can take our observations from above to provide an equivalence relation on multi-shape occurrences identifying those occurrences which represent the same informal occurrence (according to our intuition). The definition of this equivalence relation can be understood as the formalisation of our informal concept of representation.

11.3.1 Introduction of the Equivalence of Occurrences

We provide the formal definition of the equivalence of occurrences.

11.2 DEF (Equivalence of Occurrences): Two multi-shape occurrences $\mathfrak{o}$ and $\mathfrak{o}'$ are *equivalent* (formally, $\mathfrak{o} \equiv \mathfrak{o}'$), if the following both conditions are satisfied:

1. *context:* Both multi-shape occurrences have a common context (formally, $\mathbf{con}(\mathfrak{o}) \mathrel{\hat{=}} \mathbf{con}(\mathfrak{o}')$).

2. *position:* The same intended occurrences are marked by the respective positions (formally, $\mathbf{pos}(\mathfrak{o}) \equiv \mathbf{pos}(\mathfrak{o}')$).

Remarks (Equivalence of Occurrences):

1. *equivalence relation:* It is easily checked that the equivalence of occurrences is, indeed, an equivalence relation on multi-shape occurrences.

2. *shapes:* We do not have to formulate restrictions on the sequence of shapes, as the shapes are determined by context and position up to essential equality; the entries in the sequence of shapes, which are not determined by essential equality do not have any influence on the question, whether the same occurrence is represented.

3. *caveat:* As already seen in the examples, we cannot choose, in general, every nominal term equivalent to a given position to generate equivalent multi-shape occurrences. If n is the number of different actually eliminated entries in the sequence of shapes of a given multi-shape occurrence, then the position of an equivalent occurrence has to have at least n different free places. This restriction is, in general, not satisfied by the unification of the position, but trivially by its simplification.

   This means that we may use the unification of positions to check whether these positions are equivalent, but we may not use, in general, the unification function to generate equivalent occurrences.

Towards Normal Occurrences: There are two (incompatible) strategies to define a normal form for multi-shape occurrences with respect to the equivalence of occurrences:

1. *exhaustive use of nominal symbols:* One idea is to use as many nominal symbol in the position as possible. In such a position, each nominal symbol is only used once for the elimination of a single intended subterm. As a consequence, such positions are simple.

2. *parsimonious use of nominal symbols:* The other idea is to use as few nominal symbols in the position as possible. In such *parsimonious* positions, subterms of equal shape are eliminated by the same nominal symbol.

In order to obtain actually a normal form for occurrences, we have to eliminate additionally unnecessary entries in the sequence of shapes and to demand a distinguished order of the nominal symbols in the position.

In order to do so, we introduce subsequently "interesting" properties of multi-shape occurrences and their places and discuss transformations related to these properties. Finally, we prove the existence of uniquely determined normal forms (with respect to both principle ideas sketched above).

11.3.2 Regular Occurrences

A multi-shape occurrences is called *regular*, if every entry of the sequence of shapes is actually eliminated in the position of that occurrence; additionally, we demand that the position of such a regular occurrence is normal with respect to the isomorphism of nominal terms. We provide the formal definition.

11.3 DEF (Regular Occurrence): Let $\mathfrak{o} = \langle t, \mathfrak{s}, \mathfrak{t} \rangle$ be an n-place multi-shape occurrence (for $n \in \omega$).

1. *vacuous places:* The place k of the occurrence $\mathfrak{o}$ is *vacuous* (for $k \in n$), if the respective nominal symbol $*_k$ does not occur in the position $\mathbf{pos}(\mathfrak{t})$ of $\mathfrak{o}$ (formally, if $k \notin \mathbf{place}(\mathfrak{t})$); otherwise, the place k is called *non-vacuous*.

2. *vacuous occurrence:* The occurrence $\mathfrak{o}$ is called *partially vacuous*, if there is a place $k \in n$ such that $\mathfrak{o}$ is vacuous at the place k, and *completely vacuous*, if all places $k \in n$ are vacuous.

3. *regular occurrence:* The occurrence $\mathfrak{o}$ is called *regular*, if there is no vacuous place $k \in n$ (formally, if $\mathbf{place}(\mathfrak{t}) = n$) and if the position $\mathfrak{t}$ of $\mathfrak{o}$ is normal with respect to the isomorphism of nominal terms.

Remarks (Regular Occurrences):

1. *normality:* We included in the definition of regular occurrences that they have a normal position. This way, we avoid subsequently to demand explicitly that an occurrences under discussion has a normal position. Furthermore, we avoid the explicit ambiguity of discussing occurrences having a normal position (with respect to the isomorphism of nominal terms) without being normal (with respect to the equivalence of occurrences).

2. *empty occurrences:* An empty occurrence $\mathfrak{o} = \langle t, \epsilon, t \rangle$ is regular, as $\mathfrak{o}$ has no vacuous places and a normal position; nevertheless, such an occurrence $\mathfrak{o}$ is also completely vacuous. Empty occurrences are the only occurrences being both regular and completely vacuous.

3. *vacuous places:* Vacuous places do not affect which informal occurrence is represented by a multi-shape occurrence. In other words: if we eliminate vacuous places in a multi-shape occurrence, then the same informal occurrence is represented as before.

In the next proposition, we show that we can eliminate vacuous places in an occurrence.

11.4 Proposition (Regular Occurrences): Let $n \in \omega$. Every n-place multi-shape occurrence $\mathfrak{o} = \langle t, \mathfrak{s}, \mathfrak{t} \rangle$ can be transformed into an equivalent occurrence $\mathfrak{o}'$, which is regular.
Proof.

1. *k-ary and normal position:* In a first step, we transform the position $\mathfrak{t}$ of $\mathfrak{o}$ into its normal form $\mathfrak{t}'$ (with respect to the isomorphism of nominal terms) via a suitable isomorphism F. In particular, $\mathfrak{t}$ is k-ary for a natural number $k = |\mathbf{place}(\mathfrak{t})| \leq n$.

   Additionally, we rearrange the sequence of shapes according to the isomorphism F. Let $\mathfrak{s}''$ be the result of this rearrangement. The triple $\mathfrak{o}'' = \langle t, \mathfrak{s}'', \mathfrak{t}' \rangle$ is an occurrence and, in particular, equivalent to $\mathfrak{o}$.

2. *regular:* In order to transform $\mathfrak{o}''$ into a regular occurrence, we have to eliminate the vacuous entries of the sequence $\mathfrak{s}''$ (which are the entries at the positions l such that $k \leq l < n$): let $\mathfrak{s}' = k(\mathfrak{s}'')$ be the initial segment of $\mathfrak{s}''$ of length k. The resulting occurrence $\mathfrak{o}' = \langle t, \mathfrak{s}', \mathfrak{t}' \rangle$ is still equivalent to $\mathfrak{o}$ and, in particular, regular. Q.E.D.

Convention (Attribution of Properties): In order to avoid limit cases and undesired exceptions, we agree upon the convention that all properties subsequently attributed to a place of a multi-shape occurrence are only attributed to non-vacuous places. Furthermore, properties of multi-shape occurrences, which we introduce subsequently, depend only on the non-vacuous places.

11.3.3 Simple Normal Form

We define the first kind of canonical representatives of an equivalence class with respect to the equivalence of occurrences, namely the simple normal forms.

11.5 DEF (Simple Normal Form): An occurrence $\mathfrak{o} = \langle t, \mathfrak{s}, \mathfrak{t} \rangle$ is in *simple normal form*, if $\mathfrak{o}$ is regular and simple.

Remarks (Simple Normal Form):

1. *simple occurrence:* Recall: an occurrence is simple, if its position is so.

2. *empty occurrences:* Recall that standard terms are, in particular, simple nominal terms; therefore, an empty occurrence $\mathfrak{o} = \langle t, \epsilon, t \rangle$ is in simple normal form. As the equivalence class of the position t is the singleton $\{t\}$, we obtain that exactly the occurrence $\mathfrak{o}' = \langle t, \mathfrak{s}, t \rangle$ are equivalent to $\mathfrak{o}$ (for arbitrary finite sequences $\mathfrak{s} \in \mathtt{T}_0{}^{<\omega}$ of standard terms). If $\mathfrak{s}$ is not the empty sequence, then $\mathfrak{o}'$ is not regular, as there are vacuous places.

3. *standard occurrences:* A standard occurrence $\mathfrak{o} = \langle t, s, \mathfrak{t} \rangle$ is in simple normal form, if and only if $\mathfrak{t}$ is simple. As $\mathfrak{t}$ is also unary, the latter is equivalent to the condition that $\mathfrak{o}$ is a single occurrence.

We show in the next proposition that every multi-shape occurrence can be transformed into a uniquely determined equivalent occurrence in simple normal form.

11.6 Proposition (Simple Normal Form): Let $\mathfrak{o} = \langle t, \mathfrak{s}, \mathfrak{t} \rangle$ be a multi-shape occurrence. There is a uniquely determined multi-shape occurrence $\mathfrak{o}'$ in simple normal form and equivalent to $\mathfrak{o}$.
Proof. We first transform $\mathfrak{o}$ into its simple normal form $\mathfrak{o}'$ and then we show that $\mathfrak{o}'$ is uniquely determined.

1. *position:* The position $\mathfrak{t}'$ of $\mathfrak{o}'$ is constructed via the simplification function: $\mathfrak{t}' \mathrel{\widehat{=}} \mathbf{simp}(\mathfrak{t})$.

   Recall that the nominal term $\mathfrak{t}'$ is, by construction, normal with respect to the isomorphism of nominal terms and simple. Furthermore, $\mathfrak{t}' \equiv \mathfrak{t}$, as the simplification function is idempotent.

2. *sequence of shapes:* Due to the proposition about elimination forms (in the section about simplification), $\mathfrak{t} \leq t$ implies $\mathfrak{t}' \mathrel{\widehat{=}} \mathbf{simp}(\mathfrak{t}) \leq t$. Hence, $\mathfrak{t}'$ is an elimination form of the context t of $\mathfrak{o}$.

   As a consequence, there is a sequence $\mathfrak{r}$ of standard terms eliminated in $\mathfrak{t}'$ with respect to t. As $\mathfrak{t}'$ is m-ary (for $m = \mathbf{weight}(\mathfrak{t})$), the uniquely determined initial segment $\mathfrak{s}' = m(\mathfrak{r})$ of $\mathfrak{r}$ is essentially equal to $\mathfrak{r}$ with respect to $\mathfrak{t}'$. Therefore, $\mathfrak{t}'[\mathfrak{s}'] \mathrel{\widehat{=}} \mathfrak{t}'[\mathfrak{r}] \mathrel{\widehat{=}} t$.

3. *occurrence:* Let $\mathfrak{o}' = \langle t, \mathfrak{s}', \mathfrak{t}' \rangle$. As $\mathfrak{t}'[\mathfrak{s}'] \eqsim t$ and as $\mathbf{lng}(\mathfrak{s}') = m$, $\mathfrak{o}'$ is an m-place multi-shape occurrence. As $\mathfrak{t}$ is m-ary and normal, $\mathfrak{o}'$ is regular. As $\mathfrak{t}$ is simple, $\mathfrak{o}'$ is already in simple normal form.

4. *uniqueness:* Let $\mathfrak{o}'' = \langle t'', \mathfrak{s}'', \mathfrak{t}'' \rangle$ be an occurrence in simple normal form and equivalent to $\mathfrak{o}$. As the equivalence of occurrences is an equivalence relation, we obtain $\mathfrak{o}' \equiv \mathfrak{o}''$. As equivalent occurrences have the same context, we obtain $t \eqsim t''$. As equivalent occurrences have equivalent positions, we obtain that $\mathfrak{t}' \equiv \mathfrak{t}''$. Therefore, we have:

$$\mathfrak{t}' \eqsim \mathbf{simp}(\mathfrak{t}') \eqsim \mathbf{simp}(\mathfrak{t}'') \eqsim \mathfrak{t}''$$

   The second equation holds, as $\mathfrak{t}' \equiv \mathfrak{t}''$, the other two equations, as both $\mathfrak{t}'$ and $\mathfrak{t}''$ are simple and normal (with respect to the isomorphism of nominal terms). Due to uniqueness of the actually eliminated sequence, we obtain finally $\mathfrak{s}' = \mathfrak{s}''$. Therefore, $\mathfrak{o}' = \mathfrak{o}''$ and $\mathfrak{o}'$ is, indeed, the uniquely determined normal form of $\mathfrak{o}$. Q.E.D.

Remarks (Simple Normal Form):

1. *actually eliminated sequence:* In the proof above, the sequence $\mathfrak{s}'$ of actually eliminated shapes of the normal occurrence $\mathfrak{o}'$ was obtained via the argument that if $\mathfrak{t}$ is an elimination form of t, then also its simplification $\mathbf{simp}(\mathfrak{t})$.

   Alternatively, we could have argued that there is a simple homomorphism F mapping $\mathbf{simp}(\mathfrak{t})$ to $\mathfrak{t}$. Due to the proposition about expansions and contraction, we could have obtained $\mathfrak{s}'$ as an F-expansion of the sequence $\mathfrak{s}$. In particular, if we analyse the application of **simp** on $\mathfrak{t}$, we may obtain concretely the homomorphism F (up to essential equality) and, therefore, also the concrete sequence $\mathfrak{s}'$ of shapes of the normal form $\mathfrak{o}'$.

11.3.4 Redundant and Parsimonious Occurrences

We call two places in a multi-shape occurrence *uniform*, if the same shape is eliminated in these places; if such places are different, then they are *redundant*. If there are no redundant places in a multi-shape occurrence, then this occurrence is called *parsimonious*. We provide the formal definitions.

11.7 DEF (Parsimonious Occurrences): Let $\mathfrak{o} = \langle t, \mathfrak{s}, \mathtt{t} \rangle$ be an n-place multi-shape occurrence (for $n \in \omega$).

1. *uniform places:* Two places $k, l \in \mathbf{place}(\mathtt{t})$ are called *uniform*, if the respective shapes s_k and s_l (contained in the sequence $\mathfrak{s}$ of shapes) are equal (formally, if $s_k \backsimeq s_l$).

2. *redundant // parsimonious places:* A place $k \in \mathbf{place}(\mathtt{t})$ is called *redundant*, if there is a place $l \in \mathbf{place}(\mathtt{t})$ different from k ($k \neq l$) such that k and l are uniform; otherwise, k is called *parsimonious*.

3. *uniform occurrence:* The occurrence $\mathfrak{o}$ is called *uniform*, if all of its places $k, l \in \mathbf{place}(\mathtt{t})$ are pairwise uniform.

4. *redundant // parsimonious occurrence:* The occurrence $\mathfrak{o}$ is called *redundant*, if there are redundant places $k \in \mathbf{place}(\mathtt{t})$; otherwise $\mathfrak{o}$ is called *parsimonious*.

Remarks (Redundant and Parsimonious Occurrences):

1. *uniformity of places:* The uniformity of places is an equivalence relation on the set $\mathbf{place}(\mathtt{t})$ of free places of the position $\mathtt{t}$ of an occurrence $\mathfrak{o}$. The equivalence classes with respect to this relation are denoted as follows:
$$[k]_u \;=\; \{l \in \mathbf{place}(\mathtt{t});\; k \text{ and } l \text{ are uniform}\}$$
A place $k \in \mathbf{place}(\mathtt{t})$ of an occurrence $\mathfrak{o}$ is parsimonious, if and only if its equivalence class $[k]_u$ is a singleton.

2. *sequence of shapes:* A regular occurrence $\mathfrak{o}$ is parsimonious, if and only if all entries s_k of the sequence $\mathfrak{s}$ of shapes are pairwise different. More generally: an arbitrary occurrence $\mathfrak{o}$ is parsimonious, if and only if the entries s_k of the sequence $\mathfrak{s}$ of shapes at non-vacuous places k are pairwise different.

3. *limit case:* Unary occurrence $\mathfrak{o} = \langle t, \mathfrak{s}, \mathtt{t} \rangle$ (these are the standard occurrences) are uniform and parsimonious. The empty occurrences $\mathfrak{o} = \langle t, \epsilon, t \rangle$ are trivially uniform and parsimonious.

In the next proposition, we show that we can eliminate redundant places of a multi-shape occurrence and that we can transform this way every occurrence into an equivalent and parsimonious occurrence.

11.8 Proposition (Parsimonious Occurrences): Let $n \in \omega$ be arbitrary. Every n-place multi-shape occurrence $\mathfrak{o} = \langle t, \mathfrak{s}, \mathfrak{t} \rangle$ can be transformed into an equivalent occurrence $\mathfrak{o}'$, which is parsimonious.
Proof.

1. *simple homomorphism:* Let $F \in \mathbf{Hom}_s(\mathtt{T})$ be the simple homomorphism induced by the following function on the set of all nominal terms:

$$F : *_k \mapsto \begin{cases} *_{\min([k]_u)} & \text{if } k \in \mathbf{place}(\mathfrak{t}) \\ *_k & \text{otherwise} \end{cases}$$

   Due to the definition of the uniformity relation the following statement holds for all $k \in \mathbf{place}(\mathfrak{t})$: $s_k \eqsim s_l$ for all $l \in [k]_u$, in particular for $l = \min([k]_u)$. Therefore, $s_k \eqsim s_{F(k)}$ for all $k \in \mathbf{place}(\mathfrak{t})$. The latter means that the sequence $\mathfrak{t}$ is an F-expansion of itself with respect to the position $\mathfrak{t}$ of $\mathfrak{o}$.

   As $\mathfrak{s}$ is finite, we have to check two side conditions: the conditions $\mathbf{rank}(\mathfrak{t}) \leq \mathbf{lng}(\mathfrak{s})$ is satisfied by the fact that $\mathfrak{o}$ is an occurrence and the condition $\mathbf{rank}(F(\mathfrak{t})) \leq \mathbf{lng}(\mathfrak{s})$, as $F(k) \leq k$ for all $k \in \mathbf{place}(\mathfrak{t})$.

   Due to the proposition about expansions and contractions, we obtain:

$$t \eqsim \mathfrak{t}[\mathfrak{s}] \eqsim F(\mathfrak{t})[\mathfrak{s}]$$

2. *parsimonious occurrence:* We define as follows: $\mathfrak{o}' = \langle t, \mathfrak{s}, F(\mathfrak{t}) \rangle$. As $F(\mathfrak{t})$ is an elimination form of t, in which the sequence $\mathfrak{s}$ is eliminated, $\mathfrak{o}'$ is an n-place multi-shape occurrence.

   Furthermore, $\mathfrak{o}'$ is by construction parsimonious, as all uniform places of $\mathfrak{o}$ are mapped to the same place, namely to the minimum of their equivalence class. Finally, as F is a simple homomorphism, $\mathfrak{t}$ and $F(\mathfrak{t})$ are equivalent (with respect to the equivalence of nominal terms). Therefore, $\mathfrak{o} \equiv \mathfrak{o}'$. Q.E.D.

Remarks (Parsimonious Occurrences): We discuss briefly some special cases of the proposition above:

1. *vacuous places:* If $\mathfrak{o}$ is not already parsimonious, then the transformation of $\mathfrak{o}$ into the parsimonious occurrence $\mathfrak{o}'$ generates vacuous places. More precisely, if $k \neq \min([k]_u)$, then $k \notin \mathbf{place}(F(\mathfrak{t}))$ for all $k \in \mathbf{place}(\mathfrak{t})$; as we do not change the sequence of shapes, such k become vacuous places in $\mathfrak{o}'$.

2. *simple occurrence:* Only if $\mathfrak{o}$ is simple and parsimonious, the transformed occurrence $\mathfrak{o}'$ is simple. More precisely, for all $k \in \mathbf{place}(F(\mathfrak{t}))$:

$$\mathbf{mult}(*_k, F(\mathfrak{t})) \;=\; \sum_{l \in [k]_u} \mathbf{mult}(*_l, \mathfrak{t})$$

3. *m-ary positions:* If the position $\mathfrak{t}$ of the n-place occurrence $\mathfrak{o}$ is m-ary (for $m \leq n$), then we can provide the position $\mathfrak{t}' \doteq F(\mathfrak{t})$ of $\mathfrak{o}'$ via the following application of the general substitution function:

$$\mathfrak{t}' \;\doteq\; \mathfrak{t}\big[*_{\min([0]_u)}, \ldots *_{\min([m-1]_u)}\big]$$

In general, $\mathfrak{t}'$ is not k-ary for no $k \in \omega$.

11.3.5 Parsimonious Normal Form

We introduce the second kind of normal occurrences (with respect to the equivalence of occrences), namely the parsimonious normal forms.

11.9 DEF (Parsimonious Normal Form): An occurrence $\mathfrak{o} = \langle t, \mathfrak{s}, \mathfrak{t} \rangle$ is in parsimonious normal form, if $\mathfrak{o}$ is regular and parsimonious.

Remarks (Parsimonious Normal Form):

1. *empty occurrence:* The empty occurrences $\langle t, \epsilon, t \rangle$ are trivially in parsimonious normal form.

2. *standard occurrences:* By definition, standard occurrences $\langle t, s, \mathfrak{t} \rangle$ are in parsimonious normal form.

In the next proposition, we show that every multi-shape occurrence can be transformed into a uniquely determined equivalent occurrence in parsimonious normal form.

11.10 Proposition (Parsimonious Normal Form): Let $\mathfrak{o} = \langle t, \mathfrak{s}, \mathfrak{t} \rangle$ be a multi-shape occurrence. There is a uniquely determined occurrence $\mathfrak{o}'$ equivalent to $\mathfrak{o}$ and in parsimonious normal form.
Proof. We first transform $\mathfrak{o}$ into its parsimonious normal form $\mathfrak{o}'$ and then we show that this normal form is uniquely determined.

1. *parsimonious occurrence:* In a first step, we transform $\mathfrak{o}$ into an occurrence $\mathfrak{o}''$ according to the proposition about parsimonious occurrences.

2. *regularity:* In a second step, we transform $\mathfrak{o}''$ into a multi-shape occurrence $\mathfrak{o}' = \langle t, \mathfrak{s}', \mathfrak{t}' \rangle$ according to the proposition of regular occurrences.

   By construction, $\mathfrak{o} \equiv \mathfrak{o}'$ and $\mathfrak{o}'$ is regular. In order to see that $\mathfrak{o}'$ is parsimonious, we have to recall the construction used in the proposition about regular occurrences: in a first step, the places are rearranged according to an isomorphism. An application of an isomorphism does not change the size of equivalence classes with respect to the uniformity of the places. Therefore, the rearranged occurrence is parsimonious as $\mathfrak{o}''$ is so. In the second step, vacuous places are eliminated by reducing the length of the sequence of shapes. Again, this does not change the fact, whether an occurrence is parsimonious or not. Therefore, $\mathfrak{o}'$ is, indeed, parsimonious.

3. *uniqueness:* Let $\mathfrak{o}'' = \langle t, \mathfrak{s}'', \mathfrak{t}'' \rangle$ be another occurrence in parsimonious normal form and equivalent to $\mathfrak{o}$.

   We first discuss the case that $\mathfrak{o}$ has a standard position $\mathfrak{t} \in \mathtt{T}_0$. In this case, the equivalence of the three positions implies that they are already equal, formally: $\mathfrak{t}' \doteq \mathfrak{t} \doteq \mathfrak{t}''$. Due to regularity of $\mathfrak{o}'$ and $\mathfrak{o}''$, the shapes of $\mathfrak{o}'$ and $\mathfrak{o}''$ are the empty sequence, formally: $\mathfrak{s}' = \epsilon = \mathfrak{s}''$. Therefore, $\mathfrak{o}' = \mathfrak{o}''$.

   We investigate the case that the position $\mathfrak{t}$ of $\mathfrak{o}$ is a proper nominal term. Furthermore, we may assume, without loss of generality, that $\mathfrak{o}$ is in simple normal form. (If $\mathbf{simp}(\mathfrak{o})$ is the simple normal form of $\mathfrak{o}$, then $\mathfrak{o} \equiv \mathbf{simp}(\mathfrak{o})$. As the equivalence of occurrences is an equivalence relation, both $\mathfrak{o}'$ and $\mathfrak{o}''$ are equivalent to $\mathbf{simp}(\mathfrak{o})$.)

   As $\mathfrak{o}$ is simple, there are simple homomorphisms F' and F'' such that:

$$F'(\mathfrak{t}) \doteq \mathfrak{t}' \quad ; \quad F''(\mathfrak{t}) \doteq \mathfrak{t}''$$

   Recalling that $\mathfrak{t}$, $\mathfrak{t}'$ and $\mathfrak{t}''$ are proper nominal terms, we argue as follows: as $\mathfrak{t}'$ and $\mathfrak{t}''$ are normal with respect to the isomorphism of nominal terms, we have both that $\min(\mathfrak{t}') = 0$ and $\min(\mathfrak{t}'') = 0$. As $\min(\mathfrak{t}) = 0$, we may calculate as follows:

$$\begin{aligned} 0 &= \min(\mathfrak{t}') = \min(F'(\mathfrak{t})) = \min(F'(*_0)) \\ 0 &= \min(\mathfrak{t}'') = \min(F''(\mathfrak{t})) = \min(F''(*_0)) \end{aligned}$$

   As a consequence, $F'(*_0) \doteq *_0$ and $F''(*_0) \doteq *_0$.

   Furthermore, as $\mathfrak{t}'$ and $\mathfrak{t}''$ are the positions of parsimonious occurrences, we have that $F'(*_k) \doteq *_0 \doteq F''(*_k)$ for all $k \in [0]_u$.

Iterating the argumentation (discussing the next leftmost nominal symbol and the equivalence class of its label with respect to the uniformity relation) finitely many times, we obtain that F' and F'' are essentially equal with respect to $\mathfrak{t}$, and therefore:

$$\mathfrak{t}' \;\hat{=}\; F'(\mathfrak{t}) \;\hat{=}\; F''(\mathfrak{t}) \;\hat{=}\; \mathfrak{t}''$$

The shape $\mathfrak{s}'$ is the uniquely determined initial segment of all sequences eliminated in $\mathfrak{t}'$ with respect to the context t. As $\mathfrak{t}'' \;\hat{=}\; \mathfrak{t}'$, the same holds for $\mathfrak{s}'' = \mathfrak{s}'$. Therefore, $\mathfrak{o}' = \mathfrak{o}''$. The latter means that the parsimonious normal form is, indeed, uniquely determined. Q.E.D.

12 Independence of Nominal Terms

Two occurrences (in the same context) are *independent*, if the respective shapes do not overlap. Crucial aspect of this concept of independence is the position of occurrences. In order to prepare the introduction of independent occurrences, we discuss in this section the independence of nominal terms.

Two sufficiently similar nominal terms are called *independent*, if the nominal symbols of one nominal term are locally covered by a standard term in the other nominal term and vice versa. We investigate first a weak version of this relation, where it is additionally permitted that a nominal symbol is locally covered by the same nominal symbol; for the strong version (discussed in the next section), the latter is excluded.

12.1 Formal Introduction of Independence

We introduce the notion of independent nominal terms.

12.1 DEF (Independent Nominal Terms): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ be arbitrary. The nominal terms $\mathtt{t}$ and $\mathtt{s}$ are *independent* (formally, $\mathtt{t} \parallel \mathtt{s}$), if one of the following conditions is satisfied:

1. *atomic nominal terms:* The nominal terms $\mathtt{t}$ and $\mathtt{s}$ are both atomic and equal.

2. *nominal symbol:* One nominal term is a nominal symbol and covered by the other nominal term which is standard.

   Formally: ($\mathtt{t} \in \mathbf{V}_*$ and $\mathtt{s} \in \mathtt{T}_0$) or ($\mathtt{t} \in \mathtt{T}_0$ and $\mathtt{s} \in \mathbf{V}_*$).

3. *complex nominal terms:* The nominal terms $\mathtt{t}$ and $\mathtt{s}$ are similar and the respective direct subterms are independent.

   Formally: $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n) \sim f(\mathtt{s}_0, \ldots \mathtt{s}_n) \doteq \mathtt{s}$ and $\mathtt{t}_k \parallel \mathtt{s}_k$ for all $k \in n'$.

Furthermore, a set $\mathtt{S} \subseteq \mathtt{T}$ is called *independent*, if all pairs $\mathtt{t}, \mathtt{s} \in \mathtt{S}$ of nominal terms in $\mathtt{S}$ are independent.

Remarks (Independent Nominal Terms):

1. *atomic nominal terms:* In clause (1) of the definition, we only demand that $\mathtt{t}$ and $\mathtt{s}$ are atomic. This means that the following both cases are subsumed: $\mathtt{t}$ and $\mathtt{s}$ are equal standard atomic terms (variables or constant symbols) as well as they are equal nominal symbols.

2. *locally covered-by:* Independence of nominal terms is related with the strong covered-by relation, as in both relations nominal symbols may be covered by themselves or by standard terms. Due to clause (2) of the definition, the independence relation becomes a local symmetric version of the strong covered-by relation. As symmetry is given locally, independent nominal terms are, in general, not covering each other. Observe the following example:

$$* + x \parallel x + * \quad \text{but} \quad * + x \not\ll_* x + * \text{ and } x + * \not\ll_* * + x$$

3. *sufficiently similar:* Independent nominal terms are "sufficiently" similar. The latter means that if we neglect the covered nominal symbols and their standard covering in both nominal terms, then they are equal. This similarity is guaranteed in the definition above by clause (1), demanding equality, and by clause (3), demanding similarity. In other words: if we replace the standard subterms found in clause (2) by the respective nominal symbol, then the results of these replacements are equal nominal terms.

Basic Properties (Independence): We communicate some basic properties of the independence relation.

1. *standard terms:* The restriction of the independence relation to standard terms equals to the syntactic equality. More formally, $t \parallel s$ implies $t \doteq s$ for all standard terms t and s.

2. *reflexivity:* The independence relation is reflexive. More formally, $\mathtt{t} \parallel \mathtt{t}$ for all nominal terms $\mathtt{t}$.

3. *symmetry:* The independence relation is symmetric. More formally, $\mathtt{t} \parallel \mathtt{s}$ implies $\mathtt{s} \parallel \mathtt{t}$ for all nominal terms $\mathtt{t}$ and $\mathtt{s}$.

4. *non-transitive:* The independence relation is not transitive. There are nominal terms $\mathtt{t}$ and $\mathtt{s}$ and $\mathtt{r}$ such that $\mathtt{t} \parallel \mathtt{s}$, $\mathtt{s} \parallel \mathtt{r}$, but $\mathtt{t} \not\parallel \mathtt{r}$.

   Investigate, for example, $\mathtt{t} \doteq x_0$, $\mathtt{s} \doteq *_0$ and $\mathtt{r} \doteq x_1$.

5. *independent set:* Subsets of independent sets are independent.

12.2 Independence of Covered Nominal Terms

As already mentioned, the notion of independence is closely related with the strong covered-by relation. As a first proposition, we show that strongly covered nominal terms are independent.

12.2 Proposition (Covered Nominal Terms): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$. If $\mathtt{t} \ll_* \mathtt{s}$, then $\mathtt{t} \parallel \mathtt{s}$.
Proof. By induction over the structure of $\mathtt{t}$.

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* $\mathtt{t} \ll_* \mathtt{s}$ means that $\mathtt{s} \doteq \mathtt{t}$ or $\mathtt{s} \in \mathtt{T}_0$. In both cases, $\mathtt{t} \parallel \mathtt{s}$ (according to different clauses).

2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* $\mathtt{t} \ll_* \mathtt{s}$ means that $\mathtt{s} \doteq \mathtt{t}$. Therefore, $\mathtt{t} \parallel \mathtt{s}$ (according to the first clause of definition).

3. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* $\mathtt{t} \ll_* \mathtt{s}$ means that $\mathtt{t} \sim \mathtt{s}$ (therefore, there are nominal terms $\mathtt{s}_k$ such that $\mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$) and $\mathtt{t}_k \ll_* \mathtt{s}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain $\mathtt{t}_k \parallel \mathtt{s}_k$ for all $k \in n'$. Therefore, $\mathtt{t} \parallel \mathtt{s}$. Q.E.D.

We improve the result and show that intermediate nominal terms with respect to the strong covered-by relation are independent.

12.3 Proposition (Intermediate Nominal Terms): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ be two nominal terms such that $\mathtt{t} \ll_* \mathtt{s}$. The following statement holds for all nominal terms $\mathtt{r}, \mathtt{r}' \in \mathtt{T}$: if $\mathtt{t} \ll_* \mathtt{r} \ll_* \mathtt{s}$ and $\mathtt{t} \ll_* \mathtt{r}' \ll_* \mathtt{s}$, then $\mathtt{r} \parallel \mathtt{r}'$.
Proof. By induction over the structure of $\mathtt{t}$.

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* As $\mathtt{t} \ll_* \mathtt{s}$, we can distinguish the following two cases:

   (a) $\mathtt{s} \doteq \mathtt{t}$*:* As $\mathtt{r}, \mathtt{r}' \ll_* \mathtt{s}$, both $\mathtt{r}, \mathtt{r}' \doteq \mathtt{s}$. Therefore, $\mathtt{r} \parallel \mathtt{r}'$.

   (b) $\mathtt{s} \in \mathtt{T}_0$*:* Again as $\mathtt{r}, \mathtt{r} \ll_* \mathtt{s}$, both $\mathtt{r}, \mathtt{r}' \in \{\mathtt{t}, \mathtt{s}\}$. Investigating all possible combinations we obtain $\mathtt{r} \parallel \mathtt{r}'$.

2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* $\mathtt{t} \ll_* \mathtt{r}, \mathtt{r}'$ implies $\mathtt{r} \doteq \mathtt{t} \doteq \mathtt{r}'$. Therefore, $\mathtt{r} \parallel \mathtt{r}'$.

3. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* As $\mathtt{t} \ll_* \mathtt{r} \ll_* \mathtt{s}$ and $\mathtt{t} \ll_* \mathtt{r}' \ll_* \mathtt{s}$, we obtain that all involved nominal terms are similar. Furthermore, we have that the respective direct subterms are all strongly covered-by as determined by the complex nominal terms. More formally and denoting the direct subterms as expected, for all $k \in n'$:

   $$\mathtt{t}_k \ll_* \mathtt{r}_k \ll_* \mathtt{s}_k \quad \text{and} \quad \mathtt{t}_k \ll_* \mathtt{r}'_k \ll_* \mathtt{s}_k$$

   Applying n'-many times induction hypothesis, we obtain $\mathtt{r}_k \parallel \mathtt{r}'_k$ for all $k \in n'$. Therefore, we have $\mathtt{r} \parallel \mathtt{r}'$. Q.E.D.

An immediate consequence of the proposition above is that the set of all intermediate nominal term between two nominal terms (with respect to the strong covered-by relation) is independent. We provide the formal statement:

12.4 Corollary (Intermediate Nominal Terms): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$. The set $\mathtt{S} = \{\mathtt{r} \in \mathtt{T};\ \mathtt{t} \ll_* \mathtt{r} \ll_* \mathtt{s}\}$ of intermediate nominal terms between $\mathtt{t}$ and $\mathtt{s}$ is independent.

12.3 The Merge Functions

We introduce two operations on nominal terms, the *merge function* and the *dual merge function*. Their intended application is the transformation of independent nominal terms into a common nominal term representing both arguments. More precisely, the merge function is intended to result in a nominal term, in which the nominal symbol of both argument are all present, and the dual merge function in a nominal term, in which only the common nominal symbols of both arguments are present.[90]

We provide the formal definition of both merge functions.

12.5 DEF (Merge Functions):

1. *merge function:* The binary *merge function* $\mu : \mathtt{T} \times \mathtt{T} \to \mathtt{T}$ is defined recursively (in the first argument) as follows:

   (a) $\mathtt{t}$ *atomic:* $\mu(\mathtt{t}, \mathtt{s}) \doteqdot \begin{cases} \mathtt{t} & \text{if } \mathtt{t} \in \mathbf{V}_* \\ \mathtt{s} & \text{otherwise} \end{cases}$

   (b) $\mathtt{t} \doteqdot f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:*

   $$\mu(\mathtt{t}, \mathtt{s}) \doteqdot \begin{cases} f(\mu(\mathtt{t}_0, \mathtt{s}_0), \ldots \mu(\mathtt{t}_n, \mathtt{s}_n)) & \text{if } \mathtt{t} \sim \mathtt{s} \doteqdot f(\mathtt{s}_0, \ldots \mathtt{s}_n) \\ \mathtt{s} & \text{otherwise} \end{cases}$$

2. *dual merge function:* The binary *dual merge function* $\overline{\mu} : \mathtt{T} \times \mathtt{T} \to \mathtt{T}$ is defined recursively (in the first argument) as follows:

   (a) $\mathtt{t}$ *atomic:* $\overline{\mu}(\mathtt{t}, \mathtt{s}) \doteqdot \begin{cases} \mathtt{s} & \text{if } \mathtt{t} \in \mathbf{V}_* \\ \mathtt{t} & \text{otherwise} \end{cases}$

[90] In the course of our investigations, we are mainly interested in the merge function. The dual merge function is additionally investigated for systematic reasons.

(b) $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:*

$$\overline{\mu}(\mathtt{t}, \mathtt{s}) \doteq \begin{cases} f(\overline{\mu}(\mathtt{t}_0, \mathtt{s}_0), \ldots \overline{\mu}(\mathtt{t}_n, \mathtt{s}_n)) & \text{if } \mathtt{t} \sim \mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n) \\ \mathtt{t} \quad \text{otherwise} \end{cases}$$

Basic Properties (Merge Functions): We communicate some basic properties of both merge functions.

1. *idempotence:* Both merge functions are idempotent. More formally, for all nominal terms $\mathtt{t}$:

   $$\mu(\mathtt{t}, \mathtt{t}) \doteq \mathtt{t} \quad ; \quad \overline{\mu}(\mathtt{t}, \mathtt{t}) \doteq \mathtt{t}$$

   (Straightforward induction.)

2. *absorption:* Strongly covered nominal terms are absorbed by the merge functions. More formally, for all nominal terms $\mathtt{t}$ and $\mathtt{s}$:

   $$\mathtt{t} \ll_* \mathtt{s} \quad \Rightarrow \quad \mu(\mathtt{t}, \mathtt{s}) \doteq \mathtt{t} \text{ and } \overline{\mu}(\mathtt{t}, \mathtt{s}) \doteq \mathtt{s}$$

   (Straightforward induction.)

3. *non-commutative:* Neither merge function is commutative. Investigate the following typical counterexamples:

   (a) *clash of nominal symbols:*

   $$\begin{aligned} \mu(*_0, *_1) &\doteq *_0 \neq *_1 \doteq \mu(*_1, *_0) \\ \overline{\mu}(*_0, *_1) &\doteq *_1 \neq *_0 \doteq \overline{\mu}(*_1, *_0) \end{aligned}$$

   (b) *clash of structure:*

   $$\begin{aligned} \mu(1+2, 1\cdot 2) &\doteq 1\cdot 2 \neq 1+2 \doteq \mu(1\cdot 2, 1+2) \\ \overline{\mu}(1+2, 1\cdot 2) &\doteq 1+2 \neq 1\cdot 2 \doteq \overline{\mu}(1\cdot 2, 1+2) \end{aligned}$$

4. *non-associative:* Neither merge function is associative. Investigate the following typical counterexamples:

   $$\begin{aligned} \mu(\mu(*+*, 1\cdot 1), 1+1) &\doteq \mu(1\cdot 1, 1+1) \doteq 1+1 \\ &\neq *+* \doteq \mu(*+*, 1+1) \doteq \mu(*+*, \mu(1\cdot 1, 1+1)) \end{aligned}$$

   $$\begin{aligned} \overline{\mu}(\overline{\mu}(*+*, 1\cdot 1), 1+1) &\doteq \overline{\mu}(*+*, 1+1) \doteq 1+1 \\ &\neq *+* \doteq \overline{\mu}(*+*, 1\cdot 1) \doteq \overline{\mu}(*+*, \overline{\mu}(1\cdot 1, 1+1)) \end{aligned}$$

   Observe that both counterexamples depend on a clash of structures.

Due to the clashes of nominal symbols and of structure, the merge functions do not behave well. The situation changes, if the merge functions are applied on independent arguments:

1. *avoiding clashes:* The clash of structure is avoided, as complex independent nominal terms are similar and generated out of independent direct subterms. The clash of nominal symbols is avoided, as nominal symbols are independent, if and only if they are equal.

2. *alternative approach:* The clash of nominal symbols could be avoided by discussing the properties of the merge function modulo the equivalence of nominal terms; in such an approach, we would modify the definition of independence such that arbitrary nominal symbols would become independent.

12.4 Merging Independent Nominal Terms

We investigate the properties of both merge functions.

12.4.1 Commutativity

In the next proposition, we show that the restriction of both merge function to independent arguments is commutative.

12.6 Proposition (Commutativity): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ be arbitrary nominal terms. If $\mathtt{t} \parallel \mathtt{s}$, then $\mu(\mathtt{t}, \mathtt{s}) \eqsim \mu(\mathtt{s}, \mathtt{t})$ and $\overline{\mu}(\mathtt{t}, \mathtt{s}) \eqsim \overline{\mu}(\mathtt{s}, \mathtt{t})$.
Proof. By induction over the structure of $\mathtt{t}$.

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* We have $\mu(\mathtt{t}, \mathtt{s}) \eqsim \mathtt{t}$ and $\overline{\mu}(\mathtt{t}, \mathtt{s}) \eqsim \mathtt{s}$. We distinguish the following cases:

   (a) $\mathtt{s} \in \mathbf{V}_*$*:* As $\mathtt{t} \parallel \mathtt{s}$, we obtain that $\mathtt{t} \eqsim \mathtt{s}$. Therefore, $\mu(\mathtt{s}, \mathtt{t}) \eqsim \mathtt{t}$ and $\overline{\mu}(\mathtt{s}, \mathtt{t}) \eqsim \mathtt{s}$.

   (b) $\mathtt{s} \notin \mathbf{V}_*$*:* Independently of the actual shape of $\mathtt{s}$ (whether $\mathtt{s}$ is atomic or complex), we obtain $\mu(\mathtt{s}, \mathtt{t}) \eqsim \mathtt{t}$ and $\overline{\mu}(\mathtt{s}, \mathtt{t}) \eqsim \mathtt{s}$.

2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* We have $\mu(\mathtt{t}, \mathtt{s}) \eqsim \mathtt{s}$ and $\overline{\mu}(\mathtt{t}, \mathtt{s}) \eqsim \mathtt{t}$. We distinguish the following cases:

   (a) $\mathtt{s} \in \mathbf{V}_*$*:* Immediately, $\mu(\mathtt{s}, \mathtt{t}) \eqsim \mathtt{s}$ and $\overline{\mu}(\mathtt{s}, \mathtt{t}) \eqsim \mathtt{t}$.

   (b) $\mathtt{s} \notin \mathbf{V}_*$*:* $\mathtt{t} \parallel \mathtt{s}$ implies that $\mathtt{t} \eqsim \mathtt{s}$. Therefore, $\mu(\mathtt{s}, \mathtt{t}) \eqsim \mathtt{s}$ and $\overline{\mu}(\mathtt{s}, \mathtt{t}) \eqsim \mathtt{t}$

3. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* We distinguish as follows:

   (a) $\mathtt{s} \in \mathbf{V}_*$*:* Immediately:

   $$\mu(\mathtt{t},\mathtt{s}) \doteq \mathtt{s} \doteq \mu(\mathtt{s},\mathtt{t}) \quad \text{and} \quad \overline{\mu}(\mathtt{t},\mathtt{s}) \doteq \mathtt{t} \doteq \overline{\mu}(\mathtt{s},\mathtt{t})$$

   (b) $\mathtt{s} \notin \mathbf{V}_*$*:* $\mathtt{t} \parallel \mathtt{s}$ implies that $\mathtt{t} \sim \mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$ (for some nominal terms $\mathtt{s}_k$) and that $\mathtt{t}_k \parallel \mathtt{s}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain for all $k \in n'$:

   $$\mu(\mathtt{t}_k,\mathtt{s}_k) \doteq \mu(\mathtt{s}_k,\mathtt{t}_k) \quad ; \quad \overline{\mu}(\mathtt{t}_k,\mathtt{s}_k) \doteq \overline{\mu}(\mathtt{s}_k,\mathtt{t}_k)$$

   Therefore, we may calculate as follows:

   $$\begin{aligned} \mu(\mathtt{t},\mathtt{s}) &\doteq f(\mu(\mathtt{t}_0,\mathtt{s}_0), \ldots \mu(\mathtt{t}_n,\mathtt{s}_n)) \\ &\doteq f(\mu(\mathtt{s}_0,\mathtt{t}_0), \ldots \mu(\mathtt{s}_n,\mathtt{t}_n)) \doteq \mu(\mathtt{s},\mathtt{t}) \end{aligned}$$

   $$\begin{aligned} \overline{\mu}(\mathtt{t},\mathtt{s}) &\doteq f(\overline{\mu}(\mathtt{t}_0,\mathtt{s}_0), \ldots \overline{\mu}(\mathtt{t}_n,\mathtt{s}_n)) \\ &\doteq f(\overline{\mu}(\mathtt{s}_0,\mathtt{t}_0), \ldots \overline{\mu}(\mathtt{s}_n,\mathtt{t}_n)) \doteq \overline{\mu}(\mathtt{s},\mathtt{t}) \end{aligned}$$

Q.E.D.

12.4.2 Associativity

In the next proposition, we show that the restriction of both merge function to independent arguments is associative.

12.7 Proposition (Associativity): Let $\mathtt{S} = \{\mathtt{t},\mathtt{s},\mathtt{r}\} \subseteq \mathtt{T}$ be an independent set of nominal terms. The following both equations hold:

$$\mu(\mu(\mathtt{t},\mathtt{s}),\mathtt{r}) \doteq \mu(\mathtt{t},\mu(\mathtt{s},\mathtt{r})) \quad ; \quad \overline{\mu}(\overline{\mu}(\mathtt{t},\mathtt{s}),\mathtt{r}) \doteq \overline{\mu}(\mathtt{t},\overline{\mu}(\mathtt{s},\mathtt{r}))$$

Proof. By induction over the structure of $\mathtt{t}$.

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* We calculate as follows:

   $$\mu(\mu(\mathtt{t},\mathtt{s}),\mathtt{r}) \doteq \mu(\mathtt{t},\mathtt{r}) \doteq \mathtt{t} \doteq \mu(\mathtt{t},\mu(\mathtt{s},\mathtt{r}))$$

   $$\overline{\mu}(\overline{\mu}(\mathtt{t},\mathtt{s}),\mathtt{r}) \doteq \overline{\mu}(\mathtt{s},\mathtt{r}) \doteq \overline{\mu}(\mathtt{t},\overline{\mu}(\mathtt{s},\mathtt{r}))$$

2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* We calculate as follows:

$$\mu(\mu(\mathtt{t},\mathtt{s}),\mathtt{r}) \;\doteqdot\; \mu(\mathtt{s},\mathtt{r}) \;\doteqdot\; \mu(\mathtt{t},\mu(\mathtt{s},\mathtt{r}))$$

$$\overline{\mu}(\overline{\mu}(\mathtt{t},\mathtt{s}),\mathtt{r}) \;\doteqdot\; \overline{\mu}(\mathtt{t},\mathtt{r}) \;\doteqdot\; \mathtt{t} \;\doteqdot\; \overline{\mu}(\mathtt{t},\overline{\mu}(\mathtt{s},\mathtt{r}))$$

3. $\mathtt{t} \doteqdot f(\mathtt{t}_0,\ldots\mathtt{t}_n)$ *complex:* We distinguish two cases:

   (a) $\mathtt{t} \sim \mathtt{s} \sim \mathtt{r}$*:* If all three nominal terms are similar, then there are nominal terms $\mathtt{s}_k$ and the $\mathtt{r}_k$ (for $k \in n'$) such that:

$$\mathtt{s} \doteqdot f(\mathtt{s}_0,\ldots\mathtt{s}_n) \quad ; \quad \mathtt{r} \doteqdot f(\mathtt{r}_0,\ldots\mathtt{r}_n)$$

   As $\mathtt{t}$, $\mathtt{s}$ and $\mathtt{r}$ are all complex, their pairwise independence is given according to clause (3) of the definition. Therefore, we obtain that the set $\{\mathtt{t}_k,\mathtt{s}_k,\mathtt{r}_k\}$ is independent for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain for all $k \in n'$:

$$\mu(\mu(\mathtt{t}_k,\mathtt{s}_k),\mathtt{r}_k) \;\doteqdot\; \mu(\mathtt{t}_k,\mu(\mathtt{s}_k,\mathtt{r}_k))$$

$$\overline{\mu}(\overline{\mu}(\mathtt{t}_k,\mathtt{s}_k),\mathtt{r}_k) \;\doteqdot\; \overline{\mu}(\mathtt{t}_k,\overline{\mu}(\mathtt{s}_k,\mathtt{r}_k))$$

   A simple calculation yields:

$$\mu(\mu(\mathtt{t},\mathtt{s}),\mathtt{r}) \;\doteqdot\; \mu(\mathtt{t},\mu(\mathtt{s},\mathtt{r}))$$

$$\overline{\mu}(\overline{\mu}(\mathtt{t},\mathtt{s}),\mathtt{r}) \;\doteqdot\; \overline{\mu}(\mathtt{t},\overline{\mu}(\mathtt{s},\mathtt{r}))$$

   (b) not($\mathtt{t} \sim \mathtt{s} \sim \mathtt{r}$)*:* We first discuss the case that $\mathtt{t} \not\sim \mathtt{s}$. Immediately, $\mathtt{t} \parallel \mathtt{s}$ cannot be given according to clause (3) of the definition of independence. As $\mathtt{t}$ is complex, clause (1) is also excluded. Therefore, independence is given according to clause (2). This means that $\mathtt{t} \in \mathtt{T}_0$ is standard and $\mathtt{s} \doteqdot *_k \in \mathbf{V}_*$ is a nominal symbol. Analogous argumentation holds with respect to $\mathtt{r}$ in the case that $\mathtt{t} \not\sim \mathtt{r}$.

   If both nominal terms $\mathtt{s},\mathtt{r} \in \mathbf{V}_*$, then their independence is given according to clause (1), which means that $\mathtt{s} \doteqdot *_k \doteqdot \mathtt{r}$.

   If one of the nominal terms is similar to $\mathtt{t}$, lets say $\mathtt{s}$, then the independence of $\mathtt{s}$ and $\mathtt{r}$ must be given according to clause (2). Therefore, $\mathtt{s} \in \mathtt{T}_0$ is a standard term. As $\mathtt{t}$ is a standard term independent from $\mathtt{s}$, we obtain that $\mathtt{t} \doteqdot \mathtt{s}$.

   We summarise the situation: there is a nominal symbol $*_k$ such that $\mathtt{s}$ or $\mathtt{r}$ (or both) are equal to $*_k$. If one of both nominal terms is different from $*_k$, then this nominal term is equal to $\mathtt{t}$.

In each case, we have that $\mu(\mathtt{s},\mathtt{r}) \eqsim *_k$. Furthermore, $\mu(\mathtt{t},\mathtt{s}) \eqsim \mathtt{s}$. Therefore:

$$\mu(\mu(\mathtt{t},\mathtt{s}),\mathtt{r}) \;\eqsim\; \mu(\mathtt{s},\mathtt{r}) \;\eqsim\; *_k \;\eqsim\; \mu(\mathtt{t},*_k) \;\eqsim\; \mu(\mathtt{t},\mu(\mathtt{s},\mathtt{r}))$$

We also have that $\overline{\mu}(\mathtt{s},\mathtt{r}) \in \{\mathtt{t},*_k\}$ and both $\overline{\mu}(\mathtt{t},\mathtt{t}) \eqsim \mathtt{t}$ and $\overline{\mu}(\mathtt{t},*_k) \eqsim \mathtt{t}$. Therefore:

$$\overline{\mu}(\overline{\mu}(\mathtt{t},\mathtt{s}),\mathtt{r}) \;\eqsim\; \overline{\mu}(\mathtt{t},\mathtt{r}) \;\eqsim\; \mathtt{t} \;\eqsim\; \overline{\mu}(\mathtt{t},\overline{\mu}(\mathtt{s},\mathtt{r}))$$

Q.E.D.

12.4.3 Weight of Merged Nominal Terms

In the next proposition, we show that the weight of nominal terms can be calculated, if the merge functions are applied to independent nominal terms. More precisely, the number of nominal symbols in the merged nominal term equals to the sum of the nominal symbols in both nominal terms decreased by the number of nominal symbols locally covered by themselves. The latter number is the number of nominal symbols in the result of applying the dual merge function on both arguments.

12.8 Proposition (Weight of Merged Nominal Terms): Let $\mathtt{t},\mathtt{s} \in \mathtt{T}$ be nominal terms. If $\mathtt{t} \parallel \mathtt{s}$, then the following equations holds:

$$\mathbf{weight}(\mu(\mathtt{t},\mathtt{s})) = \mathbf{weight}(\mathtt{t}) + \mathbf{weight}(\mathtt{s}) - \mathbf{weight}(\overline{\mu}(\mathtt{t},\mathtt{s}))$$

Proof. By induction over the structure of $\mathtt{t}$.

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* Recalling that $\mu(\mathtt{t},\mathtt{s}) \eqsim \mathtt{t}$ and $\overline{\mu}(\mathtt{t},\mathtt{s}) \eqsim \mathtt{s}$, we obtain:

$$\begin{aligned}\mathbf{weight}(\mu(\mathtt{t},\mathtt{s})) &= \mathbf{weight}(\mathtt{t}) \\ &= \mathbf{weight}(\mathtt{t}) + \mathbf{weight}(\mathtt{s}) - \mathbf{weight}(\mathtt{s}) \\ &= \mathbf{weight}(\mathtt{t}) + \mathbf{weight}(\mathtt{s}) - \mathbf{weight}(\overline{\mu}(\mathtt{t},\mathtt{s}))\end{aligned}$$

2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* Independence of $\mathtt{t}$ and $\mathtt{s}$ can be given according to clause (1) of the definition, which means that $\mathtt{t} \eqsim \mathtt{s}$, or according to clause (2), which means that $\mathtt{s} \in \mathbf{V}_*$.

In the first case, the equation holds trivially, as both μ and $\overline{\mu}$ are idempotent:

$$\begin{aligned}\mathbf{weight}(\mu(\mathtt{t},\mathtt{s})) &= \mathbf{weight}(\mathtt{t}) \\ &= \mathbf{weight}(\mathtt{t}) + \mathbf{weight}(\mathtt{s}) - \mathbf{weight}(\mathtt{s}) \\ &= \mathbf{weight}(\mathtt{t}) + \mathbf{weight}(\mathtt{s}) - \mathbf{weight}(\overline{\mu}(\mathtt{t},\mathtt{s}))\end{aligned}$$

In the second case, we may use the commutativity of both merge functions and calculate as above.

3. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* Again, we have to distinguish two cases. If independence is given according to clause (2) of the definition, we obtain, in particular, that $\mathtt{s} \in \mathbf{V}_*$. Using again commutativity of both merge functions, we may again calculate as in clause (1) of this proof. Otherwise, independence is given according to clause (3) of the definition. The latter means both that $\mathtt{t} \sim \mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$ and that $\mathtt{t}_k \parallel \mathtt{s}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we calculate as follows:

$$\begin{aligned}\mathbf{weight}(\mu(\mathtt{t},\mathtt{s})) &= \mathbf{weight}(f(\mu(\mathtt{t}_0,\mathtt{s}_0), \ldots \mu(\mathtt{t}_n,\mathtt{s}_n))) \\ &= \sum_{k\in n'} \mathbf{weight}(\mu(\mathtt{t}_k,\mathtt{s}_k)) \\ &= \sum_{k\in n'} \mathbf{weight}(\mathtt{t}_k) + \mathbf{weight}(\mathtt{s}_k) - \mathbf{weight}(\overline{\mu}(\mathtt{t}_k,\mathtt{s}_k)) \\ &= \mathbf{weight}(\mathtt{t}) + \mathbf{weight}(\mathtt{s}) \\ &\qquad - \mathbf{weight}(f(\overline{\mu}(\mathtt{t}_0,\mathtt{s}_0), \ldots \overline{\mu}(\mathtt{t}_n,\mathtt{s}_n))) \\ &= \mathbf{weight}(\mathtt{t}) + \mathbf{weight}(\mathtt{s}) - \mathbf{weight}(\overline{\mu}(\mathtt{t},\mathtt{s}))\end{aligned}$$

Observe that the first and the last calculation step presuppose the similarity of $\mathtt{t}$ and $\mathtt{s}$. Q.E.D.

12.4.4 Covered-By Relation

We discuss the relationship of the merge functions (restricted to independent arguments) and the strong covered-by relation. In a first proposition, we show that independent nominal terms are between the results of both merge functions with respect to the strong covered-by relation.

12.9 Proposition (Covered-By Relation - I): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$. If $\mathtt{t} \parallel \mathtt{s}$, then the following statements hold: $\mu(\mathtt{t}, \mathtt{s}) \ll_* \mathtt{t}, \mathtt{s}$ and $\mathtt{t}, \mathtt{s} \ll_* \overline{\mu}(\mathtt{t}, \mathtt{s})$.
Proof. As both merge functions are commutative for independent nominal terms, it is sufficient to show that $\mu(\mathtt{t}, \mathtt{s}) \ll_* \mathtt{t} \ll_* \overline{\mu}(\mathtt{t}, \mathtt{s})$; this is done by induction over the structure of $\mathtt{t}$ for an arbitrary nominal term $\mathtt{s} \in \mathtt{T}$.

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* Independence of $\mathtt{t}$ and $\mathtt{s}$ means $\mathtt{s} \doteq \mathtt{t}$ (clause (1) of the definition) or $\mathtt{s} \in \mathtt{T}_0$ (clause (2) of the definition). In both cases, we have $\mathtt{t} \ll_* \mathtt{s}$. Therefore:

$$\mu(\mathtt{t}, \mathtt{s}) \doteq \mathtt{t} \ll_* \mathtt{t} \quad ; \quad \mathtt{t} \ll_* \mathtt{s} \doteq \overline{\mu}(\mathtt{t}, \mathtt{s})$$

2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* Independence of $\mathtt{t}$ and $\mathtt{s}$ means $\mathtt{s} \doteq \mathtt{t}$ (clause (1) of the definition) or $\mathtt{s} \in \mathbf{V}_*$ (clause (2) of the definition). In both cases, we have $\mathtt{s} \ll_* \mathtt{t}$. Therefore:

$$\mu(\mathtt{t}, \mathtt{s}) \doteq \mathtt{s} \ll_* \mathtt{t} \quad ; \quad \mathtt{t} \ll_* \mathtt{t} \doteq \overline{\mu}(\mathtt{t}, \mathtt{s})$$

3. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* We distinguish two cases:

   (a) $\mathtt{t} \sim \mathtt{s}$*:* As $\mathtt{t} \sim \mathtt{s}$, there are nominal terms $\mathtt{s}_k$ (for $k \in n'$) such that $\mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n)$. Independence of $\mathtt{t}$ and $\mathtt{s}$ is given according to clause (3) of the definition. Therefore, $\mathtt{t}_k \parallel \mathtt{s}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain for all $k \in n'$:

$$\mu(\mathtt{t}_k, \mathtt{s}_k) \ll_* \mathtt{t}_k \ll_* \overline{\mu}(\mathtt{t}_k, \mathtt{s}_k)$$

   As $\mathtt{t} \sim \mathtt{s}$, we also have:

$$\mu(\mathtt{t}, \mathtt{s}) \doteq f(\mu(\mathtt{t}_0, \mathtt{s}_0), \ldots \mu(\mathtt{t}_n, \mathtt{s}_n))$$

$$\overline{\mu}(\mathtt{t}, \mathtt{s}) \doteq f(\overline{\mu}(\mathtt{t}_0, \mathtt{s}_0), \ldots \overline{\mu}(\mathtt{t}_n, \mathtt{s}_n))$$

   This means both that $\mathtt{t} \sim \mu(\mathtt{t}, \mathtt{s})$ and $\mathtt{t} \sim \overline{\mu}(\mathtt{t}, \mathtt{s})$. Therefore:

$$\mu(\mathtt{t}, \mathtt{s}) \ll_* \mathtt{t} \ll_* \overline{\mu}(\mathtt{t}, \mathtt{s})$$

   (b) $\mathtt{t} \not\sim \mathtt{s}$*:* If $\mathtt{t} \not\sim \mathtt{s}$, then independence of $\mathtt{t}$ and $\mathtt{s}$ is only possible according to clause (2) of the definition of independence. Correspondingly, $\mathtt{t} \in \mathtt{T}_0$ and $\mathtt{s} \in \mathbf{V}_*$. As a consequence, $\mathtt{s} \ll_* \mathtt{t}$. Therefore:

$$\mu(\mathtt{t}, \mathtt{s}) \doteq \mathtt{s} \ll_* \mathtt{t} \quad ; \quad \mathtt{t} \ll_* \mathtt{t} \doteq \overline{\mu}(\mathtt{t}, \mathtt{s})$$

Q.E.D.

In the proposition above, we have seen that the result of an application of the merge function (restricted to independent arguments) is a lower bound of its arguments and that the result of an application of the dual merge function (again restricted to independent arguments) an upper bound. In particular, we obtain as a corollary that the result of merging two independent nominal terms is strongly covered by the result of dual merging them.

12.10 Corollary (Covered-By Relation - I): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ be nominal terms. If $\mathtt{t} \parallel \mathtt{s}$, then $\mu(\mathtt{t}, \mathtt{s}) \ll_* \overline{\mu}(\mathtt{t}, \mathtt{s})$.
Proof. Immediate consequence of the proposition above, as the strong covered-by relation is transitive. Q.E.D.

In the next proposition, we show a complementary result: if a nominal term is below two independent nominal terms, then also below the result of an application of the merge function on the independent nominal terms; analogously, if a nominal term is above two independent nominal terms, then also above the result of an application of the dual merge function on the independent nominal terms.

12.11 Proposition (Covered-By Relation - II): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ be two independent nominal terms. The following statements hold for all nominal terms $\mathtt{r} \in \mathtt{T}$.

1. *merge function:* If $\mathtt{r} \ll_* \mathtt{t}$ and $\mathtt{r} \ll_* \mathtt{s}$, then also $\mathtt{r} \ll_* \mu(\mathtt{t}, \mathtt{s})$.
2. *dual merge function:* If $\mathtt{t} \ll_* \mathtt{r}$ and $\mathtt{s} \ll_* \mathtt{r}$, then also $\overline{\mu}(\mathtt{t}, \mathtt{s}) \ll_* \mathtt{r}$.

Proof. We prove the first statement by induction over the structure of $\mathtt{t}$:

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* If $\mathtt{r} \ll_* \mathtt{t}$, then immediately $\mathtt{r} \ll_* \mathtt{t} \doteq \mu(\mathtt{t}, \mathtt{s})$.
2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* If $\mathtt{r} \ll_* \mathtt{s}$, then immediately $\mathtt{r} \ll_* \mathtt{s} \doteq \mu(\mathtt{t}, \mathtt{s})$.
3. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$*:* We distinguish some cases:
   (a) $\mathtt{t} \not\sim \mathtt{s}$*:* As in the standard atomic case, we have:
   If $\mathtt{r} \ll_* \mathtt{s}$, then immediately $\mathtt{r} \ll_* \mathtt{s} \doteq \mu(\mathtt{t}, \mathtt{s})$.
   (b) $\mathtt{t} \sim \mathtt{s}$, $\mathtt{r}$ *atomic:* Let $\mathtt{r} \ll_* \mathtt{t}$ and $\mathtt{r} \ll_* \mathtt{s}$. As $\mathtt{t}$ and $\mathtt{s}$ are complex, the strong covered-by relation must be given according to clause (1) of the definition, which means that $\mathtt{r} \in \mathbf{V}_*$ and that both $\mathtt{t}, \mathtt{s} \in \mathtt{T}_0$. As $\mathtt{t} \parallel \mathtt{s}$, the latter implies $\mathtt{t} \doteq \mathtt{s}$. Therefore:

$$\mathtt{r} \ll_* \mathtt{t} \doteq \mu(\mathtt{t}, \mathtt{s})$$

(c) $\mathtt{t} \sim \mathtt{s}$, $\mathtt{r}$ *complex:* Let $\mathtt{r} \ll_* \mathtt{t}$ and $\mathtt{r} \ll_* \mathtt{s}$. As $\mathtt{r}$ is complex, the strong covered-by relation must be given according to clause (3) of the definition. Therefore, $\mathtt{r} \sim \mathtt{t}$ and $\mathtt{r} \sim \mathtt{s}$. As a consequence, there are nominal terms $\mathtt{s}_k$ and $\mathtt{r}_k$ (for $k \in n'$) such that:

$$\mathtt{s} \doteq f(\mathtt{s}_0, \dots \mathtt{s}_n) \quad ; \quad \mathtt{r} \doteq f(\mathtt{r}_0, \dots \mathtt{r}_n)$$

As $\mathtt{r}$ is strongly covered by $\mathtt{t}$ and $\mathtt{s}$, we have also that $\mathtt{r}_k \ll_* \mathtt{t}_k$ and $\mathtt{r}_k \ll_* \mathtt{s}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain $\mathtt{r}_k \ll_* \mu(\mathtt{t}_k, \mathtt{s}_k)$ for all $k \in n'$. Therefore, the following statement holds:

$$\mathtt{r} \ll_* f(\mu(\mathtt{t}_0, \mathtt{s}_0), \dots \mu(\mathtt{t}_n, \mathtt{s}_n)) \doteq \mu(\mathtt{t}, \mathtt{s})$$

We prove the second statement by induction over the structure of $\mathtt{t}$.

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* If $\mathtt{s} \ll_* \mathtt{r}$, then immediately $\overline{\mu}(\mathtt{t}, \mathtt{s}) \doteq \mathtt{s} \ll_* \mathtt{r}$.
2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* If $\mathtt{t} \ll_* \mathtt{r}$, then immediately $\overline{\mu}(\mathtt{t}, \mathtt{s}) \doteq \mathtt{t} \ll_* \mathtt{r}$.
3. $\mathtt{t} \doteq f(\mathtt{t}_0, \dots \mathtt{t}_n)$ *complex:* We distinguish two cases:
   (a) $\mathtt{t} \not\sim \mathtt{s}$*:* As in the standard atomic case:
   If $\mathtt{t} \ll_* \mathtt{r}$, then immediately $\overline{\mu}(\mathtt{t}, \mathtt{s}) \doteq \mathtt{t} \ll_* \mathtt{r}$.
   (b) $\mathtt{t} \sim \mathtt{s}$*:* First, we observe that $\mathtt{t} \sim \mathtt{s} \sim \mathtt{r}$. (Similarity with $\mathtt{r}$ is given, as $\mathtt{r}$ is strongly covering complex nominal terms.) Therefore, there are nominal terms $\mathtt{s}_k$ and $\mathtt{r}_k$ (for $k \in n'$) such that:

$$\mathtt{s} \doteq f(\mathtt{s}_0, \dots \mathtt{s}_n) \quad ; \quad \mathtt{r} \doteq f(\mathtt{r}_0, \dots \mathtt{r}_n)$$

As $\mathtt{t} \ll_* \mathtt{r}$ and $\mathtt{s} \ll_* \mathtt{r}$, we also have $\mathtt{t}_k \ll_* \mathtt{r}_k$ and $\mathtt{s}_k \ll_* \mathtt{r}_k$ for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain $\overline{\mu}(\mathtt{t}_k, \mathtt{s}_k) \ll_* \mathtt{r}_k$ for all $k \in n'$. Therefore:

$$\overline{\mu}(\mathtt{t}, \mathtt{s}) \doteq f(\overline{\mu}(\mathtt{t}_0, \mathtt{s}_0), \dots \overline{\mu}(\mathtt{t}_n, \mathtt{s}_n)) \ll_* \mathtt{r}$$

Q.E.D.

Remarks (Proposition - Covered-By Relation - II):

1. *independence:* Observe that we do not use independence in the proof of the second statement of the proposition above. In the first statement, independence is only needed in that complex case in which $\mathtt{t} \sim \mathtt{s}$, but $\mathtt{r}$ is atomic. This case is not possible with respect to the second statement, as $\mathtt{r}$ is above $\mathtt{t}$ (and also above $\mathtt{s}$) with respect to the strong covered-by relation and, therefore, complex.

2. *complementary statements:* There are complementary statements to those of the proposition above:

   (a) *dual merge function:* If $\mathtt{r} \ll_* \mathtt{t}$ and $\mathtt{r} \ll_* \mathtt{s}$, then $\mathtt{r} \ll_* \overline{\mu}(\mathtt{t}, \mathtt{s})$.

   (b) *merge function:* If $\mathtt{t} \ll_* \mathtt{r}$ and $\mathtt{s} \ll_* \mathtt{r}$, then $\mu(\mathtt{t}, \mathtt{s}) \ll_* \mathtt{r}$.

   Both statements can even be weakened by replacing "and" by "or" and are an immediate consequence of the first proposition about the covered-by relation together with transitivity of the strong covered-by relation.

12.4.5 Preservation of Independence

The merge functions preserve independence of nominal terms. More precisely, if three nominal terms are pairwise independent, then also the result of merging two of them and the third.

12.12 Proposition (Preservation of Independence): If $\mathtt{S} = \{\mathtt{t}, \mathtt{s}, \mathtt{r}\}$ is an independent set of nominal terms, then both $\mu(\mathtt{t}, \mathtt{s}) \parallel \mathtt{r}$ and $\overline{\mu}(\mathtt{t}, \mathtt{s}) \parallel \mathtt{r}$.
Proof. By induction over the structure of $\mathtt{t}$.

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* Immediately:

$$\mu(\mathtt{t}, \mathtt{s}) \;\hat{=}\; \mathtt{t} \parallel \mathtt{r} \quad ; \quad \overline{\mu}(\mathtt{t}, \mathtt{s}) \;\hat{=}\; \mathtt{s} \parallel \mathtt{r}$$

2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* Immediately:

$$\mu(\mathtt{t}, \mathtt{s}) \;\hat{=}\; \mathtt{s} \parallel \mathtt{r} \quad ; \quad \overline{\mu}(\mathtt{t}, \mathtt{s}) \;\hat{=}\; \mathtt{t} \parallel \mathtt{r}$$

3. $\mathtt{t} \;\hat{=}\; f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* We distinguish as follows:

   (a) $\mathtt{t} \sim \mathtt{s} \sim \mathtt{r}$*:* As all nominal terms are similar, there are nominal terms $\mathtt{s}_k$ and $\mathtt{r}_k$ such that:

$$\mathtt{s} \;\hat{=}\; f(\mathtt{s}_0, \ldots \mathtt{s}_n) \quad ; \quad \mathtt{r} \;\hat{=}\; f(\mathtt{r}_0, \ldots \mathtt{r}_n)$$

   Pairwise independence of the three nominal terms means that the sets $\{\mathtt{t}_k, \mathtt{s}_k, \mathtt{r}_k\}$ are independent for all $k \in n'$. Applying n'-many times induction hypothesis, we obtain both that $\mu(\mathtt{t}_k, \mathtt{s}_k) \parallel \mathtt{r}_k$ and $\overline{\mu}(\mathtt{t}_k, \mathtt{s}_k) \parallel \mathtt{r}_k$ for all $k \in n'$ and, as a consequence, that $\mu(\mathtt{t}, \mathtt{s}) \parallel \mathtt{r}$ and $\overline{\mu}(\mathtt{t}, \mathtt{s}) \parallel \mathtt{r}$.

(b) not($\mathtt{t} \sim \mathtt{s} \sim \mathtt{r}$)*:* One of the nominal terms $\mathtt{s}$ or $\mathtt{r}$ is not similar to $\mathtt{t}$; this nominal term is a nominal symbol $*_k$ and therefore $\mathtt{t} \in \mathtt{T}_0$ a standard term (both due to independence). Furthermore, if the both nominal terms $\mathtt{s}$ and $\mathtt{r}$ are not similar to $\mathtt{t}$, then they have to be the equal. If it is not the case that both nominal terms $\mathtt{s}$ and $\mathtt{r}$ are nominal symbols, then the other nominal term is equal to the standard term $\mathtt{t}$.

This means: $\mu(\mathtt{t}, \mathtt{s}) \in \{*_k, \mathtt{t}\}$ and $\mathtt{r} \in \{*_k, \mathtt{t}\}$. In any case, $\mu(\mathtt{t}, \mathtt{s}) \parallel \mathtt{r}$.

Furthermore, $\overline{\mu}(\mathtt{t}, \mathtt{s}) \doteq \mathtt{t}$, and, therefore, immediately $\overline{\mu}(\mathtt{t}, \mathtt{s}) \parallel \mathtt{r}$.

Q.E.D.

As a consequence of the proposition above, the extensions of independent sets of nominal terms by the result of merging some elements of such a set is again independent.

12.13 Corollary (Independent Extensions): Let $\mathtt{S} \subseteq \mathtt{T}$ be an independent set of nominal terms. The sets $\mathtt{S}' = \mathtt{S} \cup \{\mu(\mathtt{t}, \mathtt{s})\}$ and $\mathtt{S}'' = \mathtt{S} \cup \{\overline{\mu}(\mathtt{t}, \mathtt{s})\}$ are independent for all nominal terms $\mathtt{t}, \mathtt{s} \in \mathtt{S}$.

Proof. We show the independence of the set $\mathtt{S}'$; the proof with respect to $\mathtt{S}''$ is analogous. Let $\mathtt{r} \in \mathtt{S}'$ be arbitrary. It is sufficient to show $\mathtt{r} \parallel \mu(\mathtt{t}, \mathtt{s})$. (All other pairs of nominal terms are contained in $\mathtt{S}$ and, therefore, independent.) The case $\mathtt{r} \doteq \mu(\mathtt{t}, \mathtt{s})$ is trivial, as independence is reflexive. We may assume, therefore, that $\mathtt{r} \in \mathtt{S}$. The latter means that $\{\mathtt{t}, \mathtt{s}, \mathtt{r}\} \subseteq \mathtt{S}$ is an independent set. According to the proposition above, we have $\mathtt{r} \parallel \mu(\mathtt{t}, \mathtt{s})$. Q.E.D.

12.5 Independent Sets of Nominal Term

The aim of this section is to show that independent sets of nominal terms are finite. In order to do so, we use the following version of Lindenbaum's Theorem:[91] every independent set of nominal terms can be extended to a maximal independent set of nominal terms.

[91] Lindenbaum's Theorem is passed down by Tarski [31] and is formulated with respect to the consistency of formula sets in (countable) formal languages. Cf. Gazzari [8] for a brief discussion of different abstract versions of this theorem (equivalent to the Axiom of Choice) and of the distinction of these theorems from other (set-theoretical) statements.

12.5.1 Maximal Independent Sets

In a first step, we introduce maximal independent sets.

12.14 DEF (Maximally Independent Sets): An independent set $\mathtt{S} \subseteq \mathtt{T}$ of nominal terms is *maximal*, if the following conditions is satisfied for all nominal terms $\mathtt{s} \in \mathtt{T}$: if $\mathtt{S} \cup \{\mathtt{s}\}$ is independent, then $\mathtt{s} \in \mathtt{S}$.

Maximal independent sets contain a least and a greatest element with respect to the strong covered-by relation.

12.15 Proposition (Maximally Independent Sets): Let $\mathtt{S} \subseteq \mathtt{T}$ be a maximal independent set. Then the following statements hold:

1. *the least element:* $\mathtt{S}$ contains a least element $\mathtt{t} \eqsim \min(\mathtt{S})$ with respect to the strong covered-by relation. This element is minimal in $\mathtt{T}$; in particular, $\overline{\mathbf{weight}}(\mathtt{t}) = 0$.

2. *the greatest element:* $\mathtt{S}$ contains a greatest element $\mathtt{s} \eqsim \max(\mathtt{S})$ with respect to the strong covered-by relation. This element is maximal in $\mathtt{T}$; in particular, $\mathbf{weight}(\mathtt{s}) = 0$ which means that $\mathtt{s} \in \mathtt{T}_0$ is standard.

Proof. We first discuss statement (1).

1. *construction:* First, we observe that $\mathtt{S}$ is not empty. (As independence is reflexive, the empty set is not maximal.) This means that there is $\mathtt{r}' \in \mathtt{S}$. As descending chains with respect to the strong-covered by relation are finite, we find after finitely many steps a minimal element $\mathtt{t} \in \mathtt{S}$ below $\mathtt{r}'$. Minimality means that we have for all $\mathtt{r} \in \mathtt{S}$: if $\mathtt{r} \ll_* \mathtt{t}$, then $\mathtt{t} \eqsim \mathtt{r}$.

2. *the least element* $\in \mathtt{S}$*:* The nominal term $\mathtt{t}$ is the least element of $\mathtt{S}$: Let $\mathtt{r} \in \mathtt{S}$ be arbitrary. As $\mathtt{t} \parallel \mathtt{r}$, the set $\mathtt{S} \cup \{\mu(\mathtt{t}, \mathtt{r})\}$ is an independent extension of $\mathtt{S}$. As $\mathtt{S}$ is maximal, we obtain that $\mu(\mathtt{t}, \mathtt{r}) \in \mathtt{S}$. As still $\mathtt{t} \parallel \mathtt{r}$, we obtain both that $\mu(\mathtt{t}, \mathtt{r}) \ll_* \mathtt{t}$ and $\mu(\mathtt{t}, \mathtt{r}) \ll_* \mathtt{r}$. As $\mathtt{t}$ is minimal with respect to $\ll_*$, we have that $\mathtt{t} \eqsim \mu(\mathtt{t}, \mathtt{r})$ and, therefore, $\mathtt{t} \eqsim \mu(\mathtt{t}, \mathtt{r}) \ll_* \mathtt{r}$. Therefore, the nominal term $\mathtt{t}$ is the least element of $\mathtt{S}$ with respect to $\ll_*$.

3. *minimal in* $\mathtt{T}$*:* The nominal term $\mathtt{t}$ is minimal in $\mathtt{T}$: Let $\mathtt{t}' \in \mathtt{T}$ such that $\mathtt{t}' \ll_* \mathtt{t}$. We have for arbitrary $\mathtt{r} \in \mathtt{S}$:

$$\mathtt{t}' \ll_* \mathtt{t} \ll_* \mathtt{r}$$

Due to transitivity of the strong covered-by relation, we obtain that $\mathtt{t}' \ll_* \mathtt{r}$. As covered nominal terms are independent, the latter implies $\mathtt{t}' \parallel \mathtt{r}$. Therefore, the set $\mathtt{S}' = \mathtt{S} \cup \{\mathtt{t}'\}$ is independent. As $\mathtt{S}$ is maximal, we obtain that $\mathtt{t}' \in \mathtt{S}$. As $\mathtt{t}$ is the least element of $\mathtt{S}$, we also have that $\mathtt{t} \ll_* \mathtt{t}'$. Due to anti-symmetry of the strong covered-by relation, we obtain $\mathtt{t} \doteq \mathtt{t}'$. The latter means that $\mathtt{t}$ is, indeed, a minimal element in $\mathtt{T}$ with respect to the strong covered-by relation. Minimal elements with respect to the strong covered-by relation have dual weight 0.

Statement (2) is proved analogously: a nominal term $\mathtt{s} \in \mathtt{S}$ is constructed via an ascending chain; it is shown that $\mathtt{s}$ is the greatest element of $\mathtt{S}$ and a maximal element in $\mathtt{T}$. Finally, we mention that maximal elements with respect to the strong covered-by relation are standard terms and that they have weight 0. Q.E.D.

12.5.2 Extensions of Independent Sets

In the next proposition, we show that every independent set can be extended to a maximal independent set.

12.16 Proposition (Lindenbaun's Theorem (with AC)): Let $\mathtt{S} \subseteq \mathtt{T}$ be an independent set of nominal terms. There is a maximal independent extension $\mathtt{S}'$ of $\mathtt{S}$. The latter means that the following three conditions are satisfied: $\mathtt{S} \subseteq \mathtt{S}'$, $\mathtt{S}'$ is independent and if a set $\mathtt{S}' \cup \{\mathtt{s}\}$ is independent, then $\mathtt{s} \in \mathtt{S}'$.

Proof. By the Axiom of Choice (AC), we may presuppose a (transfinite) enumeration of the set $\mathtt{T}$ of all nominal terms. The latter means that there is an ordinal $\gamma \in \Omega$ such that $\mathtt{T} = \{\mathtt{t}_\alpha;\ \alpha \in \gamma\}$. We construct a sequence $\mathtt{S}_\alpha$ (with $\alpha \in \gamma'$) of extensions of $\mathtt{S}$ as follows:

1. $\alpha = 0$*:* $\mathtt{S}_0 = \mathtt{S}$
2. $\alpha = \beta'$ *successor:*

$$\mathtt{S}_\alpha \ = \ \begin{cases} \mathtt{S}_\beta \cup \{\mathtt{t}_\beta\} & \text{if } \mathtt{S}_\beta \cup \{\mathtt{t}_\beta\} \text{ independent} \\ \mathtt{S}_\beta & \text{otherwise} \end{cases}$$

3. α *limit ordinal:* $\mathtt{S}_\alpha = \bigcup_{\beta \in \alpha} \mathtt{S}_\beta$

Finally, let $\mathtt{S}' = \mathtt{S}_\gamma$. We check the properties of $\mathtt{S}'$:

1. *extension:* By construction, $\mathtt{S} \subseteq \mathtt{S}'$.

2. *independence:* $\mathtt{S}_\alpha$ is independent for all $\alpha \in \gamma'$: $\mathtt{S}_0$ is independent by presupposition. If $\mathtt{S}_\beta$ is independent, then also $\mathtt{S}_{\beta'}$ (immediately, by construction). If α is a limit ordinal, such that $\mathtt{S}_\beta$ is independent for all $\beta \in \alpha$, then also $\mathtt{S}_\alpha$ (assuming that not, there are two nominal terms $\mathtt{t}, \mathtt{s} \in \mathtt{S}_\alpha$ such that $\mathtt{t} \between \mathtt{s}$. As the $\mathtt{S}_\beta$ are ascending, there is a $\beta \in \alpha$ such that $\mathtt{t}, \mathtt{s} \in \mathtt{S}_\beta$, which is a contradiction to the independence of the set $\mathtt{S}_\beta$). In particular, $\mathtt{S}' = \mathtt{S}_\gamma$ is independent.

3. *maximality:* $\mathtt{S}'$ is maximal: Let $\mathtt{s} \in \mathtt{T}$ such that the set $\mathtt{S}' \cup \{\mathtt{s}\}$ is independent. As we enumerated $\mathtt{T}$, there is $\beta \in \gamma$ such that $\mathtt{s} \doteqdot \mathtt{t}_\beta$. By construction of the $\mathtt{S}_\alpha$, we have $\mathtt{S}_\beta \cup \{\mathtt{s}\} \subseteq \mathtt{S}' \cup \{\mathtt{s}\}$. Therefore, $\mathtt{S}_\beta \cup \{\mathtt{s}\}$ is independent (as a subset of an independent set). By construction, $\mathtt{s} \doteqdot \mathtt{t}_\beta \in \mathtt{S}_{\beta'} \subseteq \mathtt{S}'$.

We conclude that $\mathtt{S}'$ is, indeed, a maximal independent extension of $\mathtt{S}$.

Q.E.D.

With the help of Lindenbaum's Theorem for independent sets of nominal terms, we prove that independent sets of nominal terms are finite.

12.17 Proposition (Finiteness of Independent Sets): Let $\mathtt{S} \subseteq \mathtt{T}$ be a set of nominal terms. If $\mathtt{S}$ is independent, then $\mathtt{S}$ is finite. In particular, if $\mathtt{S}$ is maximal independent, then $|\mathtt{S}| = 2^{\mathbf{weight}(\mathtt{t})}$ for the least element $\mathtt{t} \in \mathtt{S}$ with respect to the strong covered-by relation.

Proof. We first discuss maximal independent sets $\mathtt{S} \subseteq \mathtt{T}$: According to the proposition about maximal independent sets, there are a nominal term $\mathtt{t} \doteqdot \min(\mathtt{S})$ a nominal term $\mathtt{s} \doteqdot \max(\mathtt{S})$. Investigate the following set:

$$\mathtt{S}' \;=\; \{\mathtt{r} \in \mathtt{T};\; \mathtt{t} \ll_* \mathtt{r} \ll_* \mathtt{s}\}$$

As $\mathtt{t}$ is the least and $\mathtt{s}$ the greatest element of $\mathtt{S}$ with respect to the strong covered-by relation, $\mathtt{S} \subseteq \mathtt{S}'$. As intermediate nominal terns are independent, $\mathtt{S}'$ is independent. As $\mathtt{S}$ is maximal independent, $\mathtt{S}' \subseteq \mathtt{S}$. Therefore, $\mathtt{S} = \mathtt{S}'$. The number of intermediate nominal terms between a minimal nominal term $\mathtt{t}$ and a maximal nominal term $\mathtt{s}$ equals to $2^{\mathbf{weight}(\mathtt{t})}$, which is the size of $\mathtt{S}$ and, in particular, finite.

If $\mathtt{S} \subseteq \mathtt{T}$ is an arbitrary independent set, then $\mathtt{S}$ is according to Lindenbaum's Theorem a subset of a maximal independent set $\mathtt{S}'$. As $\mathtt{S}'$ is finite, $\mathtt{S}$ is also finite. Q.E.D.

Avoiding the Axiom of Choice: In Lindenbaum's Theorem, the existence of maximal independent extensions of independent sets of nominal terms is proved under the presupposition of the Axiom of Choice. Nevertheless, it seems that this result can be proved constructively, without the strong presupposition of the Axiom of Choice:

1. *constructing an extension:* Let $\emptyset \neq \mathtt{S} \subseteq \mathtt{T}$ be a non-empty independent set of nominal terms. (If $\mathtt{S} = \emptyset$, then $\{*_0, x_0\}$ is a maximal independent extension.) Crucial aspect of the constructive proof is an explicit construction of the least element $\mathtt{t}$ and the greatest element $\mathtt{s}$ of a maximal independent extension $\mathtt{S}'$ of $\mathtt{S}$.

   There are nominal terms $\mathtt{t}' \in \mathtt{S}$ with minimal dual weight. We have to show that merging two different of these nominal terms results in a nominal term with strictly lower dual weight. We add all nominal terms generated this way to $\mathtt{S}$ and repeat the procedure until we obtain a nominal term with dual weight zero. This nominal term will be the least element of the maximal independent extension. (If the procedure terminates too early, as there is a uniquely determined nominal term $\mathtt{t}'$ with minimal dual weight, then we can use an arbitrary nominal term below $\mathtt{t}'$ having dual weight zero.)

   Analogously, a nominal term $\mathtt{s}$ above $\mathtt{S}$ is constructed with the help of the dual merge function satisfying that its weight equals to zero.

   We define $\mathtt{S}' = \{\mathtt{r} \in \mathtt{T};\ \mathtt{t} \ll_* \mathtt{r} \ll_* \mathtt{s}\}$.

2. *properties:* We have to show that $\mathtt{S}'$ has all desired properties. This proof succeeds, as we know already that the sets $\mathtt{S}'$ are exactly the maximal independent sets of nominal terms.

12.5.3 Merging Independent Sets

As independent sets of nominal terms are finite, we are motivated to introduce the set versions of the merge functions.

12.18 DEF (Set Version - Merge Functions): The set versions of both merge functions are defined recursively (over the cardinality of independent sets $\mathtt{S}$ of nominal terms) and in parallel as follows:

1. $\mathtt{S} = \{\mathtt{s}\}$ *singleton:* $\mu(\{\mathtt{S}\}) \doteq \mathtt{s}$ and $\overline{\mu}(\{\mathtt{S}\}) \doteq \mathtt{s}$

2. $\mathtt{s} \in \mathtt{S} \neq \{\mathtt{s}\}$ *not a singleton:*

$$\mu(\mathtt{S}) \;=\; \mu(\mu(\mathtt{S}\backslash\{\mathtt{s}\}), \mathtt{s}) \quad \text{and} \quad \overline{\mu}(\mathtt{S}) \;=\; \overline{\mu}(\overline{\mu}(\mathtt{S}\backslash\{\mathtt{s}\}), \mathtt{s})$$

Remarks (Set Version - Merge Functions):

1. *well-defined:* As the restrictions of both merge functions (their version for nominal terms) to independent sets of nominal terms are associative and commutative, both merge functions are defined in the complex case independently of the choice of the nominal term $\mathtt{s}$.

2. *empty sets:* The set versions of both merge functions are not defined on the empty set; the reason is that there are no suitable distinguished nominal terms in the set $\mathtt{T}$ of nominal terms.

   The situation is different, if we presuppose an underlying maximal independent set $\mathtt{S}'$ such that the arguments $\mathtt{S}$ for the merge functions are subsets of $\mathtt{S}'$. The set $\mathtt{S}'$ contains a least element $\mathtt{t}$ and a greatest element t (both with respect to the strong covered-by relation). In such a context, we presuppose:

   $$\mu(\emptyset) \;\hat{=}\; t \quad ; \quad \overline{\mu}(\emptyset) \;\hat{=}\; \mathtt{t}$$

   Under such circumstances, the recursive clauses in the definition of the merge functions also hold with respect to singletons. More formally, for all $\mathtt{s} \in \mathtt{S}'$:

   $$\mu(\{\mathtt{s}\}) \;\hat{=}\; \mu(\mu(\emptyset), \mathtt{s}) \quad ; \quad \overline{\mu}(\{\mathtt{s}\}) \;\hat{=}\; \overline{\mu}(\overline{\mu}(\emptyset), \mathtt{s})$$

3. *extreme elements:* Merging an independent set $\mathtt{S}$ results in a bound to the set $\mathtt{S}$. More precisely:

   (a) *greatest lower bound:* $\mu(\mathtt{S})$ is the greatest lower bound of $\mathtt{S}$ with respect to the strong covered-by relation; in particular, if $\mathtt{S}$ is maximal, then $\mu(\mathtt{S}) = \min(\mathtt{S})$.

   (b) *least upper bound:* $\overline{\mu}(\mathtt{S})$ is the least upper bound of $\mathtt{S}$ with respect to the strong covered-by relation; in particular, if $\mathtt{S}$ is maximal, then $\overline{\mu}(\mathtt{S}) = \max(\mathtt{S})$.

12.6 Formal Introduction of Strong Independence

Two nominal terms are strongly independent, if they are independent and if they satisfy additionally that no nominal symbol is locally covered by itself. We provide the formal definition of this version of independence of nominal terms.

12.19 DEF (Strong Independence): Two nominal terms $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ are *strongly independent* (formally, $\mathtt{t} \parallel_* \mathtt{s}$), if one of the following conditions is satisfied:

1. *atomic nominal terms:* The nominal terms $\mathtt{t}$ and $\mathtt{s}$ are atomic, standard and equal.
2. *nominal symbols:* ($\mathtt{t} \in \mathbf{V}_*$ and $\mathtt{s} \in \mathtt{T}_0$) or ($\mathtt{t} \in \mathtt{T}_0$ and $\mathtt{s} \in \mathbf{V}_*$).
3. *complex nominal term:* $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n) \sim f(\mathtt{s}_0, \ldots \mathtt{s}_n) \doteq \mathtt{s}$ and $\mathtt{t}_k \parallel_* \mathtt{s}_k$ for all $k \in n'$.

A set $\mathtt{S} \subseteq \mathtt{T}$ of nominal terms is called *strongly independent*, if all pairs of different elements are strongly independent. Formally, if for all $\mathtt{t}, \mathtt{s} \in \mathtt{S}$:

$$\mathtt{t} \neq \mathtt{s} \quad \Rightarrow \quad \mathtt{t} \parallel_* \mathtt{s}$$

Basic Properties (Strong Independence): We communicate some basic properties of the strong independence relation.

1. *independence:* Strongly independent nominal terms are, in particular, independent. More formally, for all nominal terms $\mathtt{t}$ and $\mathtt{s}$:

   $$\mathtt{t} \parallel_* \mathtt{s} \quad \Rightarrow \quad \mathtt{t} \parallel \mathtt{s}$$

   (Immediate, as both relations are defined identically, besides the additional demand of being standard in clause (1) of strong independence.)

2. *reflexivity:* Strong independence is not reflexive: a nominal term is strongly independent of itself, if and only if it is a standard term. More formally, for all nominal terms $\mathtt{t}$:

   $$\mathtt{t} \parallel_* \mathtt{t} \quad \Leftrightarrow \quad \mathtt{t} \in \mathtt{T}_0$$

   (Proof of "$\Rightarrow$" by contraposition, of "$\Leftarrow$" directly; both directions by straightforward induction.)

3. *strongly independent sets:* Subsets of strongly independent sets are strongly independent. (Immediate.)

4. *symmetry:* Strong independence is symmetric. More formally, for all nominal terms $\mathtt{t}$ and $\mathtt{s}$:

   $$\mathtt{t} \parallel_* \mathtt{s} \quad \Rightarrow \quad \mathtt{s} \parallel_* \mathtt{t}$$

   (Straightforward induction.)

5. *standard terms:* If a standard term t and a nominal term $\mathtt{t}$ are independent, then they are already strongly independent. More formally, for all standard terms t and all nominal terms $\mathtt{t}$:

$$t \parallel \mathtt{t} \quad \Rightarrow \quad t \parallel_* \mathtt{t}$$

(Straightforward induction.)

Recalling that strongly covered nominal terms are independent, we obtain that if the nominal term $\mathtt{t}$ is (strongly) covered by the standard term t, then they are strongly independent. More formally:

$$\mathtt{t} \ll_* t \quad \Rightarrow \quad \mathtt{t} \parallel_* t$$

6. *equivalence of nominal terms:* Strong independence is compatible with the equivalence of nominal terms. The latter means that the following condition is satisfied for all nominal terms $\mathtt{t}, \mathtt{s}$ and $\mathtt{r}$:

$$\mathtt{t} \parallel_* \mathtt{s} \text{ and } \mathtt{s} \equiv \mathtt{r} \quad \Rightarrow \quad \mathtt{t} \parallel_* \mathtt{r}$$

(Straightforward induction.)

12.7 Merging Strongly Independent Nominal Terms

We investigate the relationship between strong independence of nominal terms and both merge functions.

12.7.1 Weight of Merged Nominal Terms

If two strongly independent nominal terms are merged, the number of nominal symbols in the result equals the sum of the number of nominal symbols in both arguments; if they are dual merged, then all nominal symbols vanish.

12.20 Proposition (Weight of Merged Nominal Terms): Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ be nominal terms. If $\mathtt{t}$ and $\mathtt{s}$ are strongly independent (formally, $\mathtt{t} \parallel_* \mathtt{s}$), then the following both equations hold:

1. *merge function:* $\mathbf{weight}(\mu(\mathtt{t}, \mathtt{s})) = \mathbf{weight}(\mathtt{t}) + \mathbf{weight}(\mathtt{s})$
2. *dual merge function:* $\mathbf{weight}(\overline{\mu}(\mathtt{t}, \mathtt{s})) = 0$

Proof. Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ be strongly independent. First, we observe that $\overline{\mu}(\mathtt{t}, \mathtt{s}) \in \mathtt{T}_0$ is a standard term. (This is easily checked by a straightforward induction.) As a consequence, we obtain immediately the second equation,

namely that $\mathbf{weight}(\overline{\mu}(\mathtt{t},\mathtt{s})) = 0$. We recall the general result with respect to independent nominal terms:

$$\mathbf{weight}(\mu(\mathtt{t},\mathtt{s})) \;=\; \mathbf{weight}(\mathtt{t}) + \mathbf{weight}(\mathtt{s}) - \mathbf{weight}(\overline{\mu}(\mathtt{t},\mathtt{s}))$$

Due to the second equation, the first equation follows immediately. Q.E.D.

12.7.2 Unique Covering

In the next proposition, we show that independent sets containing strongly independent nominal terms are strongly covered by a uniquely determined standard term.

12.21 Proposition (Unique Covering): Let $\mathtt{S} \subseteq \mathtt{T}$ be an independent set of nominal terms. If $\mathtt{S}$ contains strongly independent elements $\mathtt{t}$ and $\mathtt{s}$, then $t \eqsim \overline{\mu}(\mathtt{t},\mathtt{s})$ is the uniquely determined standard term strongly covering all elements of $\mathtt{S}$.

Proof. Let $\mathtt{S} \subseteq \mathtt{T}$ independent and $\mathtt{t},\mathtt{s} \in \mathtt{S}$ such that $\mathtt{t} \parallel\!\!\!\parallel_* \mathtt{s}$. Let $t \eqsim \overline{\mu}(\mathtt{t},\mathtt{s})$. According to the proposition above, $t \in \mathtt{T}_0$ is a standard term.

As $\mathtt{S}$ is independent, $\mathtt{S} \cup \{\overline{\mu}(\mathtt{t},\mathtt{s})\}$ is also independent. Therefore, there is a maximally independent set $\mathtt{S}'$ extending the latter set. $\overline{\mu}(\mathtt{S}')$ is a standard term covering all elements of $\mathtt{S}'$; in particular, $t \ll_* \overline{\mu}(\mathtt{S}')$. As standard terms are maximal with respect to the strong covered-by relation, we obtain $t \eqsim \overline{\mu}(\mathtt{S}')$. As $\mathtt{S}$ is a subset of $\mathtt{S}'$, we have $\mathtt{r} \ll_* t$ for all $\mathtt{r} \in \mathtt{S}$.

We still have to show that t is uniquely determined. Let $t' \in \mathtt{T}_0$ such that $\mathtt{r} \ll_* t'$ for all $\mathtt{r} \in \mathtt{S}$. Therefore, $t' \parallel \mathtt{r}$ for all $\mathtt{r} \in \mathtt{S}$ and, as a consequence, $\mathtt{S} \cup \{t'\}$ is independent. As both $\mathtt{t}$ and $\mathtt{s}$ are contained in this independent set, we obtain that its extension $\mathtt{S} \cup \{t', \overline{\mu}(\mathtt{t},\mathtt{s})\} = \mathtt{S} \cup \{t, t'\}$ is independent. Therefore, $t \parallel t'$ and, as both are standard terms, also $t \eqsim t'$. Q.E.D.

As a corollary, we obtain that we may extend strongly independent sets (containing two strongly independent elements) by their unique covering.

12.22 Corollary (Unique Covering): Let $\mathtt{S} \subseteq \mathtt{T}$ be a strongly independent set. If $\mathtt{S}$ contains strongly independent elements $\mathtt{t}$ and $\mathtt{s}$, then $\mathtt{S} \cup \{\overline{\mu}(\mathtt{t},\mathtt{s})\}$ is strongly independent.

Proof. Let $t \eqsim \overline{\mu}(\mathtt{t},\mathtt{s}) \in \mathtt{T}_0$ for strongly independent $\mathtt{t},\mathtt{s} \in \mathtt{S}$. It is sufficient to check that $\mathtt{r} \parallel\!\!\!\parallel_* t$ for all $\mathtt{r} \in \mathtt{S}$. Let $\mathtt{r} \in \mathtt{S}$ arbitrary. Due to the proposition above, we obtain $\mathtt{r} \ll_* t$. As a consequence, we obtain $\mathtt{r} \parallel t$. As $t \in \mathtt{T}_0$ is standard, the latter implies $\mathtt{r} \parallel\!\!\!\parallel_* t$. Q.E.D.

12.7.3 Preservation of Strong Independence

We discuss the conditions such that a transformation of a strongly independent set remains strongly independent.

Remark (Extensions of Strongly Independent Sets): It follows almost immediately from the corollary above that an extension $\mathtt{S}' = \mathtt{S} \cup \{\overline{\mu}(\mathtt{t}, \mathtt{s})\}$ of a strongly independent set $\mathtt{S}$ by the result of an application of the dual merge function on elements of $\mathtt{S}$ is again strongly independent.

This is, in general, not true with respect to an extension $\mathtt{S}' = \mathtt{S} \cup \{\mu(\mathtt{t}, \mathtt{s})\}$ by the result of an application of the merge function. Investigate, for example:

$$\mathtt{S} = \{(* + 0), (0 + *)\} \quad ; \quad \mathtt{r} \doteq \mu(* + 0, 0 + *) \doteq * + *$$

We have neither $\mathtt{r} \parallel_* (* + 0)$ nor $\mathtt{r} \parallel_* (0 + *)$. In particular, $\mathtt{S} \cup \{\mathtt{r}\}$ is not strongly independent (but independent).

Nevertheless, the situation changes, if we replace in $\mathtt{S}$ the strongly independent arguments by the result of an application of the merge function on these arguments: the resulting set is still strongly independent. In order to show this result, it is convenient to investigate first some cases under which the merge function preserves strong independence.

12.23 Proposition (Preservation of Strong Independence): Let $\mathtt{S} \subseteq \mathtt{T}$ be strongly independent. Let $\mathtt{t}, \mathtt{s}, \mathtt{r} \in \mathtt{S}$ such that $\mathtt{t} \not\equiv \mathtt{r} \not\equiv \mathtt{s}$. If $\mu(\mathtt{t}, \mathtt{s}) \not\equiv \mathtt{r}$, then $\mu(\mathtt{t}, \mathtt{s}) \parallel_* \mathtt{r}$.

Proof. In a first step, we discuss some observations.

1. If $\mathtt{t} \doteq \mathtt{s}$, then $\mu(\mathtt{t}, \mathtt{s}) \doteq \mathtt{t} \in \mathtt{S}$. By presupposition, $\mathtt{t} \not\equiv \mathtt{r}$ and, therefore, $\mu(\mathtt{t}, \mathtt{s}) \not\equiv \mathtt{r}$. As $\mathtt{S}$ is strongly independent, we obtain $\mu(\mathtt{t}, \mathtt{s}) \parallel_* \mathtt{r}$. As a consequence, we can presuppose subsequently $\mathtt{t} \not\equiv \mathtt{s}$.

2. By presupposition, $\mathtt{t} \not\equiv \mathtt{r} \not\equiv \mathtt{s}$. As all three nominal terms are contained in the strongly independent set $\mathtt{S}$, we immediately obtain that $\mathtt{t} \parallel_* \mathtt{r}$ and $\mathtt{r} \parallel_* \mathtt{s}$.

   In general, we do not have $\mathtt{t} \not\parallel_* \mathtt{s}$, as we do not demand $\mathtt{t} \not\equiv \mathtt{s}$. (Recall that the restriction of strong independence to proper nominals is even anti-reflexive.)

We prove our statement by induction over the structure of $\mathtt{t}$.

1. $\mathtt{t} \in \mathbf{V}_*$ *atomic:* As $\mathtt{t} \parallel_* \mathtt{r}$, we obtain $\mathtt{r} \in \mathtt{T}_0$.

   Assuming that $\mathtt{s} \in \mathtt{T}_0$, we obtain $\mathtt{s} \doteq \mathtt{r}$, as $\mathtt{s} \parallel_* \mathtt{r}$. This is a contradiction to the presupposition $\mathtt{s} \not\equiv \mathtt{r}$. Therefore $\mathtt{s} \notin \mathtt{T}_0$. As $\mathtt{s} \parallel_* \mathtt{r}$, we

obtain $\mathtt{s} \in \mathbf{V}_*$. Therefore, $\mathtt{t} \not\parallel_* \mathtt{s}$. As $\mathtt{S}$ is strongly independent, the latter implies $\mathtt{t} \doteq \mathtt{s}$. According to our first observation, we already have $\mu(\mathtt{t}, \mathtt{s}) \parallel_* \mathtt{r}$.

2. $\mathtt{t} \notin \mathbf{V}_*$ *atomic:* As $\mathtt{t} \parallel_* \mathtt{r}$, we have $\mathtt{r} \in \mathbf{V}_*$. Therefore, $\mathtt{s} \parallel_* \mathtt{r}$ implies $\mathtt{s} \in \mathtt{T}_0$. As $\mathtt{S}$ is strongly independent, we have that $\mathtt{t} \doteq \mathtt{s}$ or $\mathtt{t} \parallel_* \mathtt{s}$. As both $\mathtt{t} \in \mathtt{T}_0$ and $\mathtt{s} \in \mathtt{T}_0$, strong independence implies again $\mathtt{t} \doteq \mathtt{s}$. Therefore, $\mathtt{t} \doteq \mathtt{s}$, which means according to our first observation that $\mu(\mathtt{t}, \mathtt{s}) \parallel_* \mathtt{r}$.

3. $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_n)$ *complex:* As $\mathtt{t} \parallel_* \mathtt{r}$, we have that $\mathtt{t} \sim \mathtt{r}$ or both that $\mathtt{t} \in \mathtt{T}_0$ and $\mathtt{r} \in \mathbf{V}_*$. We discuss both cases separately:

   (a) $\mathtt{t} \not\sim \mathtt{r}$*:* As already observed, $\mathtt{t} \in \mathtt{T}_0$ and $\mathtt{r} \in \mathbf{V}_*$. As $\mathtt{s} \parallel_* \mathtt{r}$, we also have that $\mathtt{s} \in \mathtt{T}_0$. As discussed above, $\mathtt{t} \in \mathtt{T}_0$ and $\mathtt{s} \in \mathtt{T}_0$ implies that $\mathtt{t} \doteq \mathtt{s}$. Again according to our first observation, the latter implies $\mu(\mathtt{t}, \mathtt{s}) \parallel_* \mathtt{r}$.

   (b) $\mathtt{t} \sim \mathtt{r}$*:* Assuming that $\mathtt{r} \not\sim \mathtt{s}$, we obtain due to strong independence that $\mathtt{s} \in \mathbf{V}_*$ and $\mathtt{r} \in \mathtt{T}_0$. As $\mathtt{t} \notin \mathbf{V}_*$, we obtain $\mathtt{t} \neq \mathtt{s}$ and, therefore, $\mathtt{t} \in \mathtt{T}_0$ (again due to strong independence). As $\mathtt{t} \parallel_* \mathtt{r}$, we obtain $\mathtt{t} \doteq \mathtt{r}$ (both nominal terms are standard). This contradicts our presuppositions.

   Therefore, $\mathtt{r} \sim \mathtt{s}$. Due to similarity, there are nominal terms $\mathtt{s}_k$ and $\mathtt{r}_k$ such that:

$$\mathtt{s} \doteq f(\mathtt{s}_0, \ldots \mathtt{s}_n) \quad ; \quad \mathtt{r} \doteq f(\mathtt{r}_0, \ldots \mathtt{r}_n)$$

   Without loss of generality, we assume that $\mathtt{t} \neq \mathtt{s}$. We have pairwise different nominal terms contained in a strongly independent set of nominal terms. Therefore, these nominal terms are pairwise strongly independent. As these nominal terms are all similar (and complex), we obtain for all $k \in n'$:

$$\mathtt{t}_k \parallel_* \mathtt{s}_k \quad ; \quad \mathtt{t}_k \parallel_* \mathtt{r}_k \quad ; \quad \mathtt{s}_k \parallel_* \mathtt{r}_k$$

   We show $\mu(\mathtt{t}_k, \mathtt{s}_k) \parallel_* \mathtt{r}_k$ for arbitrary $k \in n'$. We have to distinguish some cases:

   - If $\mathtt{t}_k \doteq \mathtt{s}_k$, then $\mu(\mathtt{t}_k, \mathtt{s}_k) \doteq \mathtt{t}_k$ and, therefore, $\mu(\mathtt{t}_k, \mathtt{s}_k) \parallel_* \mathtt{r}$.
   - Otherwise $\mathtt{t}_k \neq \mathtt{s}_k$.
     If $\mathtt{t}_k \doteq \mathtt{r}_k$, then strong independence implies that $\mathtt{t}_k \in \mathtt{T}_0$ and $\mathtt{r}_k \in \mathtt{T}_0$. As $\mathtt{t}_k \neq \mathtt{s}_k$, strong independence also implies that

$\mathtt{s} \in \mathbf{V}_*$. This means that $\mu(\mathtt{t}_k, \mathtt{s}_k) \doteq \mathtt{s}_k \in \mathbf{V}_*$ and, therefore, $\mu(\mathtt{t}_k, \mathtt{s}_k) \parallel_* \mathtt{r}_k$.
If $\mathtt{s}_k \doteq \mathtt{r}_k$, then strong independence implies that $\mathtt{s}_k \in \mathtt{T}_0$ and $\mathtt{r}_k \in \mathtt{T}_0$. Analogously to the case before, we can conclude that $\mathtt{t}_k \in \mathbf{V}_*$. Again, we obtain$\mu(\mathtt{t}_k, \mathtt{s}_k) \in \mathbf{V}_*$ and, therefore, $\mu(\mathtt{t}_k, \mathtt{s}_k) \parallel_* \mathtt{r}_k$.
Finally, we have to investigate the case that $\mathtt{t}_k \neq \mathtt{r}_k \neq \mathtt{s}_k$. Here, we can apply induction hypothesis (observe that the set $\mathtt{S}_k = \{\mathtt{t}_k, \mathtt{s}_k, \mathtt{r}_k\}$ is strongly independent) and we obtain again $\mu(\mathtt{t}_k, \mathtt{s}_k) \parallel_* \mathtt{r}_k$.

We summarise: $\mu(\mathtt{t}_k, \mathtt{s}_k) \parallel_* \mathtt{r}_k$ for all $k \in n'$ and in all subcases.

As $\mathtt{t} \sim \mathtt{s}$, we can calculate as follows:

$$\mu(\mathtt{t}, \mathtt{s}) \;\doteq\; f(\mu(\mathtt{t}_0, \mathtt{s}_0), \ldots \mu(\mathtt{t}_n, \mathtt{s}_n)) \;\sim\; \mathtt{r}$$

As all respective pairs of the direct subterms of $\mu(\mathtt{t}, \mathtt{s})$ and of $\mathtt{r}$ are strongly independent, we conclude that $\mu(\mathtt{t}, \mathtt{s}) \parallel_* \mathtt{r}$

Q.E.D.

The proposition about the replacement of two strongly independent nominal terms by the result of merging both is an immediate consequence of the proposition above. We communicate this corollary.

12.24 Corollary (Preservation of Strong Independence): Let $\mathtt{S} \subseteq \mathtt{T}$ be strongly independent. The set $\mathtt{S}' = (\mathtt{S} \backslash \{\mathtt{t}, \mathtt{s}\}) \cup \{\mu(\mathtt{t}, \mathtt{s})\}$ is strongly independent for all nominal terms $\mathtt{t}, \mathtt{s} \in \mathtt{S}$.

Proof. We have to show for all $\mathtt{r}, \mathtt{r}' \in \mathtt{S}'$: if $\mathtt{r} \neq \mathtt{r}'$, then $\mathtt{r} \parallel_* \mathtt{r}'$. Let $\mathtt{r}, \mathtt{r}' \in \mathtt{S}'$ be arbitrary, but different ($\mathtt{r} \neq \mathtt{r}'$). If $\mathtt{r}, \mathtt{r}' \in \mathtt{S}$, then $\mathtt{r} \parallel_* \mathtt{r}'$ is trivially satisfied, as $\mathtt{S}$ is strongly independent. Otherwise and without loss of generality, $\mathtt{r}' \doteq \mu(\mathtt{t}, \mathtt{s}) \in \mathtt{S}' \backslash \mathtt{S}$. As $\mathtt{r} \neq \mathtt{r}'$, we have that $\mathtt{r}' \in \mathtt{S} \backslash \{\mathtt{t}, \mathtt{s}\}$. The latter means that $\mathtt{t} \neq \mathtt{r} \neq \mathtt{s}$. Furthermore, $\{\mathtt{t}, \mathtt{s}, \mathtt{r}\} \subseteq \mathtt{S}$. Therefore, $\{\mathtt{t}, \mathtt{s}, \mathtt{r}\}$ is strongly independent. As $\mathtt{r} \neq \mathtt{r}' \doteq \mu(\mathtt{t}, \mathtt{s})$, we may apply the proposition above and obtain $\mathtt{r} \parallel_* \mathtt{r}'$. Q.E.D.

12.8 The Completion Function

We complement the results of this section by discussing briefly the *completion function* mapping pairs of nominal terms, in the intended case of strongly

independent arguments, to a relation describing, how the locally covered nominal symbols in both arguments are covered. Changing the perspective, the completion function describes, in the intended case, how to complete the arguments to their unique covering. Additionally, the image of the completion function may contain the falsum ($\bot$) indicating that the arguments are not intended. We provide the definition of this function.

12.25 DEF (Completion Function): The binary *completion function* $\mathbf{comp} : \mathtt{T} \times \mathtt{T} \to \mathfrak{p}(\omega \times \mathtt{T}) \cup \{\bot\}$ is defined as follows:

1. *atomic nominal terms:* If both $\mathtt{t}$ and $\mathtt{s}$ are standard atomic and equal:
$$\mathbf{comp}(\mathtt{t}, \mathtt{s}) \;=\; \emptyset$$
2. *nominal symbols:* If ($\mathtt{t} \doteqdot *_k$ and $\mathtt{s} \doteqdot \mathtt{r}$) or ($\mathtt{s} \doteqdot *_k$ and $\mathtt{t} \doteqdot \mathtt{r}$) for $k \in \omega$ and $\mathtt{r} \in \mathtt{T}_0$:
$$\mathbf{comp}(\mathtt{t}, \mathtt{s}) \;=\; \{\langle k, \mathtt{r}\rangle\}$$
3. *complex nominal terms:* If $\mathtt{t} \doteqdot f(\mathtt{t}_0, \ldots \mathtt{t}_n) \sim f(\mathtt{s}_0, \ldots \mathtt{s}_n) \doteqdot \mathtt{s}$
$$\mathbf{comp}(\mathtt{t}, \mathtt{s}) \;=. \bigcup_{k \in n'} \mathbf{comp}(\mathtt{t}_k, \mathtt{s}_k)$$
4. *otherwise:* $\mathbf{comp}(\mathtt{t}, \mathtt{s}) = \{\bot\}$

Observations (Completion Function): The following observations are easily checked:

1. *the falsum:* The falsum ($\bot$) contained in the result of an application of the completion function indicates that the arguments are not strongly independent. More precisely, for all nominal terms $\mathtt{t}$ and $\mathtt{s}$:
$$\bot \in \mathbf{comp}(\mathtt{t}, \mathtt{s}) \quad \Leftrightarrow \quad \mathtt{t} \not\parallel_* \mathtt{s}$$
2. *relation:* The result $\mathbf{comp}(\mathtt{t}, \mathtt{s})$ of an application of the completion function on strongly independent nominal terms $\mathtt{t}$ and $\mathtt{s}$ is a relation between natural numbers and standard terms. The ordered pair $\langle k, r\rangle$ is contained in the result, if and only if the nominal symbol $*_k$ in one argument is locally covered by the standard term r in the other argument.

3. *function:* The result $\mathbf{comp}(\mathtt{t}, \mathtt{s})$ of an application of the completion function on strongly independent nominal terms $\mathtt{t}$ and $\mathtt{s}$ is a function (on a set of natural numbers), if and only if there is no $k \in \omega$ such that $\langle k, r\rangle$ and $\langle k, r'\rangle$ are contained in $\mathbf{comp}(\mathtt{t}, \mathtt{s})$ for different terms $r \neq r'$.

   This condition is, for example, satisfied, if the sets of free places of the nominal terms $\mathtt{t}$ and $\mathtt{s}$ are disjoint and if both nominal terms are simple. More formally, if $\mathbf{place}(\mathtt{t}) \cap \mathbf{place}(\mathtt{s}) = \emptyset$ and if $\mathtt{t}, \mathtt{s} \in \mathtt{T}_s$.

   (Observe that this condition is sufficient, but not necessary.)

   The domain $\mathbf{dom}(\mathbf{comp}(\mathtt{t}, \mathtt{s}))$ of the result of such an application is the union $\mathbf{place}(\mathtt{t}) \cup \mathbf{place}(\mathtt{s})$ of the sets of free places of the arguments.

4. *sequence:* The result $\mathbf{comp}(\mathtt{t}, \mathtt{s})$ of an application of the completion function on strongly independent nominal terms $\mathtt{t}$ and $\mathtt{s}$ is a (finite) sequence, if the result is a function on a natural number $n \in \omega$.

   This condition is, for example, satisfied, if $\mathtt{t}$ is n-ary and $\mathtt{s}$ is the result of shifting the nominal symbols in an m-ary nominal term $\mathtt{r}$ by n steps (for $n, m \in \omega$). More formally, if there are $n, m \in \omega$ and a nominal term $\mathtt{r}$ such that:

$$\mathtt{t} \in \mathtt{T}_n \quad ; \quad \mathtt{r} \in \mathtt{T}_m \quad ; \quad \mathtt{s} \eqsim \mathtt{r}^{+n}$$

   (As before, this is a sufficient condition, but not necessary.)

Relevance: We provide some applications of the completion functions:

1. *covering:* Let $\mathtt{t}, \mathtt{s} \in \mathtt{T}$ be two strongly independent nominal terms. Without loss of generality, we assume additionally that both nominal terms are in simple normal form.

   The result $\mathbf{comp}(\mathtt{t}, \mathtt{s}^{+n}) = \mathfrak{t} \circ \mathfrak{s}$ (where $n = \mathbf{place}(\mathtt{t})$) of an application of the completion function on $\mathtt{t}$ and the suitably shifted version of $\mathtt{s}$ can be understood as the concatenation of two finite sequences $\mathfrak{t}$ and $\mathfrak{s}$ of standard terms satisfying the following conditions:

   (a) *suitable lengths:* $\mathbf{lng}(\mathfrak{t}) = \mathbf{place}(\mathtt{t})$ and $\mathbf{lng}(\mathfrak{s}) = \mathbf{place}(\mathtt{s})$

   (b) *eliminated sequences:* The following equations hold:

$$\overline{\mu}(\mathtt{t}, \mathtt{s}) \eqsim \mu(\mathtt{t}, \mathtt{s}^{+n})[\mathfrak{t} \circ \mathfrak{s}] \eqsim \mathtt{t}[\mathfrak{t}] \eqsim \mathtt{s}[\mathfrak{s}]$$

   Observe that $\overline{\mu}(\mathtt{t}, \mathtt{s}^{+n}) \eqsim \overline{\mu}(\mathtt{t}, \mathtt{s})$.

2. *elimination form:* Let $t \in \mathtt{T}_0$ be a standard term and $\mathtt{t} \in \mathtt{T}_n$ an n-ary nominal term (for $n \in \omega$). If $\mathtt{t}$ is an elimination form of t, then $\mathbf{comp}(\mathtt{t}, t)$ is the uniquely determined sequence of standard terms actually eliminated in $\mathtt{t}$. More formally:

$$\mathtt{t} \leq t \quad \Rightarrow \mathtt{t}[\mathbf{comp}(\mathtt{t}, t)] \doteq t$$

Observe that this is a special case of the general case discussed above, as elimination forms of a standard term are strongly independent of the standard term.

Variants (Completion Function): There are some interesting variants of the completion function:

1. *standard terms:* If we restrict the second argument to standard terms, then the falsum ($\bot$) contained in the result of an application indicates that the first argument is not covered by the second argument.

2. *independence:* If we drop the restriction in the first clause that the atomic arguments have to be standard, then the falsum ($\bot$) contained in the result of an application indicates that the arguments are not independent.

   As ordered pairs $\langle *_k, *_k \rangle$ are mapped by this version of the completion function to the empty set, completing the arguments (as discussed above) does not result in a standard term, but in a nominal term in which the nominal symbols locally covered by themselves still occur.

12.9 Excursus: Correspondence to Set Theory

We conclude our discussion of the (strong) independence of nominal terms with a brief excursus on the relationship between the theory of nominal terms and set theory.

First Observations: The size of maximal independent sets motivates to identify such sets with power sets. Our results with respect to the weight of merged and dual merged nominal terms remind on the analogous results with respect to the size of sets under union and intersection; in particular, strongly independent nominal terms behave as disjoint sets.

Correspondence: Let $\mathtt{S} \subseteq \mathtt{T}$ be a maximal independent set and X a set such that $|X| = \mathbf{weight}(\mu(\mathtt{S}))$. We may observe the following correspondence between the maximal independent set $\mathtt{S}$ and the power set $\mathfrak{p}(X)$ of X:

1. *union and intersection:* The dual merge function $\overline{\mu}$ corresponds with set intersection $\cap$, the merge function μ with set union $\cup$.

2. *empty set and full set:* The greatest element $\overline{\mu}(\mathtt{S})$ corresponds with the empty set, the least element $\mu(\mathtt{S})$ with the full set X.

3. *singletons and subsets:* The single nominal terms $\mathtt{t}$ correspond with singletons $\{x\}$; once such a concrete correspondence Φ is established, this correspondence can be extended to all subsets $Y \subseteq X$ by demanding that this extension respects the merge function with respect to strong independence. More formally, by demanding for all $\mathtt{t}, \mathtt{s} \in \mathtt{S}$:

$$\mathtt{t} \parallel_* \mathtt{s} \quad \Rightarrow \quad \Phi(\mu(\mathtt{t}, \mathtt{s})) = \Phi(\mathtt{t}) \cup \Phi(\mathtt{s})$$

4. *disjointness:* Strong independence corresponds with disjointness.

5. *subset relation:* The strong covered-by relation corresponds inversely with set inclusion. This means for all nominal terms $\mathtt{t}, \mathtt{s} \in \mathtt{S}$:

$$\mathtt{t} \ll_* \mathtt{s} \quad \Rightarrow \quad \Phi(\mathtt{s}) \subseteq \Phi(\mathtt{t})$$

6. *complement:* A straightforward idea for constructing the complement in a maximal independent set would be to investigate the completion of a nominal term $\mathtt{t}$ to the greatest element $\overline{\mu}(\mathtt{S})$ and complete the least element $\mu(\mathtt{S})$ partially according to that completion. More formally, for arbitrary nominal term $\mathtt{t} \in \mathtt{S}$:

$$\mathfrak{s} = \mathbf{comp}(\mathtt{t}, \overline{\mu}(\mathtt{S})) \cup \{\langle k, *_k \rangle;\ k \in \omega \backslash \mathbf{place}(\mathtt{t})\} \quad ; \quad \nu(\mathtt{t}) \doteq \mu(\mathtt{S})[\mathfrak{s}]$$

   Here, $\nu(\mathtt{t})$ is meant to be the complement of $\mathtt{t}$ satisfying the following both equations:

$$\mu(\mathtt{t}, \nu(\mathtt{t})) \doteq \mu(\mathtt{S}) \quad ; \quad \overline{\mu}(\mathtt{t}, \nu(\mathtt{t})) \doteq \overline{\mu}(\mathtt{S})$$

   Unfortunately, the direct construction given above only works, if the least element $\mu(\mathtt{S})$ of $\mathtt{S}$ is simple; otherwise, there are clashes of nominal symbols leading to undesired results.

   In the general case, we may use well-known equalities to define the complement. For example, for all $\mathtt{t} \in \mathtt{S}$:

$$\nu(\mathtt{t}) \doteq \mu(\mathtt{s};\ \overline{\mu}(\mathtt{t}, \mathtt{s}) \doteq \overline{\mu}(\mathtt{S})) \quad \text{or} \quad \nu(\mathtt{t}) \doteq \overline{\mu}(\mathtt{s};\ \mu(\mathtt{t}, \mathtt{s}) \doteq \mu(\mathtt{S}))$$

The sketched correspondence yields that $\langle \mathtt{S}, \mu, \overline{\mu}, \nu, \overline{\mu}(\mathtt{S}), \mu(\mathtt{S}) \rangle$ is a (non-trivial) finite boolean algebra for all maximal independent sets $\mathtt{S} \subseteq \mathtt{T}$.

13 Independent Occurrences

The concept of independence is carried over to (multi-shape) occurrences and sets of occurrences. Applications of this concept are discussed.

13.1 Introduction of Independent Occurrences

We provide the formal definition of independent occurrences.

13.1 DEF (Independence of Occurrences):

1. *independent occurrences:* Two (multi-shape) occurrences $\mathfrak{o}$ and $\mathfrak{o}'$ are *independent* (formally, $\mathfrak{o} \parallel_* \mathfrak{o}'$), if they have the same context and if their positions are strongly independent. Formally, if the following both conditions are satisfied:

$$\mathbf{con}(\mathfrak{o}) \doteq \mathbf{con}(\mathfrak{o}') \quad \text{and} \quad \mathbf{pos}(\mathfrak{o}) \parallel_* \mathbf{pos}(\mathfrak{o}')$$

2. *independent sets:* A set $\mathfrak{O} \subseteq \mathcal{O}$ of multi-shape occurrences is *independent*, if pairwise different occurrences contained in $\mathfrak{O}$ are independent. Formally, if the following condition is satisfied for all $\mathfrak{o}, \mathfrak{o}' \in \mathfrak{O}$:

$$\mathfrak{o} \neq \mathfrak{o}' \quad \Rightarrow \quad \mathfrak{o} \parallel_* \mathfrak{o}'$$

Remarks (Independence of Occurrences):

1. *context:* If two occurrences with strongly independent positions have a common context, then this context is the uniquely determined covering of the positions. Nevertheless, it is possible that the contexts are different, even if the respective positions are strongly independent. Investigate the following example:

$$\mathfrak{t}_0 \doteq *+1 \quad ; \quad \mathfrak{t}_1 \doteq 0+* $$

The nominal terms $\mathfrak{t}_0$ and $\mathfrak{t}_1$ are strongly independent, their unique covering is the standard term $t \doteq 0+1$. There are infinitely many occurrences with these positions, but a context different from t. For example:

$$\mathfrak{o}_0 = \langle 1+1, 1, \mathfrak{t}_0 \rangle \quad ; \quad \mathfrak{o}_1 = \langle 0+2, 2, \mathfrak{t}_1 \rangle$$

The occurrences are not independent, as their contexts are different.

Typical Examples (Independent Occurrences): We mention some typical examples of independent occurrences.

1. *atomic shape:* Single occurrences in the same context having an atomic standard term as shape are independent or equal.

2. *equal shape:* Single occurrences in the same context having an equal shape are independent or equal.

Independence of Occurrences: We discuss the relevant aspects of the independence of occurrences.

1. *overlapping shapes:* The intended shapes of independent occurrences do not overlap, as nominal symbols in the respective positions have to be covered locally by standard terms and cannot be covered by a nominal symbol marking the same position.

2. *locally lying within:* The intended shapes of independent occurrences do not lie locally within each other, as nominal symbols marking their positions are locally covered by a standard term.

   In particular, the positions of independent occurrences are not related by the less-structured relation, which means that the occurrences do not lie within each other. (Observe that this condition is, in general strictly weaker than its local version mentioned before.)

3. *symmetry:* Independence of occurrences is symmetric, as the strong independence of nominal terms is so.

4. *reflexivity:* An occurrence $\mathfrak{o}$ is independent of itself, if and only if its position is trivial (which means that it is equal to the context). In particular, the empty occurrences $\langle t, \epsilon, t\rangle$ are independent of themselves. The restriction of independence to occurrences having proper nominal terms as position is anti-reflexive. More formally, for all occurrences $\mathfrak{o}$:

$$\mathbf{pos}(\mathfrak{o}) \in \mathtt{T}_0 \quad \Leftrightarrow \quad \mathfrak{o} \parallel_* \mathfrak{o}$$

5. *transitivity:* Independence of occurrences is not transitive. This follows from symmetry and anti-reflexivity of occurrences with proper nominal terms as position.

6. *equivalence of occurrences:* As strong independence is compatible with the equivalence of nominal terms (and determining their unique covering), independence of occurrences is also compatible with the equivalence of occurrences. The latter means that the following statement holds for all occurrences $\mathfrak{o}$, $\mathfrak{o}'$ and $\mathfrak{o}''$:

$$\mathfrak{o} \parallel_* \mathfrak{o}' \text{ and } \mathfrak{o}' \equiv \mathfrak{o}'' \quad \Rightarrow \quad \mathfrak{o} \parallel_* \mathfrak{o}''$$

We summarise our observations: the formal notion of independence of occurrences captures faithfully our intuitions about the informal concept of independent occurrences.

Besides of having a good formal representation of the informal concept of independence, the independence of occurrences is of specific interest in the theory of occurrences: the methods introduced so far, allow to relate independent occurrences with a single occurrence representing these independent occurrences. Subsequently, we discuss this relationship in some details.

13.2 Merging Independent Occurrences

First, we investigate, how independent occurrences can be merged into one occurrence representing the independent occurrences.

13.2 DEF (Merged Occurrences): Let $\mathfrak{O} \neq \emptyset$ be a non-empty set of independent occurrences. Furthermore, let t be the uniquely determined context of the elements $\mathfrak{o}$ of $\mathfrak{O}$. We define as follows:

1. *position:* The position of the merged occurrence is given as follows:

$$\mathfrak{t} \;\doteq\; \mathbf{simp}(\mu(\mathbf{pos}(\mathfrak{o});\ \mathfrak{o} \in \mathfrak{O}))$$

2. *sequence of shapes:* The sequence of shapes is given as follows:

$$\mathfrak{s} \;=\; \mathbf{comp}(\mathfrak{t}, t)$$

As $\mathfrak{t}$ is n-ary and simple and as $\mathfrak{t}$ is strongly covered by t, we have that $\mathfrak{s}$ is, indeed, a sequence of standard terms; in particular, the sequence $\mathfrak{s}$ is actually eliminated in $\mathfrak{t}$ with respect to t.

3. *merged occurrence:* The merged occurrence $\mu(\mathfrak{O})$ is given as follows:

$$\mu(\mathfrak{O}) \;=\; \langle t, \mathfrak{s}, \mathfrak{t} \rangle$$

By construction, $\mu(\mathfrak{O})$ is an occurrence representing all occurrences contained in $\mathfrak{O}$.

Remarks (Merged Occurrences):

1. *simplification:* The simplification function is used in the construction of the position to avoid clashes of nominal symbols.

   Due to such clashes, the nominal term $\mathfrak{s} \doteq \mu(\mathbf{pos}(\mathfrak{o});\ \mathfrak{o} \in \mathfrak{O})$ is, in general, strongly covered by the common context t, but not an elimination form of t. Investigate, for example, the following occurrences in the standard term $t \doteq 0 + 1$:

   $$\mathfrak{o} \ = \ \langle t, 0, * + 1\rangle \ ; \ \mathfrak{o}' \ = \ \langle t, 1, 0 + *\rangle \quad \rightsquigarrow \quad \mathfrak{s} \ \doteq \ * + * \ \not\leq \ t$$

2. *commutativity:* Due to the use of the simplification function, merging occurrences is commutative. Using other methods of avoiding clashes of nominal symbols can result in a function, which is only commutative modulo the equivalence of occurrences.

Alternative Construction (Merged Occurrences): If we presuppose that the positions of the involved independent occurrences are all n-ary nominal (not necessarily each for the same $n \in \omega$), then we may construct alternatively the merged occurrence (of two occurrences $\mathfrak{o}$ and $\mathfrak{o}'$) as follows:

1. *alternative position:* The alternative position of the merged occurrence is defined as follows:

   $$\mathfrak{t}_a \ \doteq \ \mu(\mathbf{pos}(\mathfrak{o}), \mathbf{pos}(\mathfrak{o}')^{+n})$$

2. *alternative sequence of shapes:* The alternative sequence of shapes can be given directly as follows:

   $$\mathfrak{s} \ = \ \mathbf{shape}(\mathfrak{o}) \circ \mathbf{shape}(\mathfrak{o}')$$

3. *alternative occurrence:* The alternative is $\mathfrak{o}_a = \langle t, \mathfrak{s}_a, \mathfrak{t}_a\rangle$.

   Observe that the resulting merged occurrences $\mu(\mathfrak{o}, \mathfrak{o}')$ and $\mu(\mathfrak{o}, \mathfrak{o}')$ are not equal, but equivalent.

As independent sets are finite, the alternative construction can be generalised canonically to arbitrary sets of independent occurrences by merging the occurrences successively. Again, the concrete result depends on the order of merging, but the different results are all equivalent.

Application (Merging Occurrences): There are applications of the method of merging occurrences, but the interesting applications are found in theories of other kinds of occurrences. We mention some:[92]

1. *congruence:* Two single occurrences in a derivation (of Natural Deduction) are *congruent*, if their shape is necessarily equal due to inference rules. Congruence classes can be simultaneously replaced; this replacement can be described formally with the help of the merged occurrence with respect to the congruence class.

2. *assumptions:* Assumptions in a derivation (of Natural Deduction) can be understood as single occurrences of subderivations with atomic shape (which is a formula). Inference rules, as the introduction of the implication, allow to discharge some assumptions. In order to describe adequately the discharge of assumptions, it is convenient to deal with the occurrence representing all actually discharged assumptions. This occurrence can be obtained by merging the actually discharged assumptions.

13.3 Splitting up Occurrences

We discuss, how to split up an occurrence into independent occurrences such that each free place of the position of the original occurrence determines one independent occurrence representing this place:

13.3 DEF (Splitting up Occurrences): Let $\mathfrak{o} = \langle t, \mathfrak{s}, \mathfrak{t} \rangle$ be an arbitrary multi-shape occurrence.

1. *generating sequence:* Let $k \in \mathbf{place}(\mathfrak{t})$ arbitrary. We define a sequence $\mathfrak{s}_k$ of nominal terms as follows:

$$\mathfrak{s}'_l \;\widehat{=}\; \begin{cases} * & \text{if } k = l \\ \mathfrak{s}_k & \text{if } l \in \mathbf{place}(\mathfrak{t}) \backslash \{k\} \\ *_l & \text{otherwise} \end{cases} \quad ; \quad \mathfrak{s}_k \;=\; \langle \mathfrak{s}'_l;\; l \in \omega \rangle$$

2. *position:* For every $k \in \mathbf{place}(\mathfrak{t})$: $\mathfrak{r}_k \;\widehat{=}\; \mathfrak{t}[\mathfrak{s}_k]$.

3. *separated occurrences:* For every $k \in \mathbf{place}(\mathfrak{t})$: $\mathfrak{o}_k = \langle t, \mathfrak{s}_k, \mathfrak{r}_k \rangle$.

[92] More details about these applications are mentioned in the section about future work.

Remarks (Splitting up Occurrences):

1. *independence of positions:* By construction, $\mathtt{r}_k$ is a unary elimination form of t in which the k^{th} entry $\mathtt{s}_k$ of the sequence $\mathfrak{s}$ is actually eliminated. The nominal symbols $*_l$ in $\mathtt{r}_k$ are locally covered by $\mathtt{s}_l$ for $k \neq l \in \mathbf{place}(\mathtt{t})$ and by $*$ for $k = l$. As a consequence, the set

$$\mathtt{S} \;=\; \{\mathtt{t}_k;\; k \in \mathbf{place}(\mathtt{t})\}$$

   is strongly independent.

2. *representation:* The occurrence $\mathfrak{o}_k$ represents the k^{th} place of the occurrence $\mathfrak{o}$ and, therefore, the set

$$\mathtt{O} \;=\; \{\mathfrak{o}_k;\; k \in \mathbf{place}(\mathtt{t})\}$$

   represents the occurrence $\mathfrak{o}$ separated with respect to the places of $\mathfrak{o}$. In particular, $\mathtt{O}$ is an independent set of occurrences such that $\mu(\mathtt{O}) \equiv \mathfrak{o}$.

   Observe, that equality does not hold in general, as $\mu(\mathtt{O})$ is, by construction, in simple normal form.

Application (Splitting up Occurrences):

1. *generalisation of occurrences:* The methods of splitting up an occurrence into independent occurrences and merging independent occurrences provide a formal representation of the informal concept that *a set of occurrences represents an occurrence* or, conversely, that *an occurrence subsumes some occurrences*. This way, we can describe formally the (informal) relationship between single occurrences and their generalisations, the standard occurrences and the multi-shape occurrences.

14 Formal Notion of Substitutions

The formal notion of substitutions (according to the introduction) and basic methods related to this notion are introduced; the equivalence of substitutions is introduced and the existence of normal substitutions (with respect to the equivalence of substitutions) is proved.

14.1 Introduction of Substitutions

We provide the formal definition of the notion of substitutions capable to represent the simultaneous replacement of finitely many term occurrences in a term according to a given sequence of terms.

14.1 DEF (Substitution): Let $n \in \omega$.

1. *substitution:* A quintuple $\mathbf{s} = \langle t, \mathfrak{s}, \mathfrak{t}, \mathfrak{s}', t' \rangle$ is called a *(formal) substitution* of n-many terms, if the following both conditions are satisfied:

   (a) $\mathfrak{o} = \langle t, \mathfrak{s}, \mathfrak{t} \rangle$ is an n-place multi-shape occurrence.

   (b) $\mathfrak{o}' = \langle t', \mathfrak{s}', \mathfrak{t} \rangle$ is an n-place multi-shape occurrence.

   In this case, a number $k \in n$ is called a *place* in the substitution $\mathbf{s}$.

2. *projections:* The standard term $t \doteq \mathbf{con}(\mathbf{s})$ is called the *context*, the finite sequence $\mathfrak{s} = \mathbf{aff}(\mathbf{s})$ of standard terms is called the *sequence of affected terms*, the nominal term $\mathfrak{t} \doteq \mathbf{pos}(\mathbf{s})$ is called the *position*, the finite sequence $\mathfrak{s}' = \mathbf{ins}(\mathbf{s})$ of standard terms is called the *sequence of inserted terms* and the standard term $t' \doteq \mathbf{res}(\mathbf{s})$ is called the *result* of the substitution $\mathbf{s}$.

   The occurrence $\mathfrak{o} = \langle t, \mathfrak{s}, \mathfrak{t} \rangle$ is also denoted as the occurrence *affected* by the substitution $\mathbf{s}$.

3. *standard properties:* We attribute (meaningful) properties of the position $\mathfrak{t}$ of a substitution $\mathbf{s}$ also to the substitution $\mathbf{s}$ itself. In particular, we distinguish n-ary, simple, multiple and single substitutions.

4. *sets of substitutions:* The set $\mathcal{S}$ is the set of all substitutions.

   We use the label $n \in \omega$ to restrict sets S of substitutions to substitutions of n-many terms. This means for a set $S \subseteq \mathcal{S}$ of substitutions:

$$S_n \;=\; \{\mathbf{s} \in S;\ \mathbf{lng}(\mathbf{aff}(\mathbf{s})) = n\}$$

Observe that this notation is similar to the restrictions of sets of nominal terms; but instead of referring to the arity of the relevant nominal term, we refer here to the length of the relevant sequences.

Remarks (Substitution):

1. *restrictions:* According to the definition of substitutions of n-many terms (based on the definition of multi-shape occurrences), we have the following a posteriori restrictions: both sequences $\mathfrak{s}$ and $\mathfrak{s}'$ of standard terms are of equal length, namely of length n; the position $\mathtt{t}$ of a substitution satisfies $\mathbf{rank}(\mathtt{t}) \leq n$.

2. *redundancies:* The redundancies mentioned with respect to the multi-shape occurrences can be observed analogously with respect to substitutions $\mathbf{s}$. We provide the details:

   - The sequence $\mathfrak{s}$ of affected terms and the position $\mathtt{t}$ determine the context t; the sequence $\mathfrak{s}'$ of inserted terms and the position $\mathtt{t}$ determine the result t'.
   - The context t and the position $\mathtt{t}$ determine partially the sequence $\mathfrak{s}$ of affected terms, the result t' and the position $\mathtt{t}$ determine partially the sequence $\mathfrak{s}'$ of inserted terms.
   - Neither context t and sequence $\mathfrak{s}$ of affected terms nor result t' and sequence $\mathfrak{s}'$ of inserted terms determine (neither separately nor together) the position $\mathtt{t}$.

3. *simplified substitutions:* The definition of substitutions is based on the notion of multi-shape occurrences; using the standard notion of occurrences instead, results in the *simplified* substitutions:

   - A quintuple $\mathbf{s} = \langle t, s, \mathtt{t}, s', t' \rangle$ is called a *simplified* substitution, if both $\langle t, s, \mathtt{t} \rangle$ and $\langle t', s', \mathtt{t} \rangle$ are (standard) occurrences.

   In contrast to the definition of standard occurrences, we drop the demand that the position is a unary nominal terms. As a consequence, $\mathbf{s} = \langle t, s, t, s', t \rangle$ is a limit case of simplified substitutions.

   Identifying single terms with sequences of length one, such a simplified substitutions can be understood as a (standard) substitutions of one term.

4. *limit cases:* The limit case of a substitution $\mathbf{s}$ of zero terms is subsumed in the definition of substitutions. In this case, $\mathbf{s} = \langle t, \epsilon, t, \epsilon, t \rangle$. Such a substitution is also called an *empty substitution*.

   Another limit case is a substitution $\mathbf{s}$ (of one term) in which the full context is replaced. In this case, $\mathbf{s} = \langle t, t, *, s, s \rangle$. Such a substitution is also called a *total substitution*.

5. *substitution relation:* The idea to define a *substitution relation* relating all such pairs of standard terms t and s such that there is a substitution transforming the first term into the second term does not seem to be useful: due to the existence of total substitutions, such a relation would be the full cartesian product $\mathtt{T}_0 \times \mathtt{T}_0$.

   This means: it is not interesting, whether we can transform a term into another term via a substitution, but it is interesting, how such a transformation is given.

14.2 Equivalence of Substitutions

We already observed that all pairs of standard terms can be related by a substitution transforming the first term into the second. The aim of this section is to identify all substitutions relating two given standard terms and to identify (non-trivial) representatives of the equivalence classes with respect to this equivalence of substitutions.

14.2.1 Introduction of the Equivalence of Substitutions

We provide the formal definition of the equivalence of substitutions.

14.2 DEF (Equivalence of Substitutions): Two substitutions $\mathbf{s} \in \mathcal{S}$ and $\mathbf{s}' \in \mathcal{S}$ are *equivalent* (formally, $\mathbf{s} \equiv \mathbf{s}'$), if they have the same context and the same result. More formally, if:

$$\mathbf{con}(\mathbf{s}) \;\doteq\; \mathbf{con}(\mathbf{s}') \quad \text{and} \quad \mathbf{res}(\mathbf{s}) \;\doteq\; \mathbf{res}(\mathbf{s}')$$

Remarks (Equivalence of Substitutions):

1. *equivalence relation:* It is easily checked that the equivalence of substitutions is, indeed, an equivalence relation on the set of all substitutions.

2. *multi-shape occurrences:* Analogously to the equivalence of multi-shape occurrences, the equivalence of substitutions demands that the context

and result (the two standard terms) are equal. Nevertheless, there is an obvious difference between both concepts of equivalence: there is no restriction on the position of equivalent substitutions.

As a consequence, we may, in principle, apply the transformations discussed with respect to multi-shape occurrences also on substitutions (in parallel to both multi-shape occurrences constituting a substitution), as the transformations do not change context and result. But we have to consider more conversions, namely conversion transforming the position into a position not equivalent to the former position.

14.2.2 Towards Normal Substitutions

We discuss briefly our approach to identify interesting normal substitutions.

Total Substitutions: It is immediate that the total substitution

$$\langle \mathbf{con}(\mathbf{s}), \mathbf{con}(\mathbf{s}), *, \mathbf{res}(\mathbf{s}), \mathbf{res}(\mathbf{s}) \rangle$$

is a trivial representatives of the equivalence class $[\mathbf{s}]$ of a substitution $\mathbf{s}$. Subsequently, we identify two more types of distinguished elements of the equivalence class $[\mathbf{s}]$, namely the simple and the parsimonious normal form of $\mathbf{s}$.

Principle Strategy: As in the case of multi-shape occurrences, we identify "interesting" properties of substitutions and we provide conversions transforming arbitrary substitutions into equivalent substitutions having these interesting properties.

Simple Properties and Associated Conversions: The demands on equivalent multi-shape occurrences are stronger than on substitutions. They have to have the same context (which corresponds to the demand of having the same context and the same result) and equivalent positions (what we do not demand in the case of substitutions).

As a consequence, we can carry over, in principle, the properties and the conversions discussed with respect to multi-shape occurrences to substitutions (applying them in parallel to both multi-shape occurrences constituting together a substitution).

We call these properties simple, as the associated transformations of arbitrary substitutions into substitutions having these properties are, essentially, only a relabelling of the nominal symbols of the position together with the corresponding parallel rearrangement of the entries of both sequences and

the elimination of superfluous entries (or the duplication of entries). In particular, the transformations do not alter the position up to the equivalence of nominal terms.

New Conversions: Due to the more complex situation (of two multi-shape occurrences with common position), we can identify a new "interesting" properties not discussed with respect to multi-shape occurrences:

1. *trivial substitutions:* A substitution, in which some corresponding entries in both sequences of standard terms are equal, is called (partially) *trivial*, as in such a substitution a subterm of the context is replaced by itself. The conversion eliminating such trivial places is the transformation of these places into vacuous places. Observe that this transformation changes the positions in a way that the equivalence of nominal terms is lost.

2. *reducible substitutions:* A substitution, in which a pair of affected and inserted term is similar, is called *reducible*. Instead of replacing the affected term by a similar term, we can split up the substitution and replace the direct subterms of the affected term by the direct subterms of the inserted term. We provide an example:

   - Let $t \doteq f(t_0, \dots t_n)$ be a complex term, which is transformed by a total substitution $\mathbf{s}$ into a similar term $s \doteq f(s_0, \dots s_n)$. This means:
   $$\mathbf{s} \;=\; \langle t, t, *, s, s\rangle$$
   - Instead of this total substitution, we can transform t into s by a substitution $\mathbf{s}'$ affecting each direct subterm separately:
   $$\mathbf{s}' \;=\; \langle t, \langle t_0, \dots t_n\rangle, f(*_0, \dots *_n), \langle s_0, \dots s_n\rangle, s\rangle$$

   Obviously, the positions are not equivalent (with respect to the equivalence of nominal terms), but the substitutions are (according to the definition above).

   We have two strategies to reach a distinguished representative with respect to the phenomenon observed above: the first strategy would be to demand that the number of affected terms is as small as possible. This results in the total substitution as the trivial representative of each equivalence class. The other strategy is to demand that the complexity of the affected terms is as small as possible; in other words, to reduce pairs of affected and inserted terms until they become *irreducible*.

Two Different Normal Forms: As already observed with respect to the multi-shape occurrences, we find two different normal forms, namely the simple and the parsimonious normal forms.[93]

Subsequently, we discuss the mentioned properties and their associated conversions.

14.2.3 Regular Substitutions

As a first kind of substitutions, we introduce the *regular substitutions* having a normal position (with respect to the isomorphism of nominal terms) and no vacuous places.

14.3 DEF (Regular Substitution): Let $\mathbf{s} = \langle t, \mathfrak{s}, \mathtt{t}, \mathfrak{s}', t' \rangle$ be a substitution of n-many terms (for $n \in \omega$).

1. *vacuous place:* A place $k \in n$ of the substitution $\mathbf{s}$ is called *vacuous*, if the respective nominal symbol $*_k$ does not occur in the position $\mathtt{t}$ (formally, if $k \notin \mathbf{place}(\mathtt{t})$); otherwise, the place k is called *non-vacuous*.

2. *vacuous substitution:* The substitution $\mathbf{s}$ is called *partially vacuous*, if there is a vacuous place $k \in n$; $\mathbf{s}$ is called *completely vacuous*, if all places $k \in n$ are vacuous.

3. *regular substitution:* The substitution $\mathbf{s}$ is called *regular*, if there is no vacuous position $k \in n$ (formally, if $\mathbf{place}(\mathtt{t}) = n$) and if the position $\mathtt{t}$ of $\mathbf{s}$ is normal with respect to the isomorphism of nominal terms.

Remarks (Regular Substitutions):

1. *principle observations:* The principle observations stated with respect to multi-shape occurrences also hold with respect to substitutions.

In the next proposition, we state that every substitution can be transformed into an equivalent substitution, which is regular and normal.

[93] Observe that the total substitution is both simple and parsimonious; as a consequence, the total substitution is a simple as well as a parsimonious normal form.

14.4 Proposition (Regular Substitution): Every substitution $\mathbf{s}$ can be transformed into equivalent substitution $\mathbf{s}'$ such that $\mathbf{s}'$ is regular.
Proof. The transformation is done analogously to the case of multi-shape occurrences; but instead of transforming only one sequence of shapes, we have to transform both sequences in parallel in the same way as the sequence of shapes. Q.E.D.

Convention (Attribution of Properties): As in the case of multi-shape occurrences, we presuppose that all properties attributed subsequently to a place in a substitution are only attributed to non-vacuous places. Properties attributed subsequently to substitutions only depend on the non-vacuous places.

14.2.4 Trivial Substitutions

A place in a substitution is *trivial*, if the respective entries of both sequences agree. Similarly to vacuous places, trivial places do not influence a substitution, as the respective term affected by the substitution is replaced, but replaced only by itself. We provide the formal definition.

14.5 DEF (Trivial Substitutions): Let $\mathbf{s} = \langle t, \mathfrak{s}, \mathtt{t}, \mathfrak{s}', t' \rangle$ be a substitution of n-many terms (for $n \in \omega$).

1. *trivial place:* A place $k \in \mathbf{place}(\mathtt{t})$ is called *trivial*, if the respective entries of the sequence of affected terms and of the sequence of inserted terms are equal (formally, if $s_k \doteq s'_k$).

2. *trivial substitution:* The substitution $\mathbf{s}$ is called *partially trivial*, if there is a trivial place $k \in \mathbf{place}(\mathtt{t})$; $\mathbf{s}$ is called *completely trivial*, if all places $k \in \mathbf{place}(\mathtt{t})$ are trivial.

   If $\mathbf{s}$ has no trivial positions $k \in \mathbf{place}(\mathtt{t})$, then $\mathbf{s}$ is called *non-trivial*.

In the next proposition, we show that we can eliminate trivial positions.

14.6 Proposition (Trivial Substitutions): Every substitution $\mathbf{s}$ can be transformed into a equivalent substitution $\mathbf{s}'$ such that $\mathbf{s}'$ is non-trivial.
Proof. We eliminate the trivial positions by replacing the nominal symbols marking trivial positions by the respective terms contained in both sequences.

For this purpose, investigate the following function $F_0 : \mathbf{V}_* \to \mathtt{T}$ on the set of all nominal symbols:

$$F_0(*_k) \;\hat{=}\; \begin{cases} s_k & \text{if } s_k \;\hat{=}\; s'_k \\ *_k & \text{otherwise} \end{cases}$$

Let F be the homomorphism induced by F_0. It is easily checked that $F(\mathfrak{t})$ is still an elimination form of both standard terms t and t' in which the sequences $\mathfrak{s}$ and $\mathfrak{s}'$, respectively, are eliminated.

In the substitution $\mathbf{s}'' = \langle t, \mathfrak{s}, F(\mathfrak{t}), \mathfrak{s}', t' \rangle$, the trivial places of $\mathbf{s}$ are now vacuous. Obviously, $\mathbf{s} \equiv \mathbf{s}''$. Q.E.D.

14.2.5 Simple Substitutions

In the next proposition, we show that every substitution can be transformed into an equivalent and simple substitution.

14.7 Proposition (Simple Substitution): Let $n \in \omega$. Every substitution $\mathbf{s} = \langle t, \mathfrak{s}, \mathfrak{t}, \mathfrak{s}', t' \rangle$ of n-many terms can be transformed into an equivalent substitution $\mathbf{s}'$ such that $\mathbf{s}'$ is simple and regular.
Proof. Analogously to the proof of the existence of simple normal forms of multi-shape occurrences.

1. *simple position:* Let $\mathfrak{t}' \;\hat{=}\; \mathbf{simp}(\mathfrak{t})$. By construction, the nominal term $\mathfrak{t}'$ is simple and normal with respect to the isomorphism of nominal terms.

2. *sequences:* Due to the proposition about elimination forms (in the section about simplification) we obtain both that $\mathfrak{t}' \leq t$ and $\mathfrak{t}' \leq t'$. As a consequence, there are uniquely determined sequences $\mathfrak{r}$ and $\mathfrak{r}'$ actually eliminated in $\mathfrak{t}'$ with respect to t and t', respectively. (In particular, $\mathbf{lng}(\mathfrak{r}) = \mathbf{place}(\mathfrak{t}') = \mathbf{lng}(\mathfrak{r}')$.)

3. *substitution:* As a consequence, $\mathbf{s}' = \langle t, \mathfrak{r}, \mathfrak{t}', \mathfrak{r}', t' \rangle$ is a regular and simple substitution equivalent to $\mathbf{s}$. Q.E.D.

Remark (Simple Substitutions):

1. *rearrangement of sequences:* Recall that there is a simple homomorphism F such that $\mathfrak{t} \;\hat{=}\; F(\mathfrak{t}')$, as $\mathfrak{t}'$ is the simplification of $\mathfrak{t}$. Furthermore, the following both equations hold:

$$t \;\hat{=}\; \mathfrak{t}'[\mathfrak{r}] \;\hat{=}\; F(\mathfrak{t}')[\mathfrak{s}] \quad ; \quad t' \;\hat{=}\; \mathfrak{t}'[\mathfrak{r}'] \;\hat{=}\; F(\mathfrak{t}')[\mathfrak{s}']$$

Therefore, $\mathfrak{r}$ is an F-expansion of $\mathfrak{s}$ and $\mathfrak{r}'$ an F-expansion of $\mathfrak{s}'$.

The latter means that the entries of $\mathfrak{r}$ are a rearrangement of the relevant entries of $\mathfrak{s}$, where possibly some entries of $\mathfrak{s}$ become duplicated; analogously, $\mathfrak{r}'$ of $\mathfrak{s}$.

2. *simple normal form:* In the corresponding proof with respect to multi-shape occurrences, we additionally argued that the positions $\mathtt{t}$ and $\mathtt{t}'$ are equivalent (with respect to the equivalence of nominal terms) and that the resulting normal occurrence is unique.

   Both statements hold, in principle, also here. But: we do not need the equivalence of positions for the equivalence of substitutions and uniqueness only holds up to the equivalence of the positions. As we have substitutions in an equivalence class with non-equivalent positions, simplicity (together with regularity) is not sufficient to determine a normal form.

14.2.6 Redundant and Parsimonious Substitutions

We introduce redundant substitutions. In contrast to the corresponding property of multi-shape occurrences, it is not sufficient that the same term occurs twice in the sequence of affected terms or in the sequence of inserted terms. In order to attribute redundancy, we have to demand that pairs of affected and inserted terms together occur twice. We provide the formal definition.

14.8 DEF (Redundant and Parsimonious Substitution): Let $n \in \omega$ and $\mathbf{s} = \langle t, \mathfrak{s}, \mathtt{t}, \mathfrak{s}', t' \rangle$ be a substitution of n-many terms.

1. *uniform places:* Two places $k, l \in \mathbf{place}(\mathtt{t})$ are called *uniform*, if the same term is replaced by the same term in both places (formally, if both $s_k \doteq s_l$ and $s'_k \doteq s'_l$).

2. *redundant // parsimonious places:* A place $k \in \mathbf{place}(\mathtt{t})$ is called *redundant*, if there is a place $l \in \mathbf{place}(\mathtt{t})$ different from k ($k \neq l$) such that the places k and l are uniform; otherwise, k is called *parsimonious*.

3. *uniform substitution:* The substitution $\mathbf{s}$ is called *uniform*, if all places $k, l \in \mathbf{place}(\mathtt{t})$ are pairwise uniform.

4. *redundant substitutions:* The substitution $\mathbf{s}$ is called *redundant*, if there is a redundant place $k \in \mathbf{place}(\mathtt{t})$; otherwise, $\mathbf{s}$ is called *parsimonious*.

Remarks (Redundant Substitution):

1. *uniformity:* The uniformity of places is obviously an equivalence relation on the set $\mathbf{place}(\mathtt{t})$ of free places of the position of a substitution. Equivalence classes with respect to this relation are denoted as follows:

$$[k]_u \;=\; \{l \in \mathbf{place}(\mathtt{t});\; k \text{ and } l \text{ uniform}\}$$

   We already introduced the analogous relation with respect to the position of multi-shape occurrences. Observe that both relations are, in general, different; more precisely, the uniformity relation with respect to substitutions is finer than the uniformity relation with respect to occurrences, as two equations have to be satisfied instead of one.

2. *substitutions of one term:* A non-vacuous substitution of one term (which is not necessarily a simple substitution) is uniform and parsimonious. If the position $\mathtt{t}$ has more than one free places, then uniformity implies redundancy.

In the next proposition, we show that we can eliminate redundant places in a substitution.

14.9 Proposition (Parsimonious Substitutions): Let $n \in \omega$. Every substitution $\mathbf{s} = \langle t, \mathfrak{s}, \mathtt{t}, \mathfrak{s}', t' \rangle$ of n-many terms can be transformed into an equivalent substitution $\mathbf{s}'$ such that $\mathbf{s}'$ is parsimonious.
Proof. Analogously to the corresponding proof with respect to multi-shape occurrences.

1. *simple homomorphism:* Let $F \in \mathbf{Hom}_s(\mathtt{T})$ be the simple homomorphism induced by the following function on the set of all nominal terms:

$$F :\; *_k \;\mapsto \begin{cases} *_{\min([k]_u)} & \text{if } k \in \mathbf{place}(\mathtt{t}) \\ *_k & \text{otherwise} \end{cases}$$

   As in the case of the multi-shape occurrences, we obtain that both $\mathfrak{s}$ and $\mathfrak{s}'$ are F-expansions of themselves. Due to the proposition about expansions and contractions, we obtain:

$$t \;\doteq\; \mathtt{t}[\mathfrak{s}] \;\doteq\; F(\mathtt{t})[\mathfrak{s}] \quad ; \quad t' \;\doteq\; \mathtt{t}[\mathfrak{s}'] \;\doteq\; F(\mathtt{t})[\mathfrak{s}']$$

2. *parsimonious substitution:* We define as follows: $\mathbf{s}' = \langle t, \mathfrak{s}, F(\mathtt{t}), \mathfrak{s}', t' \rangle$. As $F(\mathtt{t})$ is an elimination form of t and of t', in which the sequence $\mathfrak{s}$ and $\mathfrak{s}'$ are eliminated, respectively, $\mathbf{s}'$ is a substitution of n-many terms.

Furthermore, $\mathbf{s}'$ is by construction parsimonious, as all uniform places of $\mathbf{s}$ are mapped to the same place, namely to the minimum of their equivalence class. As we did not change context and result, $\mathbf{s} \equiv \mathbf{s}'$.

Q.E.D.

14.2.7 Reducible Substitutions

A substitution is *reducible*, if it is possible to simplify pairs of standard terms contained in both sequences of affected terms and inserted terms due to similarity. We provide the formal definition.

14.10 DEF (Reducible Substitution): Let $n \in \omega$ and $\mathbf{s} = \langle t, \mathfrak{s}, \mathtt{t}, \mathfrak{s}', t' \rangle$ be a substitution of n-many terms.

1. *reducible place:* A place $k \in \mathbf{place}(\mathtt{t})$ is called *reducible*, if the corresponding entries in the sequences of affected terms and inserted terms are similar (formally, if $s_k \sim s'_k$); otherwise, k is called *irreducible*.

2. *reducible substitution:* The substitution $\mathbf{s}$ is *reducible*, if there are reducible places $k \in \mathbf{place}(\mathtt{t})$; otherwise, $\mathbf{s}$ is called *irreducible*.

Recall that the strategy of eliminating reducible places in a substitution is to split up reducible places into many places, in which the direct subterms of the respective affected term are replaced by the direct subterms of the corresponding inserted term.

Such a reduction step reduces the complexity of the involved terms, but raises the number of terms replaced by the substitution. In order to prove that a transformation based on such a reduction step terminates, we have to introduce a suitable rank function for substitutions.

14.11 DEF (Rank Function): We define as follows:

1. *rank of pairs:* The rank $\mathbf{rank}(t, s)$ of an ordered pair of standard terms $t, s \in \mathtt{T}_0$ is defined as follows:

$$\mathbf{rank}(t, s) \;=\; \begin{cases} 1 + \sum_{k \in n'} \mathbf{rank}(t_k, s_k) & \text{if } t \sim s \\ 0 \quad \text{otherwise} \end{cases}$$

Recalling that similar terms are complex and have the same arity, we presuppose (as usually) in the case $t \sim s$ that n' is the common arity

of t and s and that the t_k and the s_k are the direct subterms of t and s, respectively.

2. *rank of a substitution:* Let $n \in \omega$ and $\mathbf{s} = \langle t, \mathfrak{s}, \mathtt{t}, \mathfrak{s}', t' \rangle$ a substitution of n-many terms. The rank $\mathbf{rank}(\mathbf{s})$ is defined as follows:

$$\mathbf{rank}(\mathbf{s}) \;=\; \sum_{k \in \mathbf{place}(\mathtt{t})} \mathbf{rank}(s_k, s'_k)$$

Remarks (Rank Function):

1. *vacuous places:* The rank of a substitution does not depend on vacuous places; only the rank of pairs of entries of the sequences of affected terms and of inserted terms at non-vacuous places are summed up to the rank of a substitution.

2. *reducibility:* Let $k \in \mathbf{place}(\mathtt{t})$ be a non-vacuous place in a substitution $\mathbf{s}$. The rank $\mathbf{rank}(s_k, s'_k)$ of the respective entry s_k of the sequences of affected terms and of the entry s'_k of the sequences of inserted terms equals zero, if and only if the entries are not similar. The latter means that the place k is irreducible.

   As a consequence: the rank $\mathbf{rank}(\mathbf{s})$ of a substitution $\mathbf{s}$ equals zero, if and only if all non-vacuous places $k \in \mathbf{place}(\mathtt{t})$ are irreducible. The latter means that $\mathbf{s}$ is irreducible.

In the next proposition, we show that we may transform every substitution into an irreducible substitution.

14.12 Proposition (Irreducible Substitutions): Let $n \in \omega$. Every substitution $\mathbf{s} = \langle t, \mathfrak{s}, \mathtt{t}, \mathfrak{s}', t' \rangle$ of n-many terms can be transformed into an equivalent substitution $\mathbf{s}'$ such that $\mathbf{s}'$ is irreducible.
Proof. Essentially, by induction over the rank of a substitution $\mathbf{s}$.

1. *reduction step:* Let $\mathbf{s} = \langle t, \mathfrak{s}, \mathtt{t}, \mathfrak{s}', t' \rangle$ be a reducible substitution of n-many terms. This means:

   - There is $k \in \mathbf{place}(\mathtt{t}) \subseteq n$ such that $s_k \sim s'_k$.
   - There are $m \in \omega$, an m'-ary function symbol f and standard terms $r_l, r'_l \in \mathtt{T}_0$ (for $l \in m'$) such that:

$$s_k \;\doteq f(r_0, \ldots r_m) \quad ; \quad s'_k \;\doteq f(r'_0, \ldots r'_m)$$

Let $\mathfrak{t}''$ be the result of replacing the nominal symbol $*_k$ by the nominal term $\mathfrak{s} \doteq f(*_{n+0}, \ldots *_{n+m})$. More formally:

$$\mathfrak{t}'' \doteq \mathfrak{t}[*_0, \ldots *_{k-1}, \mathfrak{s}]$$

Observe that $k \notin \mathbf{place}(\mathfrak{t}'') \subseteq n+m$. Furthermore: replacing the new nominal symbols $*_{n+l}$ in $\mathfrak{t}''$ by standard terms a_l has the same result as replacing $*_k$ in $\mathfrak{t}$ by $f(a_0, \ldots a_m)$ (†).

We extend the sequences of the affected terms and of the inserted tersm by the direct subterms of t_k and of t'_k, respectively. This means:

$$\mathfrak{r} = \langle \mathfrak{s}, r_0, \ldots r_m \rangle \quad ; \quad \mathfrak{r}' = \langle \mathfrak{s}', r'_0, \ldots r'_m \rangle$$

By the observation (†), the following both equations hold:

$$\mathfrak{t}''[\mathfrak{r}] \doteq t \quad ; \quad \mathfrak{t}''[\mathfrak{r}'] \doteq t'$$

This means that $\mathbf{s}'' = \langle t, \mathfrak{r}, \mathfrak{t}'', \mathfrak{r}', t' \rangle$ is a substitution of $n+m$-many terms and equivalent to $\mathbf{s}$. By the definition of the rank function, it is clear that the rank of $\mathbf{s}''$ is the rank of $\mathbf{s}$ lowered by one. More precisely:

- As the place k becomes vacuous in the reduction step, the rank of the substitution is lowered by $\mathbf{rank}(s_k, s'_k) = 1 + \sum_{l \in m'} \mathbf{rank}(r_l, r'_l)$.
- By the new non-vacuous places $n+0, \ldots n+m$, the rank is increased by $\sum_{l \in m'} \mathbf{rank}(r_l, r'_l)$

2. *reduction:* Let $\mathbf{s}$ be an arbitrary substitution. If $\mathbf{s}$ is reducible, then we can apply the reduction step as discussed above. After finitely many applications of this reduction step, we obtain a substitution $\mathbf{s}'$, equivalent to $\mathbf{s}$ and of rank 0. The latter means that $\mathbf{s}'$ is irreducible.

Q.E.D.

Remarks (Irreducible Substitution):

1. *properties of conversed substitutions:* The transformation of a substitution into an irreducible substitution does not result in a substitution having good properties: in the reduction step, a reducible place is transformed into a vacuous place; furthermore, if the respective pairs of direct subterms of the affected term and the inserted term are equal, then the reduction step results in a trivial place; also redundancies may easily occur.

2. *uniqueness:* The irreducible substitution generated in the proof above is not uniquely determined, but depends on the order of the places to which the reduction step is applied.

3. *alternative account:* The reduction step provided in the proof can not only result in trivial places, but can, in general, also be applied to trivial places. From the perspective of efficiency, it seems better not to treat such trivial places, but to eliminate them (later).

In the next proposition, we show that the transformations discussed before result in irreducible substitutions, if applied to such substitutions.

14.13 Proposition (Compatibility of Transformations): Let $\mathbf{s}$ be an irreducible substitution and $\mathbf{s}'$ the result of a transformation of $\mathbf{s}$ associated with a property discussed before. The substitution $\mathbf{s}'$ is irreducible.
Proof. We check each transformation; observe that each pair of $\langle s_k, s'_k \rangle$ of affected term and inserted term have rank zero for all $k \in \mathbf{place}(\mathtt{t})$, where $\mathtt{t}$ is the position of an irreducible substitution $\mathbf{s}$.

1. *regularity:* The transformation into a regular substitution rearranges the arguments of the sequences of intended terms and arguments; this does not change the rank of the substitution, as the nominal symbols are relabelled according to this rearrangement. Additionally, vacuous places are eliminated, which also does not change the rank of a substitution, as only non-vacuous places determine the rank.

2. *triviality:* The elimination of trivial positions lowers, in principle, the rank of the substitution under discussion. If we presuppose that the rank is already zero, then only places with rank zero are eliminated. The latter means that the transformation into a non-trivial substitution does not alter the rank.

3. *elimination of redundancies:* Redundant places are calculated multiple times, when calculating the rank of a substitution. Eliminating such redundancies lowers, in principle, the rank; but as in the case of the elimination of trivial places, only places with rank zero are combined into a single place. Therefore, again, the rank is not affected by this transformation.

4. *simplification:* Transforming a substitution into a simple substitution can increase the rank of that substitution, as places are duplicated. But again, these places have rank zero, and therefore, the rank of the substitution is not altered. Q.E.D.

14.2.8 Normal Substitutions

As in the case of normal occurrences, we can introduce two incompatible variants of normal substitutions. We provide the formal definition.

14.14 DEF (Normal Substitution): Let $n \in \omega$ and $\mathbf{s} \in \mathcal{S}$ be a substitution of n-many terms.

1. *simple normal form:* The substitution $\mathbf{s}$ is in *simple normal form*, if $\mathbf{s}$ is regular, non-trivial, irreducible and simple.

2. *parsimonious normal form:* The substitution $\mathbf{s}$ is in *parsimonious normal form*, if $\mathbf{s}$ is regular, non-trivial, irreducible and parsimonious.

Remarks (Normal Substitutions):

1. *existence:* Due to the propositions proved in this section, it is clear that every substitution can be transformed into an equivalent substitutions in simple normal forms as well as in parsimonious normal form.

In the next proposition, we show that the simple normal form of substitutions is uniquely determined.

14.15 Proposition (Simple Normal Substitution): The simple normal form of a substitutions is uniquely determined.
Proof. We prove by induction over the structure of the context t of substitutions that equivalent substitutions $\mathbf{s}$ and $\mathbf{s}'$ in simple normal form are equal.

1. *t atomic:* As $\mathbf{s} \equiv \mathbf{s}'$, we immediately obtain:

   - $t \doteq \mathbf{con}(\mathbf{s}) \doteq \mathbf{con}(\mathbf{s}')$ and $t' \doteq \mathbf{res}(\mathbf{s}) \doteq \mathbf{res}(\mathbf{s}')$.

   Let $\mathtt{t}$ and $\mathtt{t}'$ be the positions of $\mathbf{s}$ and $\mathbf{s}'$, respectively. We show that $\mathtt{t} \doteq \mathtt{t}'$. As t is atomic, we first obtain that both $\mathtt{t}, \mathtt{t}' \in \{t\} \cup \mathbf{V}_*$ (both nominal terms are elimination forms of t). We distinguish two cases:

   (a) *$t \doteq t'$:* We can exclude $\mathtt{t}, \mathtt{t}' \in \mathbf{V}_*$, as the resulting substitution would have a trivial place. This means that both $\mathtt{t} \doteq t \doteq \mathtt{t}'$.

   (b) *$t \not\equiv t'$:* In this case, we can exclude that $\mathtt{t}, \mathtt{t}' \in \{t\}$ (both nominal terms are elimination forms of t'). Therefore, $\mathtt{t}, \mathtt{t}' \in \mathbf{V}_*$. Due to regularity, we obtain that $\mathtt{t} \doteq * \doteq \mathtt{t}'$.

Again due to regularity, we obtain that the sequences of affected terms and of inserted terms in both substitutions are (pairwise) equal. (If $t \doteqdot t'$, then all four sequences are the empty sequence, otherwise, the sequences of affected terms both are $\langle t \rangle$ and the sequences of inserted terms $\langle t' \rangle$.) As a consequence, $\mathbf{s} = \mathbf{s}'$.

2. $t \doteqdot f(t_0, \ldots t_n)$ *complex:* Again as $\mathbf{s} \equiv \mathbf{s}'$, we immediately obtain:

   - $t \doteqdot \mathbf{con}(\mathbf{s}) \doteqdot \mathbf{con}(\mathbf{s}')$ and $t' \doteqdot \mathbf{res}(\mathbf{s}) \doteqdot \mathbf{res}(\mathbf{s}')$.

   Let $\mathtt{t}$ and $\mathtt{t}'$ be the positions of $\mathbf{s}$ and $\mathbf{s}'$, respectively. We show that $\mathtt{t} \doteqdot \mathtt{t}'$. As t is complex, we obtain that $\mathtt{t} \sim t$ or $\mathtt{t} \in \mathbf{V}_*$ (analogously with respect to $\mathtt{t}'$). We distinguish two cases:

   (a) $t \not\sim t'$*:* We have $\mathtt{t} \not\sim t \not\sim \mathtt{t}'$, as both nominal terms are also elimination forms of t'. Therefore, both nominal terms are contained in $\mathbf{V}_*$ and due to regularity equal to $*$.

   (b) $t \sim t'$*:* We can exclude that $\mathtt{t}, \mathtt{t}' \in \mathbf{V}_*$. Otherwise, the places determined by the labels of $\mathtt{t}$ and $\mathtt{t}'$, respectively, would be reducible, as the respective entries in the sequences of intended terms and of the arguments would be similar.

   As $\mathtt{t}, \mathtt{t}' \notin \mathbf{V}_*$, we have $\mathtt{t} \sim t \sim \mathtt{t}'$. This means that there are suitable nominal terms $\mathtt{s}_k, \mathtt{s}'_k \in \mathtt{T}$ (for $k \in n'$) such that:

   $$\mathtt{t} \doteqdot f(\mathtt{s}_0, \ldots \mathtt{s}_n) \quad ; \quad \mathtt{t}' \doteqdot f(\mathtt{s}'_0, \ldots \mathtt{s}'_n)$$

   As both $\mathtt{t}$ and $\mathtt{t}'$ are simple and normal (with respect to the isomorphism of nominal terms), there are simple and normal nominal terms $\mathtt{t}_k, \mathtt{t}'_k \in \mathtt{T}$ (for $k \in n'$) such that for all $k \in n'$:

   $$\mathtt{s}_k \doteqdot \mathtt{t}_k^+ \quad ; \quad \mathtt{s}'_k \doteqdot {\mathtt{t}'_k}^+$$

   Recall that $\cdot^+$ indicates a suitable right-shift of the labels of nominal symbols.

   The nominal terms $\mathtt{t}_k$ and $\mathtt{t}'_k$ are both elimination forms of both standard terms t_k and t'_k. Using suitable segments of the sequences of affected terms and of the sequences of inserted terms of $\mathbf{s}$ and $\mathbf{s}'$, respectively, we easily obtain regular substitutions $\mathbf{s}_k$ and $\mathbf{s}'_k$ both with context t_k and result t'_k. We can exclude trivial places in all of these substitutions, as neither $\mathbf{s}$ nor $\mathbf{s}'$ have such. This means that all constructed substitutions are in simple normal form.

   Applying n'-many times the induction hypothesis, we obtain that $\mathbf{s}_k = \mathbf{s}'_k$ for all $k \in n'$ and, in particular, $\mathtt{t}_k \doteqdot \mathtt{t}'_k$ for all $k \in n'$.

The suitable right-shift of the labels in $\mathtt{s}_k$ and $\mathtt{s}'_k$ only depends on the weight of the nominal terms $\mathtt{t}_l$, where $l < k$. Therefore, we already have $\mathtt{s}_k \mathrel{\widehat{=}} \mathtt{s}'_k$. The latter means that $\mathtt{t} \mathrel{\widehat{=}} \mathtt{t}'$.

Due to regularity, the sequences of affected terms and of inserted terms are uniquely determined by the position $\mathtt{t}$ and by context and result, respectively. Therefore, $\mathbf{s} = \mathbf{s}'$. Q.E.D.

In the next proposition, we show that the parsimonious normal form of substitutions is uniquely determined.

14.16 Proposition (Parsimonious Normal Substitution): The parsimonious normal form of a substitution is uniquely determined.
Proof. Let $\mathbf{s}$ and $\mathbf{s}'$ be two equivalent substitutions, both in parsimonious normal form of n-many and n'-many terms, respectively. We can transform both substitutions into equivalent substitutions $\mathbf{s}''$ and $\mathbf{s}'''$, respectively, both simple and regular. As $\mathbf{s}$ and $\mathbf{s}'$ are irreducible and without trivial places, $\mathbf{s}''$ and $\mathbf{s}'''$ are irreducible and without trivial places. The latter means that $\mathbf{s}''$ and $\mathbf{s}'''$ are both in simple normal form and therefore equal.

- Let $\mathtt{t}$, $\mathtt{t}'$ and $\mathtt{t}''$ be the positions of the respective substitutions. We obtain:
$$\mathtt{t}'' \mathrel{\widehat{=}} \mathbf{simp}(\mathtt{t}) \quad ; \quad \mathtt{t}'' \mathrel{\widehat{=}} \mathbf{simp}(\mathtt{t}')$$
- Let $\mathfrak{s}$ the sequence of intended terms and $\mathfrak{s}'$ the sequence of inserted terms, both of $\mathbf{s}''$. This means:
$$\mathbf{s}'' = \langle t, \mathfrak{s}, \mathtt{t}'', \mathfrak{s}', t' \rangle$$

Investigate the equivalence class $[0]_u$ with respect to the simple position $\mathtt{t}''$. This equivalence class contains all labels of nominal symbols in $\mathtt{t}''$ in which the term s_0 is eliminated and replaced by the term s'_0.

Both nominal terms $\mathtt{t}$ and $\mathtt{t}'$ have to be the result of a homomorphism F applied on $\mathtt{t}''$ satisfying $F(*_k) = *_0$ for all $k \in [0]_u$. (The leftmost nominal symbol of $\mathtt{t}$ as well as of $\mathtt{t}'$ have to be $*_0$.) Furthermore, in $\mathtt{t}$ and in $\mathtt{t}'$ the same standard term must be eliminated, namely the standard term s_0 and be replaced by the same standard term, namely by s'_0. As a consequence, these standard terms have to be the first entries of the sequences of intended terms and of the arguments, in both substitutions $\mathbf{s}$ and $\mathbf{s}'$. Iterating this argumentation, we obtain that $\mathtt{t} \mathrel{\widehat{=}} \mathtt{t}'$, the equality of the respective sequences and, finally, that $\mathbf{s} = \mathbf{s}'$.

Q.E.D.

14.3 Construction of Simple Normal Substitutions

We have proved that a substitution can be transformed stepwise into a normal substitution. In this section, we provide a direct construction method for a simple normal substitutions out of two standard terms.

In a first step, we determine the *difference* between to standard terms. This difference is that nominal term, which is covered by both standard terms and in which the nominal symbols represent the positions of as simple as possible standard terms.

14.17 DEF (Difference Function): We define recursively (in the first argument) the *difference function* $\delta : \mathtt{T}_0 \times \mathtt{T}_0 \to \mathtt{T}$ as follows:

1. t *atomic:* $\delta(t, t') \doteq \begin{cases} t & \text{if } t \doteq t' \\ * & \text{otherwise} \end{cases}$

2. $t \doteq f(t_0, \ldots t_n)$ *complex:*

$$\delta(t, t') \doteq \begin{cases} f(\delta(t_0, t'_0), \ldots \delta(t_n, t'_n)) & \text{if } t \sim s \doteq f(t'_0, \ldots t'_n) \\ * & \text{otherwise} \end{cases}$$

Remarks (Difference Function):

1. *image:* By construction, $\delta(t, t') \in \mathtt{T}^*$ for all standard terms t and t'. More precisely, $\delta(t, t')$ is a standard term, if and only if $t \doteq t'$; in this case, $\delta(t, t') \doteq t$. Otherwise, $\delta(t, t')$ is unary.

2. *elimination form:* In general, $\delta(t, t')$ is neither an elimination form of t nor of t', as the nominal symbols can mark the positions of different subterms of t and of t', respectively.

   But it is easily seen, that $\delta(t, t')$ is covered by both standard terms t and t'. As a consequence, the simplification $\mathbf{simp}(\delta(t, t'))$ of $\delta(t, t')$ is a common elimination form of both t and t'.

3. *substitution:* As $\mathtt{t} \doteq \mathbf{simp}(\delta(t, t'))$ is n-ary (for an $n \in \omega$), there are two uniquely determined sequences $\mathfrak{s}$ and $\mathfrak{s}'$ of standard terms, both of length n, such that:

$$t \doteq \mathtt{t}[\mathfrak{s}] \quad ; \quad t' \doteq \mathtt{t}[\mathfrak{s}']$$

   As a consequence, $\mathtt{s} = \langle t, \mathfrak{s}, \mathtt{t}, \mathfrak{s}', t' \rangle$ is a substitution. Observe that we can even calculate both sequences with the help of the completion function:

$$\mathfrak{s} = \mathbf{comp}(\mathtt{t}, t) \quad ; \quad \mathfrak{s}' = \mathbf{comp}(\mathtt{t}, t')$$

By construction, the substitution $\mathbf{s}$ is regular (recall that $\mathtt{t}$ is normal with respect to the isomorphism of nominal terms) and simple.

Furthermore, it is easily seen that $\mathbf{s}$ contains neither reducible places nor trivial places. Therefore, $\mathbf{s}$ is in simple normal form.

14.4 Application: Calculations

Calculations, as found in everyday mathematics, can be represented formally as sequences of substitutions. We sketch some details of this correspondence.

Principle Terminology (Calculations):

1. *calculation step:* A *(justified) calculation step* is formally represented by a substitution; corresponding pairs of affected and inserted terms determine the equations used as justifications for the respective calculation step.

   We provide an example in the language $\mathfrak{L}_{\mathrm{PA}}$ of arithmetics to illustrate this correspondence:

   $$(5+3)\cdot(4+2) \;=\; 8+6$$

   The calculation step is, for example, represented by the following substitution:

   $$\langle (5+3)\cdot(4+2), \langle 5+3, 4+2\rangle, *_0+*_1, \langle 8,6\rangle, 8+6\rangle$$

   The following equations, determined by corresponding pairs of affected and inserted terms, are used as a justification in the calculation step:

   $$5+3 \;=\; 8 \quad ; \quad 4+2 \;=\; 6$$

2. *justifications:* We can distinguish calculation steps with respect to their justifications: a calculation step is *valid*, if its justifications are all valid, otherwise the calculation step is *false*.

   There are different (possible) sources for the validity of a justification; we mention some:

   (a) *list:* There is a previously given list of valid equations.
   For example, the set of all true equations over a given domain, in which only one function symbol occurs. In the case of addition over the natural numbers, this is the symmetric closure of the following set:

   $$\{0+0=0,\ 0+1=1,\ 1+0=0,\ 0+2=2,\ 1+1=2,\ \ldots\}$$

(b) *definitions:* All instances of definitional axioms of the underlying theory.

For example, the following both axioms defining the addition in arithmetics:

$$x + 0 = x \quad ; \quad (x + y)' \;=\; x + y'$$

(c) *lemmata:* All instances of lemmata previously proved in the underlying theory.

In arithmetics, for example, the propositions that addition and multiplication are commutative:

$$x + y \;=\; y + x \quad ; \quad x \cdot y \;=\; y \cdot x$$

(d) *side calculations:* Equations, which are previously calculated on the base of valid justifications.

3. *equivalence of substitutions:* The representing substitutions are determined by a calculation step only up to the equivalence of substitutions. In order to obtain uniquely determined substitutions, we suggest the following conventions:

   (a) *regularity:* We assume that a representing substitution is regular; this means that the position is normal (with respect to the isomorphism of nominal terms) and that the substitution does not have vacuous places.

   (b) *simple:* We assume that the representing substitutions is simple.

   This convention is convenient form a technical point of view, as each single calculation subsumed in one calculation step is represented by a separate nominal symbol in the representing substitution.

   (c) *justifications:* In the intended case, a calculation step is given together with its justifications. As a consequence, we cannot demand that the representing substitution is irreducible.

   Otherwise, if a calculation step is given without justifications, then we assume that the substitution is, indeed, irreducible. In this case, the representing substitution is in simple normal form.

4. *limit case:* It seems reasonable to demand that at least one term is affected in a calculation step.

This restriction to proper calculation steps corresponds with the demand that the representing substitution is not empty, which means that it has a proper nominal term as position.

Furthermore, it seems reasonable to demand that in a calculation step the affected terms are different from the result of the calculations.

This restriction to non-trivial calculation steps corresponds with the demand that the representing substitution has no trivial places.

5. *calculation:* A calculation is a sequence of subsequent calculation steps.

   A calculation is represented by a sequence of substitutions satisfying the condition that the result of every substitution in that sequence agrees with the context of the next substitution (if there is a next).

Properties of Calculations: Properties of informal calculation steps and calculations are reflected by the properties of the representing substitutions.

1. *total calculation step:* A calculation step is *total*, if the complete term is affected by the calculation step. In this case, the justification is the same equation as established by the calculation step.

   Total calculation steps are represented by total substitutions.

2. *single calculation step:* A calculation step is called *single*, if only a single occurrence is affected by the calculation step.

   Investigate the following example:

$$(5+3)\cdot(5+3) \;=\; 8\cdot(5+3) \quad ; \quad (5+3)\cdot(5+3) \;=\; 8\cdot 8$$

   The first calculation step is single, the second not.

   Single calculation steps correspond with substitutions of one term with a unary and simple position.

3. *uniform calculation step:* A calculation step is called *uniform*, if only one kind of subterm is replaced by the same term; otherwise, it is called *multiform*. Investigate the following examples:

$$(5+3)\cdot(5+3) \;=\; 8\cdot 8 \quad ; \quad (5+3)\cdot(3+5) \;=\; 8\cdot 8$$

   The first calculation step is uniform, the second multiform.

   Uniform calculation steps correspond with uniform substitutions, in which all corresponding pairs of affected and inserted terms are equal.

4. *elementary:* A calculation step is called *elementary*, if its justifications are of a very simple kind. It is plausible that this notion depends on the underlying concept of the validity of justifications.

   Investigate the following example:

   $$3+(3+3) \;=\; 9 \quad ; \quad 3+(3+3) \;=\; 3+6 \;=\; 9$$

   Presupposing the list of valid equations as discussed above, the first calculation is elementary. In this scenario, the second calculation is not even valid, as the justification is not contained in the list of valid equations. Permitting additionally side calculations, the second calculation becomes valid, but not elementary.

   The analogous distinction can be made in the arithmetical example based on definitions and rules:

   $$(x+x)+0 \;=\; x+x \quad ; \quad x+0 \;=\; 0+x$$

   The first calculation is elementary, as it is an instance of the first defining axiom $x+0=x$ for the addition; the second is not, as it is only an instance of the commutativity rule.

   The notion of reducibility seems related with the concept of elementary calculation steps. Investigate the following example, understood as a total calculation step:

   $$5+(2+4) \;=\; 5+6$$

   The justification of this calculation step (which is the equation itself) can be reduced to the equation $2+4=6$. Observe that such a reduction is only possible, if the reduced equations are still valid. (There are artificial scenarios, in which this is not the case.)

   It seems plausible to demand irreducibility of the justifications as a necessary condition for elementary calculation steps.

5. *two-in-one step calculation:* A calculation step is called *two-in-one*, if two (or more) subsequent calculation steps are subsumed (sloppily) into one step. Investigate the following example:

   $$3+(3+3) \;=\; 9 \quad ; \quad 3+(3+3) \;=\; 3+6 \;=\; 9$$

   Presupposing that only elementary equations are permitted as justifications, we have the reasonable intuition that the first calculation is a

two-in-one step calculation, but that the second calculation has none such steps. This intuition can be captured by the following definition:

A calculation step is called *two-in-one*, if the calculation step itself is not correct, but if it can be transformed into a sequence of correct calculation steps. Observe that non-elementary (but valid) calculation steps are not two-in-one.

15 Independent Substitutions

We discuss some operations on substitutions: in particular, we discuss how to split up a substitution into a sequence of independent substitutions and vice versa. In the light of these operations, we have to consider a new concept of the identity of substitutions, namely an identity of substitutions in different contexts and with different results.[94]

15.1 Introduction of Independent Substitutions

We provide the formal definition of independent substitutions.

15.1 DEF (Independent Substitutions):

1. *independent substitution:* Two substitutions $\mathbf{s}, \mathbf{s}' \in \mathcal{S}$ are called *independent* (formally, $\mathbf{s} \parallel_* \mathbf{s}'$), if their positions are strongly independent (formally, if $\mathbf{pos}(\mathbf{s}) \parallel_* \mathbf{pos}(\mathbf{s}')$).

2. *independent set:* As set $\mathtt{S} \subseteq \mathcal{S}$ is called *independent*, if pairwise different substitutions contained in $\mathtt{S}$ are independent. Formally, if the following condition is satisfied for all $\mathbf{s}, \mathbf{s}' \in \mathtt{S}$:

$$\mathbf{s} \neq \mathbf{s}' \quad \Rightarrow \quad \mathbf{s} \parallel_* \mathbf{s}'$$

Remarks (Independent Substitutions):

1. *context and result:* In contrast to independent occurrences, we do not demand that the contexts or the results of independent substitutions are equal.

   Observe that equality of the result would mean that they are equal to the uniquely determined covering of the positions of the independent substitutions; analogously, equal results are equal to that covering.

2. *notation:* It is convenient to introduce the following notation representing that a standard term t' is the result of a substitution $\mathbf{s}$ with context t.

$$t \blacktriangleright_{\mathbf{s}} t'$$

[94] Recall that the concept of the identity of substitutions discussed so far is based on the equivalence of multi-shape occurrences; as a consequence, the contexts of the involved substitutions are equal as well as their results.

Occasionally and if the substitutions $\mathbf{s}_k$ have labels k, we simplify the notation and write $t \blacktriangleright_k t'$ instead of $t \blacktriangleright_{\mathbf{s}_k} t'$.[95]

15.2 Merging Substitutions

We carry over the definition of the merge function for nominal terms to independent substitutions. After a brief discussion of the general case, we identify restrictions on the merge function yielding interesting interpretations.

15.2 DEF (Merged Substitution): Let $\mathbf{s}_0 = \langle t_0, \mathfrak{s}, \mathtt{t}_0, \mathfrak{r}, t'_0 \rangle$ be a substitution of n-many terms with n-ary position $\mathtt{t}_0$ and $\mathbf{s}_1 = \langle t_1, \mathfrak{s}', \mathtt{t}_1, \mathfrak{r}', t'_1 \rangle$ a substitution of m-many terms with m-ary position $\mathtt{t}_1$ (for $n, m \in \omega$) such that both substitutions are independent (formally, such that $\mathbf{s}_0 \parallel_* \mathbf{s}_1$). The *merged substitution* $\mu(\mathbf{s}_0, \mathbf{s}_1)$ is defined as follows:

1. *position:* The position of the merged substitution is the result of an application of the merge function on the positions of both arguments, the second argument suitably shifted for avoiding clashes of nominal symbols. More formally:

$$\mathtt{t} \mathrel{\widehat{=}} \mu(\mathtt{t}_0, {\mathtt{t}_1}^{+n})$$

2. *sequences of intended terms and arguments:* The sequences of intended terms and arguments of the merged substitution are the concatenation of the respective sequences.

3. *context and result:* The context and the result of the merged substitution are the standard terms determined by the position and the respective sequences. More formally:

$$t \mathrel{\widehat{=}} \mathtt{t}[\mathfrak{s} \circ \mathfrak{s}'] \quad ; \quad t' \mathrel{\widehat{=}} \mathtt{t}[\mathfrak{r} \circ \mathfrak{r}']$$

4. *merged substitution:* Finally, the merged substitution $\mu(\mathbf{s}_0, \mathbf{s}_1)$ is given as follows:

$$\mu(\mathbf{s}_0, \mathbf{s}_1) \;=\; \langle t, \mathfrak{s} \circ \mathfrak{s}', \mathtt{t}, \mathfrak{r} \circ \mathfrak{r}', t' \rangle$$

[95] The symbol $\blacktriangleright_{\mathbf{s}}$ can be understood as a relation symbol between standard terms; nevertheless, the underlying relation is trivial, as the related standard terms t and t' are the uniquely determined context and result of the substitution $\mathbf{s}$.

Remarks (Merged Substitution):

1. *substitution:* The merged substitution $\mu(\mathbf{s}_0, \mathbf{s}_1)$ is, indeed, a substitution. It is sufficient to mention:

   $$\mathbf{place}(\mathfrak{t}) \;=\; n+m \;=\; \mathbf{lng}(\mathfrak{s} \circ \mathfrak{s}') \;=\; \mathbf{lng}(\mathfrak{r} \circ \mathfrak{r}')$$

   As a consequence, both t and t' are standard terms; by construction, $\mathfrak{s} \circ \mathfrak{s}'$ is eliminated in $\mathfrak{t}$ with respect to t and $\mathfrak{r} \circ \mathfrak{r}'$ with respect to t'.

2. *commutativity:* Merging substitutions is only commutative modulo the equivalence of substitutions, as we shift the nominal symbols of the position of the second argument to avoid clashes of nominal symbols. More formally, for all independent substitutions $\mathbf{s}_0$ and $\mathbf{s}_1$:

   $$\mu(\mathbf{s}_0, \mathbf{s}_1) \;\equiv\; \mu(\mathbf{s}_1, \mathbf{s}_0)$$

3. *alternative definition:* We presupposed that the positions of the independent substitutions are n-ary and m-ary, respectively. In principle, we could drop this restriction and proceed here as in the case of occurrences: instead of demanding regularity, we could directly merge the positions and then apply the simplification function. The resulting position would be equivalent to $\mathfrak{t}$.

   The problem of such an alternative approach is the loss of control over the origin of the nominal symbols: investigate a nominal symbol $*_k$ of $\mathbf{simp}(\mu(\mathfrak{t}_0, \mathfrak{t}_1))$. There is no simple method of determining, whether $*_k$ was originally a nominal symbol in $\mathfrak{t}_0$ or in $\mathfrak{t}_1$ and, in particular, which one. But this information is needed for the definition of the affected and the inserted sequences of the merged substitution.

   In the case of occurrences, it was possible to define the suitable sequence of shapes via the completion function applied on the merged position and the previously known context. This is not possible in the case of merged substitutions, as neither the context nor the result of the merged substitution are previously given.

Analysis (Merging Substitutions): We analyse the relationship between the independent substitutions $\mathbf{s}_0$ and $\mathbf{s}_1$ being merged and the merged substitution $\mathbf{s}$.

Let $r \mathrel{\widehat{=}} \overline{\mu}(\mathfrak{t}_0, \mathfrak{t}_1)$ be the unique covering of the positions of the independent substitutions. There are (uniquely determined) sequences $\mathfrak{a}$ and $\mathfrak{b}$ of standard terms such that the following statements hold:

1. *merged substitution:* $r \doteq \mathfrak{t}[\mathfrak{a} \circ \mathfrak{b}]$
2. *first substitution:* $\mathfrak{t}_0 \doteq \mathfrak{t}[\langle *_0, \ldots *_{n-1} \rangle \circ \mathfrak{b}]$
3. *second substitution:* $\mathfrak{t}_1 \doteq \mathfrak{t}[\mathfrak{a} \circ \langle *_0, \ldots *_{m-1} \rangle]$

As a consequence, the substitutions under discussion can be given as follows:

1. *the first substitution:* $\mathfrak{t}[\mathfrak{s}_0, \mathfrak{b}] \doteq \mathfrak{t}_0[\mathfrak{s}_0] \blacktriangleright_0 \mathfrak{t}_0[\mathfrak{r}_0] \doteq \mathfrak{t}[\mathfrak{r}_0, \mathfrak{b}]$
2. *the second substitution:* $\mathfrak{t}[\mathfrak{a}, \mathfrak{s}_1] \doteq \mathfrak{t}_1[\mathfrak{s}_1] \blacktriangleright_1 \mathfrak{t}_1[\mathfrak{r}_1] \doteq \mathfrak{t}[\mathfrak{a}, \mathfrak{r}_1]$
3. *the merged substitution:* $\mathfrak{t}[\mathfrak{s}_\circ, \mathfrak{s}_1] \blacktriangleright_{\mathbf{s}} \mathfrak{t}[\mathfrak{r}_0, \mathfrak{r}_1]$

We easily identify four more substitutions transforming the context of the merged substitution into the contexts of the first substitution ($\mathbf{s}_{s,b}$) and of the second substitution ($\mathbf{s}_{a,s}$) and the results of the first and of the second substitution into the result of the merged substitution ($\mathbf{s}_{r,b}$ and $\mathbf{s}_{a,r}$, respectively). Therefore, we obtain the following situation:

$$\mathfrak{t}[\mathfrak{s}_\circ, \mathfrak{s}_1] \left\{ \begin{array}{cccc} \blacktriangleright_{s,b} & \mathfrak{t}[\mathfrak{s}_0, \mathfrak{b}] & \blacktriangleright_0 & \mathfrak{t}[\mathfrak{r}_0, \mathfrak{b}] \quad \blacktriangleright_{r,b} \\ \\ \blacktriangleright_{a,s} & \mathfrak{t}[\mathfrak{a}, \mathfrak{s}_1] & \blacktriangleright_1 & \mathfrak{t}[\mathfrak{a}, \mathfrak{r}_1] \quad \blacktriangleright_{a,r} \end{array} \right\} \mathfrak{t}[\mathfrak{r}_\circ, \mathfrak{r}_1]$$

Under this perspective, $\mu(\mathbf{s}_0, \mathbf{s}_1)$ is a minimal substitution subsuming both independent substitutions $\mathbf{s}_0$ and $\mathbf{s}_1$.

Special Cases (Merging Substitutions): More interesting as the complete picture given above are some special cases:

1. *common context ($\bullet \rightrightarrows$):* Both arguments of the merge function share the same context. In other words:

   $$\mathfrak{a} = \mathfrak{s}_0 \quad \text{and} \quad \mathfrak{b} = \mathfrak{s}_1$$

   In this case, the merged function is a substitution on the common context and results in a standard term, in which both substitutions are applied in parallel.

2. *common result ($\rightrightarrows \bullet$):* The special case that both substitutions share the same result is dual to the first case. We can characterise this case as follows:

   $$\mathfrak{a} = \mathfrak{r}_0 \quad \text{and} \quad \mathfrak{b} = \mathfrak{r}_1$$

In this case, the merged substitution is a substitution applied on a standard term and resulting in the common result, such that both substitutions are applied in parallel.

3. *intermediate term ($\rightarrow \bullet \rightarrow$):* In the third interesting case, the result of the first substitution is the context of the second. This case can be characterised as follows:

$$\mathfrak{a} = \mathfrak{r}_0 \quad \text{and} \quad \mathfrak{b} = \mathfrak{s}_1$$

In contrast to the previous special cases, here the merged substitution represents a sequence of substitutions, in which the first argument is applied first and then the second.

4. *missing case ($\rightarrow \circ \rightarrow$):* The last interesting case is dual to the third: the context of the first substitution is the result of the second substitution. This case can be characterised as follows:

$$\mathfrak{a} = \mathfrak{s}_0 \quad \text{and} \quad \mathfrak{b} = \mathfrak{r}_1$$

Again, the merged substitution represents a sequence of substitutions, but here a substitution, in which the second argument is applied first and then the first argument.

We summarise our observations: whenever we have wto independent substitutions satisfying the conditions given in one of the interesting cases, we can apply the merge function on both substitutions and obtain the characterised merged substitution.

15.3 Splitting up Substitutions

We intend to split up a given substitution into a sequence of independent substitutions. In order to discuss this operation on substitutions, we introduce the so called *intermediate substitutions*. These are substitutions possibly occurring in a sequence, in which a given substitution is split up. These intermediate substitutions are defined with the help of a signature determining them.

15.3 DEF (Signature): Let $\mathbf{s} \in \mathcal{S}$ be a substitution of n-many terms with an n-ary position (for $n \in \omega$). A sequence $\sigma = \langle \sigma_0, \ldots \sigma_{n-1} \rangle \in \{0, \pm 1\}^n$ of length n having entries in $\{0, \pm 1\}$ is called a *signature* with respect to the substitution $\mathbf{s}$.

We associate a uniquely determined substitution to each signature.

15.4 DEF (Associated Substitution): Let $\mathbf{s} = \langle t, \mathfrak{s}, \mathtt{t}, \mathfrak{s}', t' \rangle$ be a substitution of n-many terms with an n-ary position (for $n \in \omega$) and σ a signature with respect to the substitution $\mathbf{s}$.

1. *associated sequence:* Let $\mathtt{r}_\sigma = \langle \mathtt{r}_0, \ldots \mathtt{r}_{n-1} \rangle \in \mathtt{T}^n$ be that sequence of nominal terms with length n that satisfies the following condition for all $k \in n$:
$$\mathtt{r}_k \;\hat{=}\; \begin{cases} \mathtt{s}_k & \text{if } \sigma_k = -1 \\ *_k & \text{if } \sigma_k = 0 \\ \mathtt{s}'_k & \text{if } \sigma_k = +1 \end{cases}$$

2. *associated position:* The *associated position* $\mathtt{t}_\sigma$ is defined as follows:
$$\mathtt{t}_\sigma \;\hat{=}\; \mathtt{t}[\mathtt{r}_\sigma]$$

3. *associated substitution:* The *associated substitution* $\sigma(\mathbf{s})$ is defined as follows:
$$\sigma(\mathbf{s}) \;=\; \langle t_\sigma, \mathfrak{s}, \mathtt{t}_\sigma, \mathfrak{s}', t'_\sigma \rangle$$
Where:
$$t_\sigma \;\hat{=}\; \mathtt{t}_\sigma[\mathfrak{s}] \;\hat{=}\; \mathtt{t}[\mathtt{r}_\sigma][\mathfrak{s}] \quad ; \quad t'_\sigma \;\hat{=}\; \mathtt{t}_\sigma[\mathfrak{s}'] \;\hat{=}\; \mathtt{t}[\mathtt{r}_\sigma][\mathfrak{s}']$$

Furthermore, a substitution $\mathbf{s}'$ is called an *intermediate substitution* with respect to $\mathbf{s}$, if there is a signature σ such that $\mathbf{s}'$ is the respective associated substitution (formally, if $\mathbf{s}' = \sigma(\mathbf{s})$).

Remarks (Signature):

1. *associated substitution:* The associated substitution $\sigma(\mathbf{s})$ is, by construction, a substitution.

2. *entries of the signature:* The free places of the position $\mathtt{t}_\sigma$ of the associated substitution $\sigma(\mathbf{s})$ are determined by the entries of σ equal to 0. More formally:
$$\mathbf{place}(\mathtt{t}_\sigma) \;=\; \{k \in n;\ \sigma_k = 0\}$$
If an entry σ_k equals to -1, the nominal symbol $*_k$ of the position $\mathtt{t}$ of $\mathbf{s}$ are replaced in $\mathtt{t}_\sigma$ by the respective subterm of the context t of $\mathbf{s}$; otherwise, if σ_k equals to $+1$, by the respective subterm of the result t' of $\mathbf{s}$.

Therefore, an entry -1 represents that the respective place is *not yet* replaced, an entry $+1$ that the respective entry is *already* replaced and an entry 0 that the respective place is actually replaced (all replacements according to the substitution $\mathbf{s}$). In other words, $\sigma(\mathbf{s})$ is, indeed, an intermediate substitution with respect to $\mathbf{s}$.

3. *trivial signature:* We investigate some limit cases:

   (a) *no entry equals zero:* If no entry σ_k equals zero, then the position $\mathtt{t}_\sigma$ of the associated substitution is a standard term. As a consequence, context and result are equal, the associated substitution $\sigma(\mathbf{s})$ represents *no replacements* of a term.

   (b) *all entries equal zero:* If all entries σ_k equal zero, then the position $\mathtt{t}_\sigma$ of the associated substitution is equal to the position $\mathtt{t}$ of $\mathbf{s}$. Therefore, the associated substitution $\sigma(\mathbf{s})$ equals to $\mathbf{s}$. More precisely:

$$\sigma(\mathbf{s}) = \mathbf{s} \quad \Leftrightarrow \quad \sigma = \langle 0, \ldots 0 \rangle$$

4. *independence:* Two intermediate substitutions $\sigma(\mathbf{s})$ and $\sigma'(\mathbf{s})$, determined by the signatures σ and σ', are independent, if the following both conditions are satisfied for all $k \in n$:

   (a) *no common free places:* It is not the case that $\sigma_k = 0$ and $\sigma'_k = 0$.

   (b) *no clash of structure:* If both $\sigma_k \neq 0$ and $\sigma'_k \neq 0$, then $\sigma_k = \sigma'_k$.

   We call such signatures also *independent*.

In order to merge independent intermediate substitutions, it is convenient to provide a slight variation μ_i of the merge function for substitutions.

15.5 DEF (Merged Intermediate Substitutions): Let $\mathbf{s} = \langle t, \mathfrak{s}, \mathtt{t}, \mathfrak{s}', t' \rangle$ be a substitution of n-many terms with an n-ary position (for $n \in \omega$). Furthermore, let σ and σ' be two signatures with respect to $\mathbf{s}$ as well as $\mathbf{s}' = \sigma(\mathbf{s})$ and $\mathbf{s}'' = \sigma'(\mathbf{s})$ the respective substitutions.

1. *position:* The position $\mathtt{t}_i$ of the merged substitution is given as follows:

$$\mathtt{t}_i \doteq \mu(\mathtt{t}_\sigma, \mathtt{t}_{\sigma'})$$

2. *merged substitution:* The merged substitution $\mu_i(\mathbf{s}', \mathbf{s}'')$ is given as follows:

$$\mu_i(\mathbf{s}', \mathbf{s}'') \;=\; \langle \mathtt{t}_i[\mathfrak{s}], \mathfrak{s}, \mathtt{t}_i, \mathfrak{s}', \mathtt{t}_i[\mathfrak{s}'] \rangle$$

Merging Intermediate Substitutions:

1. *well-defined:* By construction, $\mu_i(\mathbf{s}', \mathbf{s}'')$ is a substitution for all pairs σ and σ' of signatures.

2. *merging independent substitutions:* Let σ and σ be two signatures such that the associated substitutions are independent.

3. *compatibility with signatures:* Let the signature $\sigma'' = \langle r_0, \dots r_{n-1} \rangle$ be defined as follows:

$$r_k \;=\; \begin{cases} 0 & \text{if } \sigma_k \cdot \sigma'_k = 0 \\ \sigma_k & \text{otherwise} \end{cases}$$

   If a clash of structure is avoided (guaranteed by the condition that $\sigma_k = \sigma'_k$ for all $k \in n$ satisfying $\sigma_k \neq 0$ and $\sigma'_k \neq 0$), then σ'' determines the merged substitution. More formally:

$$\mu_i(\sigma(\mathbf{s}), \sigma'(\mathbf{s})) \;=\; \sigma''(\mathbf{s})$$

   This holds, in particular, for independent signatures σ and σ'.

   Observe that σ'' represents an intermediate substitution, in which a place k is replaced, if this place is replaced in one of the arguments. The status of all other places (whether the place is already replaced or not yet) is equal to the status of both arguments, if the sequences satisfy the clash of structure condition.

4. *complementary:* Two intermediate substitutions $\sigma(\mathbf{s})$ and $\sigma'(\mathbf{s})$ are called *complementary*, if the following condition is satisfied for all $k \in n$:

$$\sigma_k = 0 \quad \Leftrightarrow \quad \sigma'_k \neq 0$$

   It is immediate that complementary intermediate substitutions are independent.

Splitting up a Substitution: Let $\mathbf{s}$ be a substitution of n-many terms with an n-ary position.

1. *sequence of signatures:* A sequence $\sigma_0, \dots \sigma_m$ of signatures is called a *split sequence*, if the following conditions are all satisfied:

   (a) *initial signature:* In the initial signature σ_0, all entries are either -1 or 0.

(b) *successor signature:* In a successor signature σ_{k+1}, all entries 1 of σ_k are preserved, all entries 0 of σ_k are 1 and entries -1 of σ_k are -1 or become 0.

(c) *final signature:* In the final signature σ_m, all entries are either $+1$ or 0.

Furthermore, we demand that at least one place is replaced in the substitution represented by a signatures:

(a) *non-triviality:* Every signature σ_k has an entry equal to 0.

Each split sequence determines a sequence of intermediate substitutions such that subsequent substitutions are independent and such that merging them successively results in the original substitution.

15.4 Identity of Intermediate Substitutions

We have the intuition that two intermediate substitutions are equal, if their signatures differ only with respect to non-zero entries: the difference between such substitutions is their position in suitable split sequences, but the *same* transformation of terms is represented. We provide an equivalence relation on intermediate substitutions capturing this intuition.

15.6 DEF (Similarity of Intermediate Substitutions): Let $\mathbf{s} \in \mathcal{S}$ be a substitution of n-many terms (and n-ary position). Two intermediate substitutions $\mathbf{s}'$ and $\mathbf{s}''$ are *similar*, if their signatures σ' and σ'' have at the same positions the entry 0. More formally, for all $k \in n$:

$$\sigma'_k \;=\; 0 \quad \Leftrightarrow \quad \sigma''_k \;=\; 0$$

Remarks (Similarity of Intermediate Substitutions):

1. *equivalence relation:* It is immediate that the similarity relation is an equivalence relation on the set of intermediate substitutions.

2. *equivalence of substitutions:* If two different intermediate substitutions are similar, then they neither have a common context nor a common result (even if some non-zero entries of their signatures are equal). As a consequence, similar, but different intermediate substitutions are not equivalent (with respect to the equivalence of substitutions).

3. *simplified variant:* If the underlying substitution $\mathbf{s}$ is simple, then each position in a signature corresponds with a single occurrence of a nominal symbol in the position of $\mathbf{s}$ (and not, as in the general case, with multiple occurrences of the same nominal symbol).

   This means: if we define signatures and intermediate substitutions with respect to the simplification $\mathbf{s}'$ of $\mathbf{s}$ (which is not the simple normal form of $\mathbf{s}$, as we do not reduce the entries in the affected and inserted sequences), then we obtain more intermediate substitutions. The resulting simplified variant of the similarity relation is finer than the similarity relation as defined above: we can identify the transformation represented by a single occurrence of a nominal symbol independently of other occurrences of the same nominal symbol.

Example (Similarity of Intermediate Substitutions): Similarity of intermediate substitutions becomes interesting, when we have two different sequences of independent substitutions satisfying the condition that merging these sequences results in the same merged substitution. Recall that substitutions represent calculation steps and investigate the following calculations:

1. *first calculation:* $(5+3)+(6+4) = 8+(6+4) = 8+10$
2. *second calculation:* $(5+3)+(6+4) = (5+3)+10 = 8+10$
3. *merged calculation:* $(5+3)+(6+4) \;=\; 8+10$

We meet the presuppositions: merging the first and the second calculations, respectively, results in the third calculation; the first two calculations are represented by sequences of independent substitutions. We provide the respective signatures:

1. *first calculation:* $\sigma_0 = \langle 0, -1\rangle$ and $\sigma_1 = \langle 1, 0\rangle$.
2. *second calculation:* $\sigma_2 = \langle -1, 0\rangle$ and $\sigma_3 = \langle 0, 1\rangle$.

This means that the substitutions determined by σ_0 and σ_3 are similar, as well as those determined by σ_1 and σ_2. This corresponds with our intuitions about the calculations: the first calculation step in (1) and the second in (2) are both "calculating $5+3$", the remaining two calculation steps are both "calculating $6+4$". In other words: the first and the same calculations are the same, besides the order of their calculation steps.

16 Application: Substitution Functions

Based on the formal notion of substitutions, we introduce a class of special functions, the so called *(explicit) substitution functions*. In a second step, we introduce *implicit* substitution functions, which are functions having an *explication method* turning them into explicit substitution functions. Having introduced both kinds of substitution functions, we illustrate that functions usually understood as substitution functions are, indeed, implicit substitution functions.

16.1 Explicit Substitution Functions

Even though the substitutions are not defined in these investigations as a specific kind of functions, there is a strong tie between both notions: a formal substitution can be understood as an element of a (set theoretical) function. This motivates to call functions consisting of substitutions *explicit substitution functions*. We provide the formal definition of these functions.

16.1 DEF (Explicit Substitution Function): A set $\mathbf{S} \subseteq \mathcal{S}$ of substitutions is called an *explicit substitution function*.

Remarks (Explicit Substitution Function):

1. *terminology:* The explicit substitution functions are called *explicit*, as the substitutions are explicitly present (as elements) in such a function.

We have to show that explicit substitution functions are, indeed, (set theoretical) functions.

16.2 Proposition (Explicit Substitution Functions): Every explicit substitution function $\mathbf{S} \subseteq \mathcal{S}$ is a function.
Proof.

1. *basic observation:* Presupposing the standard recursive definition of an n-tuple (for $2 < n \in \omega$), a substitution

$$\begin{aligned} \mathbf{s} &= \langle t, \mathfrak{s}, \mathfrak{t}, \mathfrak{s}', t' \rangle \\ &= \langle \langle \langle t, \mathfrak{s}, \mathfrak{t} \rangle, \mathfrak{s}' \rangle, t' \rangle \\ &= \langle \langle \mathfrak{o}, \mathfrak{s}' \rangle, t' \rangle \end{aligned}$$

is an ordered pair satisfying the condition that the first entry is an ordered pair of an m-place occurrence and a sequence of standard terms

(of length m) and the second entry is a standard term. This means that every set $\mathbf{S} \subseteq \mathcal{S}$ of substitutions is, indeed, a set of ordered pairs.

2. *unique image:* As the result t' of a substitution $\mathbf{s}$ is uniquely determined by its position $\mathfrak{t}$ (contained in the first argument) and the sequence $\mathfrak{s}'$ of arguments (the second argument), it is immediate that every pair of a multi-shape occurrences and sequences of arguments is mapped by an explicit substitution function $\mathbf{S}$ to a uniquely determined standard term. This means that an explicit substitution function is, indeed, a function. Q.E.D.

We discuss the domain of explicit substitution functions; in particular, we develop conditions for well-behaved explicit substitutions functions.

Domain of Explicit Substitution Functions: The domain $\mathbf{dom}(\mathbf{S})$ of an explicit substitution function $\mathbf{S}$ can easily be reconstructed:

$$\mathbf{dom}(\mathbf{S}) \;=\; \{\langle \mathfrak{o}, \mathfrak{s}' \rangle \in \mathcal{O} \times \mathtt{T}_0{}^{<\omega};\; \exists t \in \mathtt{T}_0 :\; \langle \mathfrak{o}, \mathfrak{s}', t' \rangle \in \mathbf{S}\}$$

On its domain, an explicit substitution function is (trivially) a total function.

Well-Behaved Substitution Functions: It seems natural to demand the following restriction for a substitution function $\mathbf{S}$:

- If two compatible ordered pairs $\langle \mathfrak{o}, \mathfrak{s} \rangle$ and $\langle \mathfrak{o}', \mathfrak{s}' \rangle$ are contained in the domain $\mathbf{dom}(\mathbf{S})$, then also the pairs $\langle \mathfrak{o}, \mathfrak{s}' \rangle$ and $\langle \mathfrak{o}', \mathfrak{s} \rangle$.

  Thereby, the ordered pairs $\langle \mathfrak{o}, \mathfrak{s} \rangle$ and $\langle \mathfrak{o}', \mathfrak{s}' \rangle$ are compatible, if the length of both sequences agree (formally, if $\mathbf{lng}(\mathfrak{s}) = \mathbf{lng}(\mathfrak{s}')$).

The restriction to compatible pairs is necessary: a pair $\langle \mathfrak{o}, \mathfrak{s} \rangle$ contained in the domain $\mathbf{dom}(\mathbf{S})$ of an explicit substitution function $\mathbf{S}$ is an initial segment of a substitution $\mathbf{s}$ contained in $\mathbf{S}$. This means: the length of the sequence $\mathfrak{s}$ (the sequence of inserted terms of the substitution $\mathbf{s}$) has to agree with the length of the shape $\mathbf{shape}(\mathfrak{o})$ of $\mathfrak{o}$ (the sequence of affected terms of the substitution $\mathbf{s}$). If we drop the restriction to compatible pairs, then the condition formulated above cannot be satisfied by explicit substitution functions containing substitutions of different many terms.

In order to capture this intuitive demand more precisely, we introduce the notion of *closed* subsets of cartesian products.

16.3 DEF (Closed Subsets of Cartesian Products): Let X, Y be two sets. A subset $Z \subseteq X \times Y$ of the cartesian product of X and Y is called *closed*, if there are sets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ such that $Z = X_0 \times Y_0$.

A subset of a cartesian product is closed, if it is a cartesian product of some subsets of its arguments. With the help of this notion, we categorise explicit substitution functions.

16.4 DEF (Closed Substitution Function): Let $\mathbf{S} \subseteq \mathcal{S}$ be an explicit substitution function.

1. *component:* The n-th *component* $\mathbf{dom}_n(\mathbf{S})$ of the domain $\mathbf{dom}(\mathbf{S})$ is defined as follows for every $n \in \omega$:

$$\mathbf{dom}_n(\mathbf{S}) \;=\; \{\langle \mathfrak{o}, \mathfrak{s} \rangle \in \mathbf{dom}(\mathbf{S});\ \mathbf{lng}(\mathfrak{s}) = n\}$$

2. *complete:* The explicit substitution function $\mathbf{S}$ is *closed*, if its domain $\mathbf{dom}(\mathbf{S}) \subseteq \mathcal{O} \times \mathtt{T}_0{}^{<\omega}$ is closed.

3. *weakly closed:* The explicit substitution function $\mathbf{S}$ is *weakly closed*, if each component $\mathbf{dom}_n(\mathbf{S})$ is closed (for all $n \in \omega$).

Remarks (Closed Substitution Functions):

1. *fixed number of terms:* If an explicit substitution function is closed, then it contains only substitutions of a previously fixed number of terms. This means that there is at most one non-empty component $\mathbf{dom}_n(\mathbf{S})$. More formally: there is $n \in \omega$ such that $\mathbf{dom}_m(\mathbf{S}) = \emptyset$ for all $n \neq m \in \omega$.

2. *closed substitution functions:* As a consequence of the last observation, closed substitution functions are also weakly closed (as empty components are closed cartesian products).

3. *maximal substitution function:* As the are (different) substitutions of different many terms, the maximal explicit substitution function $\mathcal{S}$ is not closed, but weakly closed.

4. *intuitive demands:* It is easily checked that the weakly closed substitution functions are exactly those substitution functions, which satisfy our informal demands for being well-behaved. In the case of explicit substitution functions containing only substitutions of a fixed number of terms, these substitution functions are even closed.

Examples (Closed Substitution Functions): We provide some examples of closed explicit substitution functions:

1. *empty set:* The empty set $\emptyset \subseteq \mathcal{S}$ is a (trivial) explicit substitution function and, in particular, closed.

2. Every singleton $\{\mathbf{s}\} \subseteq \mathcal{S}$ is a closed substitution function.

3. A set $\{\mathbf{s}, \mathbf{s}'\} \subseteq \mathcal{S}$ of two substitutions is closed, if and only if one of the following both conditions is satisfied:

   (a) *affected occurrence:* The respective affected occurrences are equal. More formally:

   $$\langle \mathbf{con}(\mathbf{s}), \mathbf{aff}(\mathbf{s}), \mathbf{pos}(\mathbf{s}) \rangle \;=\; \langle \mathbf{con}(\mathbf{s}'), \mathbf{aff}(\mathbf{s}'), \mathbf{pos}(\mathbf{s}') \rangle$$

   (b) *inserted terms:* The respective sequences of inserted terms are equal. More formally:

   $$\mathbf{ins}(\mathbf{s}) \;=\; \mathbf{ins}(\mathbf{s}')$$

   Observe that if both conditions are satisfied, then $\mathbf{s} = \mathbf{s}'$.

4. Let $\mathfrak{o} = \langle 0 + 0, \langle 0, 0 \rangle, *_0 + *_1 \rangle$ be an occurrence in the language $\mathfrak{L}_{\mathrm{PA}}$ of arithmetics. Investigate the following sets $\mathbf{S}$ and $\mathbf{S}'$ of substitutions of two terms:

   $$\begin{aligned} \mathbf{S} &= \{\langle \mathfrak{o}, \langle n, n \rangle, n + n \rangle;\ n \in \omega\} \\ \mathbf{S}' &= \{\langle \mathfrak{o}, \langle n, m \rangle, n + m \rangle;\ n, m \in \omega,\ n \neq m\} \end{aligned}$$

   Both substitution functions are closed. Additionally, we may observe that $\mathbf{S}$ and $\mathbf{S}'$ are disjoint and that their union $\mathbf{S} \cup \mathbf{S}'$ is again closed.

16.2 Implicit Substitution Functions

In order to classify functions traditionally associated with substitutions as substitution functions, we introduce the concept of the so called *implicit substitution functions*. Such implicit substitution functions can be transformed via an *explication method* into an explicit substitution function. We provide the formal definition of these notions.

16.5 DEF (Implicit Substitution Function): Let X and Y be arbitrary sets and $F : X \to Y$ a function from X into Y.

1. *explication method:* An ordered pair $\mathcal{M} = \langle G, H \rangle$ is called an *explication method* for F, if the following both conditions are satisfied:
   (a) *explication method for the domain:* $G : X \to \mathcal{O} \times \mathtt{T}_0{}^{<\omega}$ is a function from the domain X of F into the set of ordered pairs of multi-shape occurrences and finite sequences of standard terms (into the domain of an explicit substitution function).
   (b) *explication method for the codomain:* $H : Y \to \mathtt{T}_0$ is a function from the codomain Y of F into the set of standard terms (into the codomain of an explicit substitution function).
2. *explication:* The set $F_{/\mathcal{M}} = F_{/\langle G,H \rangle} = \{\langle G(x), H(F(x)) \rangle;\ x \in X\}$ is called the *explication* of F with respect to the explication method $\mathcal{M}$.
3. *implicit substitution function:* The function F is called an *implicit substitution function* with respect to the explication method $\mathcal{M}$, if the respective explication $F_{/\mathcal{M}}$ is an explicit substitution function.

Remarks (Implicit Substitution Function):

1. *codomain:* We demand (for symmetry reason) in the definition above that the explication method H for the codomain is a function on the codomain, and not on the image of an implicit substitution function F. This makes no (essential) difference, as the function F can also be understood as a function into its image. Furthermore, the definition of an explication only depends on the restriction of the method H to the image of F.
2. *terminology:* The implicit substitution functions F are called so, as their elements $\langle x, y \rangle \in F$ represent (in a way) the substitutions contained in the corresponding explicit substitution function $F_{/\mathcal{M}}$. This representation can be described, in the intended case, informally and is made explicit by the explication method $\mathcal{M}$.
3. *explicit substitution functions:* Every explicit substitution function $\mathbf{S}$ is an implicit substitution function with respect to the trivial explication method $\mathcal{M} = \langle \mathbf{id}_{\mathbf{dom(S)}}, \mathbf{id}_{\mathbf{img(S)}} \rangle$; observe that $\mathbf{S} = \mathbf{S}_{/\mathcal{M}}$.

4. *dependency of the methods:* The methods G and H for domain and codomain of an implicit substitution function F are not independent: as the position of the affected occurrence together with the inserted sequence (both determined by G) determine the result of a substitution, the method H for the codomain is (essentially) uniquely determined. More formally, for all $x \in X$:

$$H(F(x)) \;=\; \mathbf{pos}(\pi_0(G(x)))[\pi_1(G(x))]$$

   Here, π_0 and π_1 are the respective projections for ordered pairs.

5. *simplified implicit substitution functions:* Subsequently, we are mainly interested in implicit substitution functions satisfying the condition that their explication method for the codomain is the identity function. These are, in the intended case, the traditional substitution functions, which are not applied on multi-shape occurrences and sequences of standard terms, but on a simplification of them, and where the result is a already a standard term.

   In order to simplify the terminology in such cases (where $H = \mathbf{id}_{\mathbf{img}(F)}$), we write $F_{/G}$ instead of $F_{/\langle G,H\rangle}$ and call slightly sloppy the explication method G for the domain already the explication method itself. In this simplified case, the explication is given as follows:

$$F_{/G} \;=\; \{\langle G(x), F(x)\rangle;\; x \in X\}$$

Ambiguity of Explication: Due to the generality of the concept of explication, there are (besides limit cases) infinitely many explication methods $\mathcal{M}$ classifying any function $F : X \to Y$ as an implicit substitution function. Investigate, for example, the following constant functions for $t \in \mathtt{T}_0$:

$$G_t : X \to \mathcal{O} \times {\mathtt{T}_0}^{<\omega} : x \mapsto \langle\langle t, t, *\rangle, t\rangle \quad ; \quad H_t : Y \to \mathtt{T}_0 : y \mapsto t$$

Presupposing that $X \neq \emptyset$ and $Y \neq \emptyset$, we obtain:

$$F_{/\langle G_t,H_t\rangle} \;=\; \{\langle t, t, *, t, t\rangle\} \;\subseteq\; \mathcal{S}$$

There are some obvious ways of restricting the concept of implicit substitution functions:

1. *simplified substitution functions:* As mentioned above, we may demand that the method H for the codomain is the identity function. This demand would restrict the notion of an implicit substitution function to functions mapping into the set of standard terms.

2. *meaningful representation:* Another possibility is to demand that the explication method $\mathcal{M}$ provides a "meaningful" representation of the implicit substitution function. Demanding, for example, that the explication method G for the domain is injective, guarantees that different arguments x and x' of the implicit substitution function represent different substitutions. Also, it seems reasonable to demand that the explication of an implicit substitution function has to be weakly closed.

   In particular, via such demands we are able to rule out pathological explications of functions (classifying these functions in a pathological way as implicit functions) provided that the functions under discussion are not trivial. (If, for example, a function F is defined on a singleton X, then every explication $F_{/\mathcal{M}}$ has to be a singleton, which seems to be a pathological substitution function.)

3. *identifying substitution functions:* A more subtle way of restricting the concept of implicit substitution functions is to identify explication methods $\mathcal{M}$ (via a suitable equivalence relation) satisfying the condition that the argument x is explicated by equivalent methods to equivalent substitutions for all elements $x \in X$.

   This demand does not exclude functions from being an implicit substitution function, but minimises their possibility of representing explicit substitution functions.[96]

We abstain here from discussing such restrictions of the concept of explication in more details and focus on the possibilities provided by this concept; the (philosophical) task of improving this concept by providing better restrictions is left to future work.

16.3 Examples: Implicit Substitution Function

We illustrate that it is possible to describe functions traditionally associated with substitution as implicit substitution functions.

16.3.1 The Complete Substitution Function

The *complete substitution function* $\mathbf{S}_c : \mathtt{T}_0{}^3 \to \mathtt{T}_0$ maps a triple $\langle t, s, r\rangle$ of standard terms to the result of the simultaneous replacement of all occurrences of the term s in the term t by the term r.

[96] In particular, this demand motivates the identification of explicit substitution functions.

Traditional Definition: The complete substitution $\mathbf{S}_c$ is defined recursively (using the traditional notation $t[s/r]$ instead of $\mathbf{S}_c(t, s, r)$) as follows:

1. *t atomic:* $t[s/r] \backsimeq \begin{cases} r & \text{if } s \backsimeq t \\ t & \text{otherwise} \end{cases}$

2. *$t \backsimeq f(t_0, \ldots t_n)$ complex:*

$$t[s/r] \backsimeq \begin{cases} r & \text{if } s \backsimeq t \\ f(t_0[s/r], \ldots t_n[s/r]) & \text{otherwise} \end{cases}$$

Informal Analysis: In a complete substitution, the affected occurrence is that occurrence in t, in which all occurrences of the term s are intended. The latter means that the respective position is an elimination form of t in which all occurrences of s are eliminated. The sequence of inserted terms is given by the standard term r.

Theory of Occurrences: The complete substitution function $\mathbf{S}_c(t, s, r)$ can be analysed within the theory of occurrences. We mention central aspects:[97]

1. *complete occurrence:* A one-place occurrence $\mathfrak{o} = \langle t, s, \mathtt{t} \rangle$ is *complete*, if its position equals to the result of an application of the complete elimination function **elim** on the context and the shape of $\mathfrak{o}$ (formally, if $\mathtt{t} \backsimeq \mathbf{elim}(t, s)$). Alternatively, a complete occurrence can be characterised by the demand that the context does not occur in the position (formally, by $\mathbf{mult}(s, \mathtt{t}) = 0$).

   The set of all complete occurrences is denoted by $\mathcal{O}_{1,c}$.

2. *complete substitution:* A substitution $\mathtt{s} = \langle t, s, \mathtt{t}, s', t' \rangle$ of one term is *complete*, if the affected occurrence $\mathfrak{o} = \langle t, s, \mathtt{t} \rangle$ is complete.

   The set of all complete substitutions is denoted by $\mathcal{S}_{1,c}$.

3. *characterisation:* The complete substitution function can be understood as the composition of the complete elimination function and the general substitution function. More formally, for all standard terms $t, s, r \in \mathtt{T}_0$:

$$\mathbf{S}_c(t, s, r) \backsimeq \mathbf{elim}(t, s)[r]$$

[97] The complete elimination function is defined in the section about elimination forms and occurrences as a recursively definable example for the concept of elimination.

Explication Method: A (simplified) explication method for the complete substitution function $\mathbf{S}_c$ is a function $G : \mathtt{T_0}^3 \to \mathcal{O} \times \mathtt{T_0}^{<\omega}$ transforming an ordered triple $\langle t, s, r\rangle$ of standard terms into an ordered pair of a multi-shape occurrence $\mathfrak{o}$ (of one term) and a sequence of standard terms (of length one). A suitable explication method is easily found:

$$G(t, s, r) \;=\; \langle\, \langle t, s, \mathbf{elim}(t, s)\rangle, r\,\rangle$$

The image of G equals the cartesian product $\mathcal{O}_{1,c} \times \mathtt{T_0}^1$ (a closed subset of $\mathcal{O} \times \mathtt{T}^{<\omega}$), which becomes the domain of the respective explication.

Explicit Substitution Function: The corresponding explicit substitution function $\mathbf{S}_{c/G}$ is the set of all complete substitutions and given as follows:

$$\begin{aligned}\mathbf{S}_{c/G} \;&=\; \{\langle G(t, s, r), \mathbf{S}_c(t, s, r)\rangle;\; t, s, r \in \mathtt{T_0}\}\\ &=\; \{\langle t, s, \mathbf{elim}(t, s), r, \mathbf{elim}(t, s)[r]\rangle;\; t, s, r \in \mathtt{T_0}\} \;=\; \mathcal{S}_{1,c}\end{aligned}$$

Observe that the explication $\mathcal{S}_{1,c}$ of the complete substitution function $\mathbf{S}_c$ has the closed domain $\mathbf{dom}(\mathcal{S}_{1,c}) = \mathcal{O}_{1,c} \times \mathtt{T_0}^1$ of all pairs of complete occurrences and sequences of standard terms of length one.

16.3.2 The Simultaneous Substitution Function

The simultaneous substitution function $\mathbf{S}_s : \mathtt{T_0} \times \mathtt{T_0}^{<\omega} \to \mathtt{T_0}$ has a standard term t and a finite sequence $\mathfrak{s}$ of standard terns as arguments and replaces simultaneously all occurrences of the first n variables $v_0, \ldots v_{n-1}$ in t (where $n = \mathbf{lng}(\mathfrak{s})$ is the length of the second argument) by the respective entries of the second argument.

Traditional Definition: The simultaneous substitution function $\mathbf{S}_s$ is defined recursively (in the first argument, for an arbitrary finite sequence $\mathfrak{s} = \langle s, \ldots s_{n-1}\rangle$ of standard terms) as follows:

1. *t atomic:* $\mathbf{S}_s(t, \mathfrak{s}) \doteq \begin{cases} s_k & \text{if } t \doteq v_k \text{ for a } k \in \mathbf{lng}(\mathfrak{s})\\ t & \text{otherwise}\end{cases}$

2. *$t \doteq f(t_0, \ldots t_m)$ complex:*

$$\mathbf{S}_s(t, \mathfrak{s}) \;\doteq\; f(\mathbf{S}_s(t_0, \mathfrak{s}), \ldots \mathbf{S}_s(t_m, \mathfrak{s}))$$

Informal Analysis: In a simultaneous substitution, the affected occurrence is a complete occurrence in the standard term t of the first n variables v_k, where n is the length of the inserted sequence $\mathfrak{s}$.[98]

Theory of Occurrences: We analyse the simultaneous substitution function $\mathbf{S}_s$ in the light of the theory of occurrences.

1. *elimination function:* In order to provide an explication method for $\mathbf{S}_s$, we have to construct a suitable position. It is convenient to use the following generalisation $\mathbf{elim} : \mathtt{T}_0 \times \omega \to \mathtt{T}$ of the complete elimination function.

   (a) t *atomic:* $\mathbf{elim}_n(t) \doteq \begin{cases} *_k & \text{if } t \doteq v_k \text{ and } k \in n \\ t & \text{otherwise} \end{cases}$

   (b) $\mathtt{t} \doteq f(\mathtt{t}_0, \ldots \mathtt{t}_m)$ *complex:*

   $$\mathbf{elim}_n(t) \doteq f(\mathbf{elim}_n(t_0), \ldots \mathbf{elim}_n(t_m))$$

   This version of the complete elimination function eliminates the first n variables $v_0, \ldots v_{n-1}$ in a standard term.

2. *characterisation:* The simultaneous substitution function $\mathbf{S}_s$ can be characterised with the help of the generalised elimination function $\mathbf{elim}_n$. More formally, for all $t \in \mathtt{T}_0$ and $\mathfrak{s} \in {\mathtt{T}_0}^{<\omega}$:

   $$\mathbf{S}_s(t, \mathfrak{s}) \doteq \mathbf{elim}_n(t)[\mathfrak{s}]$$

   Here, $n = \mathbf{lng}(\mathfrak{s})$.

Explication Method: The explication method G for $\mathbf{S}_s$ can be given as follows:

$$G(t, \mathfrak{s}) = \langle\, \langle t, \mathfrak{v}_n, \mathbf{elim}_n(t) \rangle, \mathfrak{s}\, \rangle$$

Here, $n = \mathbf{lng}(\mathfrak{s})$ and $\mathfrak{v}_n = \langle v_0, \ldots v_{n-1} \rangle$.

Explicit Substitution Function: The corresponding explicit substitution function is given as follows:

$$\begin{aligned} \mathbf{S}_{s/G} &= \{\langle G(t, \mathfrak{s}), \mathbf{S}_s(t, \mathfrak{s}) \rangle;\ \langle t, \mathfrak{s} \rangle \in \mathbf{dom}(\mathbf{S}_s)\} \\ &= \{\langle t, \mathfrak{v}_n, \mathbf{elim}_n(t), \mathfrak{s}, \mathbf{elim}_n(t)[\mathfrak{s}] \rangle;\ \langle t, \mathfrak{s} \rangle \in \mathbf{dom}(\mathbf{S}_s)\} \end{aligned}$$

[98] Recall the generalisations of the complete elimination function discussed in the section about elimination forms and occurrences; in particular, the complete elimination function with fixed arguments.

Again, $n = \mathbf{lng}(\mathfrak{s})$. It is easily checked that $\mathbf{S}_{s/G}$ is, indeed, a set of substitutions and therefore also an explicit substitution function.

Observe that $\mathbf{S}_{s/G}$ has not a closed domain, as $\mathbf{S}_{s/G}$ is applied to finite sequences of arbitrary length. Nevertheless, the components of the domain are closed for all $n \in \omega$:

$$\mathbf{dom}_n(\mathbf{S}_{s/G}) \;=\; \{\langle t, \mathfrak{v}_n, \mathbf{elim}_n(t)\rangle;\; t \in \mathtt{T}_0\} \times {\mathtt{T}_0}^n$$

As a consequence, the explication $\mathbf{S}_{s/G}$ of the simultaneous substitution function $\mathbf{S}_s$ is a weakly closed substitution function.

16.3.3 The General Substitution Function

The general substitution function is closely related to the simultaneous substitution function. But as the image of the general substitution function is the set of all nominal terms (and not only the set of all standard terms), this function cannot be analysed as a simplified implicit substitution function.

Nevertheless, we can carry over the theory of occurrences to nominal terms (introducing a second kind of nominal symbols). In this extended realm, the general substitution function is, indeed, an implicit substitution function. In a slightly sloppy terminology: the general substitution function is a kind of a "second order" substitution function.

Simulation of the General Substitution Function: It is worth to mention that the general substitution function can even be analysed as an implicit substitution function in the realm of standard terms, if we use non-trivial explication methods for the codomain. Recalling that we can simulate nominal symbols in a standard first order language, we define two auxiliary functions as follows:[99]

1. *simulation function:* The simulation function $\sigma : \mathtt{T} \to \mathtt{T}_0$ transforms nominal terms into standard terms by replacing nominal symbols by even variables and variables by odd variables.

$$\sigma : \mathtt{t} \mapsto \begin{cases} c & \text{if } \mathtt{t} \doteqdot c \\ v_{2k} & \text{if } \mathtt{t} \doteqdot *_k \\ v_{2k+1} & \text{if } \mathtt{t} \doteqdot v_k \\ f(\sigma(t_0), \ldots \sigma(t_n)) & \text{if } \mathtt{t} \doteqdot f(t_0, \ldots t_n) \end{cases}$$

2. *complete elimination function:* We need additionally another variant $\mathbf{elim} : \mathtt{T}_0 \times \omega \to \mathtt{T}$ of the complete elimination function eliminating

[99] The possibility of simulating nominal terms is already mentioned in the section about the basic theory of nominal terms.

in the simulated nominal term (which is a standard term) the first n simulated nominal symbols (which are the first n even variables):

(a) *t atomic:* $\mathbf{elim}_n(t) \mathrel{\widehat{=}} \begin{cases} *_k & \text{if } t \mathrel{\widehat{=}} v_{2k} \text{ and } k \in n \\ t & \text{otherwise} \end{cases}$

(b) $\mathtt{t} \mathrel{\widehat{=}} f(\mathtt{t}_0, \ldots \mathtt{t}_m)$ *complex:*

$$\mathbf{elim}_n(t) \;\mathrel{\widehat{=}}\; f(\mathbf{elim}_n(t_0), \ldots \mathbf{elim}_n(t_m))$$

With the help of both auxiliary functions, we can provide a suitable explication method for the general substitution function:

1. *explication method for the domain:* The affected occurrence is given by the simulation of the first argument of the general substitution function, in which the simulations of the nominal symbols are eliminated; observe that it is sufficient to eliminate the first n simulated nominal symbols, where n is the length of the second argument. The sequence of inserted terms is given by the simulation of the second argument of the general substitution function.

   Therefore, the explication method $G : \mathtt{T} \times \mathtt{T}^{<\omega} \to \mathcal{O} \times \mathtt{T}_0{}^{<\omega}$ for the domain can be defined as follows:

   $$G : \langle \mathfrak{t}, \mathfrak{s} \rangle \mapsto \langle \langle \sigma(\mathfrak{t}), \sigma(\mathfrak{v}_n), \mathbf{elim}_n(\sigma(\mathfrak{t})) \rangle, \sigma(\mathfrak{s}) \rangle$$

   Here, $n = \mathbf{lng}(\mathfrak{s})$. Furthermore, an application of σ on a sequence of nominal terms is understood as the sequence of applications. The latter means:

   $$\sigma(\mathfrak{v}_n) \;=\; \langle \sigma(v_k);\; k \in n \rangle \quad ; \quad \sigma(\mathfrak{s}) \;=\; \langle \sigma(\mathfrak{s}_k);\; k \in n \rangle$$

   The following equation is checked easily:

   $$\mathbf{elim}_n(\sigma(\mathfrak{t}))[\sigma(\mathfrak{v}_n)] \;\mathrel{\widehat{=}}\; \sigma(\mathfrak{t})$$

   As a consequence, the arguments for the general substitution function are, indeed, mapped by the explication method G for the domain to an ordered pair of a multi-shape occurrence and a suitable sequence of standard terms.

2. *explication method for the codomain:* The explication method for the codomain can be given as the simulation of the result of the respective application of the general substitution function. Therefore:

   $$H : \mathtt{T} \to \mathtt{T}_0 : \mathtt{t} \mapsto \sigma(\mathtt{t})$$

The following equation is checked easily:

$$\mathbf{elim}_n(\sigma(\mathfrak{t}))[\sigma(\mathfrak{s})] \;\doteq\; \sigma(\mathfrak{t}[\mathfrak{s}])$$

As a consequence, $\langle G(\mathfrak{t},\mathfrak{s}), H(\mathfrak{t},\mathfrak{s})\rangle$ is, indeed, a substitution.

We conclude that the explication $\cdot[\cdot]_{/\langle G,H\rangle}$ of the general substitution function is an explicit substitution function. The latter means that the general substitution function is an implicit substitution function.

16.3.4 A Gödelised Substitution Function

The concept of explication is sufficiently general to identify recursive functions (on natural numbers), which are representing the substitutions of terms in terms, as implicit substitution functions. We provide some details.

1. *Gödel numbering:* We presuppose a Gödel numbering $\ulcorner\cdot\urcorner : \mathtt{T}_0 \to \omega$ of the terms of a (countable) formal language $\mathfrak{L}$. The number $\ulcorner t\urcorner$ is called the Gödel number of the term t.[100]

   Additionally, let $\gamma = \{\ulcorner t\urcorner;\ t \in \mathtt{T}_0\} \subseteq \omega$ be the set of all Gödel numbers of a term in the language $\mathfrak{L}$ and $\gamma_x = \{\ulcorner x\urcorner;\ x \in \mathbf{V}_x\} \subseteq \gamma$ the restriction of γ to Gödel numbers of variables..

   Furthermore, by $[\![n]\!]$ we mean the standard term t having the Gödel number n; the latter means that $[\![\cdot]\!] : \gamma \to \mathtt{T}_0$ is the inverse function of the Gödel numbering. As the Gödel numbering is not surjective, the function $[\![\cdot]\!]$ is only defined on a proper subset of ω.

2. *substitution function:* The Gödelised substitution function $\mathbf{S}_\gamma : \omega^3 \to \omega$ satisfies the following condition for all standard terms $t, s \in \mathtt{T}_0$ and for all variables $x \in \mathbf{V}_x$:

   - $\mathbf{S}_\gamma(\ulcorner t\urcorner, \ulcorner x\urcorner, \ulcorner s\urcorner) = \ulcorner t[x/s]\urcorner$, where $t[x/s]$ is the result of replacing all occurrences of x in t by s.

   We may presuppose that $0 \notin \gamma$ is not a Gödel number of a standard term and that an application of $\mathbf{S}_\gamma$ results in zero for all cases not given above.[101]

[100] The method of Gödelising the syntax was used by Kurt Gödel [15] in the proof of his seminal incompleteness theorems for formal arithmetics PA. A presentation of these theorems and of the method of Gödelising the syntax is found in every good logic textbook.

[101] The existence of such substitution functions (and their representability in the formal theory PA of arithmetics) is essential for Gödel's proof of the incompleteness theorems.

We establish that $\mathbf{S}_\gamma$ is an implicit substitution function:

1. *explication method for the domain:* We need that the following equality holds:
$$G : \ \langle \ulcorner t \urcorner, \ulcorner x \urcorner, \ulcorner s \urcorner \rangle \ \mapsto \ \langle \langle t, x, \mathbf{elim}(t, x) \rangle, s \rangle$$
Therefore:
$$G : \langle n, m, l \rangle \ \mapsto \ \begin{cases} \langle \langle [\![n]\!], [\![m]\!], \mathbf{elim}([\![n]\!], [\![m]\!]) \rangle, [\![l]\!] \rangle & \text{if } n, l \in \gamma, \ m \in \gamma_x \\ \langle \langle v_0, \epsilon, v_0 \rangle, \epsilon \rangle & \text{otherwise} \end{cases}$$

2. *explication method for the codomain:* We need that the following equality holds:
$$H : \ \ulcorner t[x/s] \urcorner \ \mapsto \ t[x/s]$$
Therefore:
$$H : n \ \mapsto \ \begin{cases} [\![n]\!] & \text{if } n \in \gamma \\ v_0 & \text{otherwise} \end{cases}$$

The explication $\mathbf{S}_{\gamma/\langle G,H \rangle}$ of the Gödelised substitution function is an explicit substitution function; correspondingly, $\mathbf{S}_\gamma$ an implicit substitution function (with non-trivial explication method for the codomain). Observe that triples $\langle n, m, l \rangle$ not representing a suitable substitution are mapped to the trivial substitution $\langle v_0, \epsilon, v_0, \epsilon, v_0 \rangle$.

16.3.5 An Artificial Substitution Function

As a last example, we discuss a function mapping ordered pairs of natural numbers to variables and which is classified by (two different) artificial explication methods as an implicit substitution function.

The Artificial Substitution Function: The artificial substitution function $\mathbf{S}_a : \omega \times \omega \to \mathbf{V}_x : \langle k, l \rangle \mapsto v_l$ maps an ordered pair $\langle k, l \rangle$ of natural numbers to the variable v_l.

Explication Methods and Substitution Functions: We provide two different explication methods transforming $\mathbf{S}_a$ into an explicit substitution function:

1. *replacement of variables:* Assuming that both arguments k and l of $\mathbf{S}_a$ represent variables and that the first variable is implicitly replaced by the second, we obtain the following explication method:
$$G : \omega^2 \to \mathcal{O} \times {\mathtt{T}_0}^{<\omega} : \ \langle k, l \rangle \mapsto \langle \langle v_k, v_k, *_0 \rangle, v_l \rangle$$

The respective explication is, indeed, an explicit substitution function:

$$\mathbf{S}_{a/G} \;=\; \{\langle v_k, v_k, *, v_l, v_l\rangle;\; k, l \in \omega\}$$

2. *redundant argument:* Assuming that the first argument is redundant, that the second argument represents a variable and that implicitly the total substitution of a fixed standard term $t \in \mathtt{T}_0$ is intended, we obtain the following explication method:

$$G' : \omega^2 \to \mathcal{O} \times {\mathtt{T}_0}^{<\omega} :\; \langle k, l\rangle \mapsto \langle\langle t, t, *_0\rangle, v_l\rangle$$

The respective explication is given as follows:

$$\mathbf{S}_{a/G'} \;=\; \{\langle t, t, *, v_l, v_l\rangle;\; l \in \omega\}$$

This last example illustrates that there are even meaningful explication methods transforming a function in non-trivial substitution function, but still via an artificial explication method.

16.4 Conceptual Remarks

We conclude our discussion of substitution functions with a brief analysis of our concept of implicit substitution functions and their explication.

1. *principle achievement:* We illustrated that the concept of explication is capable of characterising functions traditionally understood as substitution functions as such functions and to relate these functions with our formal concepts of substitutions and (explicit) substitution functions.

   In particular, the associated explicit substitution functions seem to explicate the traditional substitution functions faithfully (according to our intuitions).

2. *underdetermination:* We have seen that the explication method and the resulting explicit substitution function is not determined by the implicit substitution function itself. In particular, we have seen that there are infinitely many trivial explication methods for arbitrary functions classifying them trivially as implicit substitution functions.

   The variability of possible explication methods is inherent to the concept of implicit substitution functions (as defined here). This variability corresponds to different (informal) interpretations of the substitutions represented by an implicit substitution function.

Nevertheless, this variability also demands further (philosophical) research with the aim of formulating good restrictions on the concept of explication, ruling out pathological and undesired cases of implicit substitution functions.

17 Conclusion: Results

The main results of these investigations are the adequate formal definitions of the fundamental syntactic notions of occurrences and substitutions and related concepts, which are usually used only informally in the field of logic (and other related fields).

These investigations focus on the paradigmatic theory of occurrences of terms in terms and are easily carried over to other kinds of occurrences. We summarise central aspects of these investigations.

17.1 Theory of Nominal Terms

Occurrences and substitutions, as discussed in these investigations, are based on the notion of *nominal terms*, essentially, as introduced by Schütte. We provide a survey of the theory of nominal terms as far as developed in these investigations.

17.1.1 Basic Theory of Nominal Terms

Nominal terms are the result of a generalisation of the standard terms (of a formal first order language) by adding countable many *nominal symbols* $*_k$ (a new kind of variables) to the alphabet of the underlying formal language and by extending the inductive definition of standard terms by these nominal symbols as new atomic expression.

Relevance of Nominal Terms: The nominal terms are crucial in these investigations, as they are capable of representing formally the positions of occurrences. More precisely, intended positions of subterms of a given standard term are represented by *elimination forms* of the given standard term. These are nominal terms, in which the intended subterms are replaced by suitable nominal symbols.

Using the same nominal symbol more than once in an elimination form allows to mark simultaneously multiple positions of the same subterm in the given standard term; using different nominal symbols allows even to mark positions of different subterms. Due to this generality, the nominal terms turn out to be superior to traditional concepts of formal occurrences, which are only capable of representing single positions.

Basic Categorisation: Depending on the nominal symbols occurring in a nominal term, the following basic categorisation is given:

1. Nominal terms without nominal symbols are the *standard terms* of the underlying formal language; nominal terms with nominal symbols are also called *proper*.

2. A nominal term is *n-ary*, if exactly the first n nominal symbols $*_k$ occur in that nominal term; in particular, a nominal term is *unary*, if exactly the nominal symbol $*_0$ occurs in it.

3. A nominal term is *simple*, if no nominal symbol occurs more than once; otherwise, the nominal term is *multiple*.

Nominal terms, which are both unary and simple, are also called *single*.

General Substitution Function: The central tool for the treatment of nominal terms is the *general substitution function*. This binary function maps a nominal term and a sequence of nominal terms to the result of a simultaneous replacement of the nominal symbols in the first argument by the corresponding entries of the second argument.

The general substitution function is a *universal homomorphism* on nominal terms (a function codifying all homomorphisms) preserving their structure given by the standard symbols of the alphabet (in the case of standard terms, these are the variables, the constant symbols and the function symbols of the underlying formal language).

17.1.2 Relations Based on Homomorphisms

Two fundamental relations, both based on homomorphisms, are introduced.

Isomorphism of Nominal Terms: Two nominal terms are *isomorphic*, if there is an isomorphism (a bijective homomorphism) mapping one nominal term to the other. Isomorphic nominal terms are equal up to a relabelling of their nominal symbols according to a permutation of the set of natural numbers.

The isomorphism of nominal terms is an equivalence relation on nominal terms. A nominal term is normal (a canonical representative of its own equivalence class) with respect to the isomorphism of nominal terms, if its nominal symbols are sorted. The latter means that the labels of the leftmost occurrences of each nominal symbol occurring in that nominal term are sorted

according to the natural numbers. The isomorphism of nominal terms is used to provide uniquely determined nominal terms in various situations.

Less-Structured Relation: Referring in the definition to arbitrary homomorphisms instead of isomorphisms, we obtain the *less-structured* relation: a nominal term is less structured than another nominal term, if there is a homomorphism mapping the first to the second. The denomination of this relation reflects that an application of a homomorphism on a nominal term results, in the typical case, in a more complex nominal term (having therefore more structure).

The less-structured relation is a partial order on nominal terms modulo the isomorphism of nominal terms; nominal symbols are the least elements with respect to the less-structured relation, standard terms are maximal.

Central technical application of the less-structured relation is the definition of the *elimination forms* of a standard term: elimination forms of a standard term are those nominal terms, which are less structured than this standard term. A sequence of standard terms is *eliminated* in an elimination form, if applying the general substitution form on the elimination form and on the sequence results in the respective standard term. An entry of an eliminated sequence is *actually* eliminated, if the corresponding nominal symbol actually occurs in the elimination form.

17.1.3 Relations Beyond Homomorphisms

Using methods beyond homomorphisms, some more relations are introduced:

Equivalence of Nominal Terms: Two nominal terms are *equivalent*, if they are equal up to the labelling of their nominal symbols. The equivalence of two nominal terms can be characterised by demanding that their *unifications* or, equivalently, their *simplifications* are equal.

Thereby, the unification of a nominal term is the result of relabelling all nominal symbols with the label zero; the simplification results from relabelling all nominal symbols from the left to the right according to the natural numbers. If a nominal term is proper, then its unification is unary; the simplification of a nominal term is simple.

The equivalence of nominal terms is, indeed, an equivalence relation. The unification and the simplification of a nominal term are two, in general different representatives of the respective equivalence class. The equivalence of nominal terms is used for the identification of positions differing only in the order of representing multiple positions. In particular, this relation is

used for the identification of generalised occurrences representing the *same* informal occurrence.

Covered-By Relation: A nominal term is *covered by* another nominal term, if some nominal symbols of the covered nominal term are replaced in the other nominal term by a standard term. This replacement cannot be given, in general, by the general substitution function, as the same nominal symbol can be replaced by different standard terms at different positions. Such replacements (independent on simultaneous replacements at other positions) are called *local*.

In the *weak version* of the covered-by relation, a nominal symbol can be locally covered by a different nominal symbol. This relation is a partial order on nominal terms modulo the equivalence of nominal terms. Nominal terms, in which all atomic subterms are nominal symbols, are minimal and standard terms are maximal with respect to the weak covered-by relation. Weakly covered nominal terms are useful as a substitute for elimination forms in a context based on the equivalence of nominal terms.

In the *strong version*, nominal symbols can be covered locally by a standard term or by the same nominal symbol. In contrast to the weak version of this relation, it is excluded that a nominal symbol is locally covered by a different nominal symbol. As a consequence, the strong covered-by relation becomes (in a way) a finitistic version of the weak covered-by relation. In particular, the strong covered-by relation is a proper partial order on nominal terms (and not only modulo the equivalence of nominal terms).

Independence of Nominal Terms: Two nominal terms are *independent*, if the nominal symbols of one nominal term are locally covered by a standard term in the other nominal term and vice versa; additionally, it is permitted that nominal symbols are locally covered by the same nominal symbol. In other words, independence of nominal terms is the local-symmetric version of the strong covered-by relation.

Central tools for dealing with independent nominal terms are the *merge function* and the *dual merge function*:

1. *merge function:* The merge function maps, in the intended case of independent nominal terms, its arguments to a nominal term, in which all nominal symbols of both arguments are present.

2. *dual merge function:* The dual merge function maps, again in the intended case, its arguments to a nominal term, in which only the nominal symbols are present, which are already present in both arguments.

In the *strong version* of the independence relation, nominal symbols have to be locally covered by standard terms; it is excluded that nominal symbols are covered by the same nominal symbol. This strong independence relation captures our intuitions about the independence of positions and is used to define the analogous relations on occurrences and on substitutions.

The weak version of independence is investigated first, as this relation behaves better (and well) with respect to both merge functions.

It is worth to mention that maximal independent sets of nominal terms equipped with both merge functions form a finite boolean algebra. The extreme elements of such a boolean algebra are minimal and maximal with respect to the strong covered-by relation; strong independence corresponds with disjointness.

17.2 The Notions of Occurrences and Substitutions

We communicate essential aspects of the different generalisations of the formal notion of occurrences, as introduced in these investigations, together with the discussed applications of these notions.

17.2.1 Standard Occurrences

The formal notion of a *(standard) occurrence* (of a term in a standard term) is defined as an ordered triple of the following kind:

$$\langle \texttt{context}, \texttt{shape}, \texttt{position} \rangle$$

Context and shape are standard terms, the position is a unary elimination form of the context in which the shape is actually eliminated. Such formal occurrences are capable of representing informal occurrences of one term at single or multiple positions.

This formal notion of occurrences satisfies the following intuitions:

1. Context and shape do not, in general, determine the position.
2. Context and position determine the shape.
3. Shape and position determine the context.

The first statement is an essential property of occurrences, the other two properties are (useful) consequences of using nominal terms as the representatives of the position.

Applications: The notion of occurrences allows to solve *hard problems* with respect to occurrences, which are not solvable only on the base of the recursive structure of the context and of the shape of an occurrence. As paradigmatic examples, formal solutions to the following hard problems are presented:

1. *counting occurrences:* Counting formally the number of simple and the number of arbitrary occurrences of a given term in a given standard term.

2. *the lies-within relation:* Deciding formally for two occurrences in the same context, whether one occurrence lies within the other or not.

17.2.2 Multi-Shape Occurrences

Multi-shape occurrences are a slight generalisation of standard occurrences: the restriction to a single shape is dropped. The latter means that the multi-shape occurrences are defined as ordered triples of the following kind:

$$\langle \texttt{context},\ \texttt{sequence of shapes},\ \texttt{position} \rangle$$

As in the case of standard occurrences, the context is a standard term. The sequence of shapes is a finite sequence of standard terms and the position is an elimination form of the context in which the sequence of shapes is eliminated. Identifying sequences of length one with their only entry, the standard occurrences become a special case of the multi-shape occurrences.

The multi-shape occurrences are capable of representing informal occurrences of finitely many terms at arbitrary positions. From a technical point of view, multi-shape occurrences are interesting, as central properties of substitutions (as introduced below) can be studied, in principle, in a simpler context.

Equivalence of Occurrences: According to our intuitions, two informal occurrences in the same context are equal, if the same shapes at the same positions are intended.

This intuition is captured by an equivalence relation on multi-shape occurrences: two multi-shape occurrences are equivalent, if they have the same context and if their positions are equivalent (with respect to the equivalence of nominal terms). Essentially, such equivalent multi-shape occurrences only differ in the way, how they mark the intended positions of their shapes.

In order to provide canonical representatives of the equivalence classes, some properties of multi-shape occurrences and of their *places* (which are positions of entries in the sequence of shapes) are introduced:

1. *vacuous place:* A place of an occurrence is *vacuous*, if the respective entry in the sequence of shapes is not actually eliminated in the position (which means that the corresponding nominal symbol does not occur in the position of such a multi-shape occurrence).

2. *regular occurrence:* An occurrence is *regular*, if there are no vacuous places and if its position is normal with respect to the isomorphism of nominal terms.

3. *redundant place:* A non-vacuous place of an occurrence is *redundant*, if there is a different non-vacuous place such that the respective entries in the sequence of shapes are equal.

4. *parsimonious occurrence:* An occurrence is *parsimonious*, if there are no redundant places.

There are two, in the general case different normal forms representing each equivalence class of occurrences:

1. *simple normal form:* A multi-shape occurrence is in simple normal form, if it is regular and if its position is simple. Occurrences in simple normal form are, in general, not parsimonious.

2. *parsimonious normal form:* A multi-shape occurrence is in parsimonious normal form, if it is regular and parsimonious. Occurrences in parsimonious normal form are, in general, not simple.

Independence of Occurrences: According to our intuitions, two informal occurrences in the same context are independent, if the intended shapes do not overlap. The latter means that they have no common shapes at the same place and that no shape of one occurrences lies locally within a shape of the other occurrence.

In order to capture this intuition, an independence relation on multi-shape occurrences is introduced: two occurrences are *independent*, if they have the same context and if their positions are strongly independent. The independence relation is compatible with the equivalence of occurrences.

Two operations on independent occurrences are introduced:

1. *merging occurrences:* A set of independent occurrences can be merged into one occurrence representing all the shapes at all positions, which are represented by one of the arguments contained in the set of independent occurrences.

   The merge operation for occurrences is a useful tool of the theory of occurrences: when dealing with equivalence classes of single occurrences in a given context (which are equal or independent), then it is possible to merge the elements of the equivalence classes into one occurrence representing the full equivalence class. Applications of this method are mentioned in the section about future work.

2. *splitting up occurrences:* An occurrence can be split up into single (and independent) occurrences, each of them representing a single shape at a single position of the shapes represented by the argument.

Splitting up occurrences into independent occurrences and merging them is also of philosophical interest, as this method reflects the relationship between single occurrences, standard occurrences and multi-shape occurrences.

17.2.3 Substitutions

Formal *substitutions* are defined as quintuples of the following kind:

$$\langle \texttt{context},\ \texttt{sequence of affected terms},\ \texttt{position},\ \texttt{sequence of inserted terms}, \texttt{result}\rangle$$

The triple of context, sequence of affected terms and position is a multi-shape occurrence as well as the triple of result, sequence of inserted terms and position. Furthermore, it is demanded that both sequences have the same (finite) length. Additionally, *simplified* substitutions are introduced as substitutions based, essentially, on the notion of standard occurrences instead of multi-shape occurrences.

A formal substitution represents the simultaneous replacement of the standard terms given in the sequence of affected terms at the represented positions in the context by the corresponding entries of the sequence of inserted terms. Simplified substitutions can only represent the replacement of one term (possibly at multiple positions).

Equivalence of Substitutions: According to our intuition, two informal substitutions can be identified, if they have the same context and the same result. This intuition is captured by the *equivalence of substitutions*.

Guided by the discussion with respect to the equivalence of occurrences, normal forms for the equivalence of substitutions are provided with the help of some properties for substitutions and for their places:

1. *regular substitutions:* The definition of vacuous places and regular substitutions can be immediately carried over from the simpler case of occurrences.

2. *non-trivial substitutions:* A substitution is *non-trivial*, if there is no non-vacuous place such that the respective entries in both sequences are equal. In such non-trivial substitutions no subterm of the context is replaced by itself.

3. *simple substitution:* A substitution is *simple*, if its position is a simple nominal term.

4. *redundant place:* Redundancy can be carried over from occurrences, but has to be adapted to the more complex situation: a non-vacuous place is redundant, if there is a different non-vacuous place such that the affected terms at these places are equal as well as the inserted terms at these places. In substitutions with redundant places, the same terms at different positions are replaced by the same terms.

5. *parsimonious substitution:* A substitution is parsimonious, if there are no redundant places.

6. *reducible place:* A place is *reducible*, if the respective entries in the sequence of affected terms and inserted terms both are complex and both have the same main function symbol. Reducible places can be split up into finitely many places representing the replacement of the direct subterms of the respective affected term by the corresponding subterms of the respective inserted term.

7. *irreducible substitution:* A substitution is irreducible, if there are no reducible places.

There are three, in the general case different normal forms representing each equivalence class of substitutions:

1. *complete substitution:* A substitution of one term is complete, if the affected term equals to the context, if the inserted term equals to the

result and if the position is proper. (The latter implies that complete substitutions are regular.) This trivial normal form is both simple and parsimonious.

2. *simple normal form:* A substitution is in simple normal form, if it is regular, non-trivial, irreducible and simple. In general, a substitution in simple normal form is not parsimonious.

3. *parsimonious normal form:* A substitution is in parsimonious normal form, if it is regular, non-trivial, irreducible and parsimonious. In general, a substitution in parsimonious normal form is not simple.

Calculations: Formal substitutions are capable of representing (informal) calculations, as found in everyday mathematics. A calculation step can be represented by a substitution and a calculation by a sequence of substitutions satisfying the following condition: the result of a substitution in this sequence is equal to the context of the next substitution (as long as there is a next).

The formal terminology introduced with respect to substitution provides the capacity of reflecting different intuitively given properties of calculations on formal grounds; in particular, the informal notion of an *elementary* calculation step can be considered in this framework.

Independent Substitutions: Two substitutions are *independent*, if their positions are strongly independent. In contrast to the independence of occurrences, equality of the contexts or of the results are not demanded, as this restriction would be too strong.

Due to the strong independence of the positions, independence of substitutions captures the idea that subterms at independent positions in sufficiently similar contexts are affected by independent substitutions. Sufficient similarity of the contexts is given implicitly by the strong independence of the positions.

As in the case of independent occurrences, two operations on independent substitutions are introduced.

1. *merging substitutions:* The merge function for substitutions transforms two independent substitutions into a new substitution. The position of a merged substitution is defined, essentially, by merging the positions of the arguments. The sequences of affected and inserted terms of the merged substitutions are the concatenations of the respective sequences of the arguments. The context and the result of the merged substitution

are uniquely determined by its position together with the sequences of affected terms and of inserted terms, respectively.

There are four interesting special cases of an application of the merge function on substitutions: if the contexts of the arguments are equal, then the merged substitution is the parallel application of both arguments on the common context; dually, if the results are equal. Furthermore, if the result of the first argument is equal to the context of the second result, then the merged substitution is the sequential application of the first and then of the second argument; similarly, if the context of the first argument is equal to the result of the second argument, then the merged substitution is again the sequential application of both arguments, but in the converse order.

2. *splitting up substitutions:* A substitution can be split up with the help of *intermediate* substitutions. Such intermediate substitutions are, essentially, constructed out of the substitution under discussion by modifying the places of that substitution. Some nominal terms are replaced by the suitable subterms of the context, some nominal terms are kept and some are replaced by the suitable subterms of the result. This way, the places of the underlying substitution are *not yet*, *actually* or *already* replaced in an intermediate substitution.

   A sequence of intermediate substitutions is a *split sequence*, if in the first substitution no place is already replaced, if in the last substitution all places are already replaced and if the remaining substitutions satisfy the following condition: exactly the places already replaced and actually replaced in the previous substitution are already replaced, some not yet replaced places of the previous substitution are actually replaced and the remaining places are not yet replaced. Subsequent entries of a split sequence are independent. A split sequence represents the substitution under discussion, as merging successively the entries results in the substitution under discussion.

Similarity of Substitutions: On the set of all intermediate substitutions (with respect to a previously given substitution), another equivalence relation is introduced: two intermediate substitutions are *similar*, if they agree on their actually replaced places.

This equivalence relation allows the identification of the same replacement in different intermediate substitutions (contained in different split sequences). The similarity of intermediate substitutions is not compatible with

the equivalence of substitutions, because similar, but different substitutions have different contexts and results.

In particular, the similarity of substitutions captures the intuition that exchanging the order of two (independent) calculation steps in a calculation results in the same calculation steps (in different order).

Substitution Functions: Due to the standard recursive definition of ordered n-tuples, a substitution can be understood as an ordered pair such that the first entry is an ordered pair of a multi-shape occurrence and a sequence of standard terms of suitable length and the second entry is that standard term, which results from a replacement of the intended terms in the context by the inserted terms at the marked positions. Therefore, a set of substitutions is a function according to the set theoretical definition of functions. Consequently, sets of substitutions are called *explicit substitution functions*.

Functions traditionally understood as substitution functions are not subsumed under this definition, as the arguments of such functions are, usually, no occurrences. In order to qualify such traditional substitution functions as substitution functions, the concept of *implicit* substitution functions is introduced. Such implicit substitution functions can be transformed via an *explication method* into an explicit substitution function.

The potential and the limitation of the concept of implicit substitution functions is illustrated by discussing central examples; in particular, it is shown that (besides some typical substitution functions) the Gödelised substitution function, which is the Gödelisation of a substitution function for terms and which is, therefore, a primitive recursive function on the natural numbers, is an implicit substitution function. In these examples, the explication method is capable of transforming faithfully an implicit substitution function into an explicit substitution function.

The limitation of the concept of implicit substitution functions is a philosophical problem: explication methods are not determined by an implicit substitution function, but depend on the intuitive interpretation of the substitutions given by an implicit substitution function. It remains a philosophical task to rule out undesired and pathological explication methods.

18 Conclusion: Future Work

We provide a survey of future work related to our investigations of the notions of occurrences and substitutions.

18.1 Fundamental Proof Theory

The idea of "Fundamental Proof Theory" is the investigation of fundamental problems of proof theory on the base of the theory of occurrences and substitutions. We sketch some projects, which can be subsumed under the idea of fundamental proof theory.

18.1.1 Proper Definition of Proofs

The aim of the project "Proper Definition of Proofs" is to provide adequate definitions of the elementary objects and methods of Natural Deduction.[102] As sketched in the introduction of these investigations, an adequate definition of these notions has to be given on the base of a suitable theory of occurrences. We provide some details:

1. *formula tree:* The basic objects of the calculus of Natural Deduction are arbitrary *formula trees*; the more elaborate notion of a derivation is defined only later and based on formula trees. The *conclusion* of a formula tree, which is a formula and not an occurrence, is defined recursively as expected.

2. *inference step:* An *inference step* can be represented by that occurrence of a subtree of a formula tree, which has the conclusion of the intended inference step as its conclusion. The *premises* of an inference step can be defined as the conclusions of the direct subtrees of the respective occurrence representing the inference step.

   It is worth mentioning that no formula occurrence has to be introduced to represent occurrences of the premises and of the conclusion of an inference step. For an adequate development of the notion of a derivation, it is sufficient to use the position of the respective inference step and the positions of the direct subtrees instead.

3. *assumption:* A *single assumption* can be represented as a single occurrence of an atomic subtree of a formula tree.

[102] First ideas related with this project were presented at the sixteenth international workshop *Proof, Computation, Complexity* (PCC 2017) in Göttingen, Germany.

The representation of assumptions as formula occurrences in formula trees seems inconvenient, as we would have to distinguish the assumptions from other formula occurrences. The obvious way to do so, would be to check whether they are also occurrences of atomic subderivations.

In general, we are entitled by the inference rules to discharge more than a single assumption in an inference step. Multiple occurrences representing all single occurrences actually discharged in an inference step can be obtained by applying the merge function on the respective single occurrences.

4. *dependence on assumptions:* Inference rules allow to discharge only assumptions on which the conclusion of an inference step or some of its direct subtrees depend on. Similarly, some inference rules demand restrictions on assumptions, on which the conclusion of an inference step or some of its direct subtrees depend on.

   This dependence can be decided formally by checking whether the respective assumptions lie within the respective inference step or within the respective occurrence of a direct subtree or not.

5. *discharge function:* From a technical point of view, it is convenient to define a *discharge function* as a function mapping inference steps to the multiple occurrence representing all assumptions discharged in that inference step. Other approaches are possible. For example: mapping the multiple assumption to the inference step, mapping the inference step to the set of single assumptions etc. All of these approaches have in common that the discharge function is, essentially, a function mapping occurrences to occurrences.

6. *inference rules:* Under the perspective of a theory of occurrences, it is convenient to define an inference rule as a function mapping, in the most general case, the direct subtrees of an inference step, the respective inference step and the respective discharge function to 1, if the inference step is generated according to this rule, and to 0 otherwise.

7. *rules function:* In the general case, there is more than one possible rule according to which an inference step could be generated. In order to keep track of the actually applied inference rules, it is convenient to define a rules function mapping the inference steps to the actually applied rules.

8. *intensionally defined derivations:* A derivation can be defined intensionally as a triple $\langle \mathfrak{D}, \mathcal{L}, \mathcal{R} \rangle$, where $\mathcal{D}$ is a formula tree together with a

discharge function $\mathcal{L}$ such that each inference step is generated according to the rules given by the rule function $\mathcal{R}$. In particular, we observe that a derivation is not an elementary syntactical entity anymore.

It is worth mentioning that the intensional definition of a derivation is more natural in the theory of occurrences than the expected recursive definition.

The essential problem of a recursive definition is that the context of an occurrence is determined uniquely by that occurrence. In other words, an occurrence in a subderivation cannot be an occurrence in the derivation itself. As a consequence, a discharge function for a subderivation cannot be extended to a discharge function for the derivation itself.

In order to overcome this problem, occurrences in subderivations have first to be identified with occurrences in the derivation.

1. *update:* The generation of a complex derivation out of its direct subderivation can be described with the help of so called *generation forms*, which are a simple kind of nominal forms. Such generation forms can be used to define *updates* of occurrences in the direct subderivations. These updates are occurrences in the complex derivation, which correspond to the occurrences in the direct subderivations as prescribed by the generation forms.

   Identifying the occurrences in the subderivations with their updates captures the intuition that the occurrences in the subderivations are the *same* as their updates (in a more complex context). This identification is a philosophically interesting identity relation between occurrences in different contexts: The identification is neither reflexive nor symmetric. Furthermore, if the same subderivation is used more than once in the generation of the complex derivation, then we have several, but different updates in the complex derivation, each identified with the respective occurrence in the subderivation, though not related with each other.

2. *recursively defined derivation:* The recursive definition of a complex derivation can be given as follows: In a first step, the complex formula tree is generated as prescribed by a generation form determined by the inference rule. In accordance to this generation form, all the occurrences in the direct subtrees are updated. In particular, the discharge function and the rule function have to be redefined with respect to the updated occurrences. In a second step, the discharge function has to be extended by the discharge done in the last generation step as well as the rule function by the inference rule applied in the last generation step.

3. *conversions:* The proof conversions, as required for normalisation, can be treated analogously to the (recursive) generation of derivations. In contrast to the generation of derivations, the proof conversions are not described by the generation forms, but by the position of that (multiple) occurrence, which represents the assumptions, where the derivations are concatenated according to the conversion schema. Additionally, the maximal formula has to be eliminated in the converted derivation, which is another transformation of derivations describable with the methods introduced in these investigations.

We emphasise: in contrast to our naive intuitions (as well as to the intuitions proposed by Prawitz in [23]), we do not need formula occurrences to define the notion of a derivation.

18.1.2 The Calculus of Natural Calculation

The aim of the project "The Calculus of Natural Calculation" is the introduction and investigation of the extension of Gentzen's calculus of Natural Deduction by proper term rules.[103] Due to these term rules, a natural formal representation of calculations inside of proofs, as it is found in mathematical praxis, is possible.

1. *term rules:* Essential feature of the calculus of Natural Calculations is the availability of proper term rules, which are inference rules such that some premises or the conclusion are proper terms instead of the usual formulae. In the basic version of the calculus of Natural Calculation, a complete and sound set of term rules for the treatment of identity is given.[104]

2. *calculation:* Essentially, a calculation step can be represented formally in the extended calculus by an inference step satisfying that one premise is the context (a term), another premise is the justification (an equation or a side calculation) and the conclusion is the result (a term) of that calculation step. A calculation is a tree of such calculation steps.

[103]Basic ideas and concepts related to the calculus of Natural Calculation are communicated partly on several occasions; we mention our talk [9] at the conference "Beyond Logic" in Cerisy-la-Sale, France, 2017, and our talk [11] at the third conference on "Proof-Theoretic Semantics" in Tübingen, Germany, 2019. A first article [12] about this calculus is submitted.

[104]Natural Deduction, as introduced in Gentzen [13], has no identity rules; the term rules introduced in Natural Calculation are meant to replace the traditional identity rules, as found, for example, in the text book "Logic and Stucture" [33] by van Dalen.

3. *calculations inside proofs:* In a pure calculation, the equations used as justifications are assumptions and the conclusion is a term.

   This is different, if the calculation is part of a proper proof: the equations are the conclusions of arbitrary subderivations, in which traditional inference steps with formulae as well as calculation steps may occur. Additionally, the result of a calculation, which is an equation, can be inferred from a calculation. This result can be used as a premise for arbitrary inference steps, in particular for the traditional inference steps with formulae.

We mention some aspects of this project related with a suitable theory of occurrences:

1. *proper definition of proofs:* A proper definition of the derivations in the extended calculus would be a technically demanding project, as we would have to deal with trees, in which formulae may occur as well as proper terms. Despite the technical effort, there seems to be little insight obtainable beyond the results already achievable in the project "Proper Definition of Proofs" as discussed above.[105]

It seems reasonable to avoid these technicalities, to focus on more elaborate topics related with the extended calculus and to apply the formal methods of a theory of occurrences and substitutions only where it promises benefits. Subsequently, we concentrate on such topics.

1. *precise definitions:* Due to an adequate theory of occurrences, we are able to provide precise formal definitions. In particular, the definitions of the inference rules for formulae as well as for terms can be given without reference to intuitions.

2. *proof conversions for calculations:* In order to prove completeness and soundness of the basic version of the calculus of Natural Calculation, it is convenient to introduce some proof conversions for calculations. In particular, the `t`-*variants* of a calculation (where `t` is a unary nominal term) are defined in terms of a theory of occurrences. Due to the existence of formal objects representing the positions of occurrences, these conversions can be introduced and investigated formally.

3. *subatomic normalisation:* In contrast to argumentations with formulae, calculations with terms are more amenable to rearrangement. The

[105] Consequently, we suggest a detailed introduction of the derivations in the extended calculus, based on our corresponding investigation of the calculus of Natural Deduction, as a suitable student project in the field of logic.

order of two independent calculation steps can be alternated; independent calculation steps can be merged into a simultaneous calculation step, a simultaneous calculation step can be split up into independent calculation steps.

In contrast to this variability, the arrangement of calculations seems also to be more restricted than the argumentation with formulae, as possible calculation steps in a calculation depend on the available justifications. In particular, the complexity of the terms in a calculation cannot depend on the complexity of previous or of subsequent terms in that calculation; a property comparable to the subformula property seems to be unprovable.

As a consequence, the introduction and investigation of normal calculations seems to be a demanding problem benefitting from the precision and methods provided by the theory of occurrences and substitutions.

4. *corresponding properties:* Most of the term rules, by which the basic version of the calculus of Natural Deduction is extended, can be understood as inference rules for the equality symbol. Due to the precise formulation of these new inference rules, it is possible to associate the properties of the equality symbol with features of the calculus.

   This correspondence between the properties of the equality symbol and its inference rules has an impact on philosophical theories of meaning, as proof-theoretic semantics, which aim to establish the meaning of logical constants by their inference rules.[106]

The basic version of the calculus of Natural Calculation provides complete and sound term rules for the identity. There are more extensions of Natural Calculation worth to be investigated:

1. *smaller-than relation:* With the help of negative and positive parts of terms,[107] it is possible to provide term rules for smaller-than calculations, for example, in a theory of the integers. As in the case of identity, the properties of the smaller-than relation correspond with special features of the inference rules.

[106] According to Schroeder-Heister [28] proof-theoretic semantics is the proof-theoretic alternative to denotational semantics; Gentzen's remark [14, p. 80], that introduction rules in Natural Deduction define the meaning of logical constants, is one of the roots of this proof-theoretic approach to meaning.

[107] The analogous concept of positive and negative parts of propositional formulae is already introduced by Schütte [29, p. 11]; he uses them to identify semantic properties of formulae by applying only syntactic methods; Prawitz [23, p. 43] uses also such negative and positive parts of formulae in his discussion of the form of normal derivations.

More generally, it is an interesting question, which properties of relations can be characterised by inference rules.

2. *non-standard rules:* Motivated by the term rules already discussed, we find more interesting rules for the calculus of Natural Calculation. We mention the following:

   It is possible to introduce *transition rules*, which allow to infer the Gödelnumber of a formula from that formula and vice versa. Such transition rules could become reasonable, if the calculus of Natural Calculation is additionally extended by term rules for a meaningful calculation with Gödelnumbers.

   Analogously, it is possible to introduce transition rules into the metalanguage: if a formula is derived from some assumptions, then we may introduce the sequent representing the proved derivability statement; if the antecedents of such a sequent are inferred as side premises, then we may infer its succedent from that sequent. Extending the calculus of Natural Calculation by suitable sequent rules makes these transition rules reasonable.

   The extension of the calculus of Natural Calculations into the metalanguage is interesting from a philosophical point of view, because the sequents are natural formal representatives of the concept of *lemmata used in a proof*.

18.1.3 On the Existence of Pure Proofs

The project "On the Existence of Pure Proofs" is motivated by the philosophical discussion about pure proofs, as discussed by Arana [1], and aims to show (on formal grounds) that every proof can be transformed into an equivalent (or stronger) proof not using extraneous notions. We provide some aspects of our account to this (philosophical) problem:[108]

1. *principle approach:* In the focus of the investigations is the formal representation of informal mathematics in a calculus, which is according to Gentzen "as close as possible to actual reasoning" [14, p. 68], namely the calculus of Natural Deduction. We identify the non-logical symbols of a (faithful) formal language as the formal representatives of the notions present in the formalised proof.

[108] Some partial results are communicated in our talk [10] at the conference "Universal Logic" in Vichy, France, 2018.

2. *concept of pureness:* We distinguish between two kinds of pureness: a derivation is called *relatively pure* with respect to a given set of non-logical symbols, if only non-logical symbols contained in that set occur in the derivation. Furthermore, non-logical symbols not contained in that set are called *extraneous*.

   A derivation is called *absolutely pure*, if the derivation is relatively pure with respect to the set of non-logical symbols occurring in the undischarged essential assumptions and in the conclusion of that derivation. Thereby, an assumption is called essential, if it does not vanish under the process of normalisation.[109]

3. *central claim:* Every derivation can be transformed into an equivalent (or stronger) derivation, which is absolutely pure.

4. *partial solutions:* It is a technical lemma found in every good textbook on mathematical logic that we can replace constant symbols in a derivation by suitable variables. As a consequence, we can eliminate extraneous constant symbols in a derivation.

   Due to the normalisation results by Prawitz [23], every derivation can be transformed into an equivalent (or stronger) derivation in normal form. Such normal derivations have the subformula property. The latter means (besides some exceptions not relevant in our discussion) that every formula occurring inside the derivation is a subformula of an undischarged assumption or of the conclusion. Therefore, no extraneous relation symbol occurs in a normal derivation.

5. *new methods:* In order to prove the main theorem, we only have to show that we can eliminate every extraneous function symbol in a derivation. Doing so, we introduce two new methods in the theory of occurrences (of terms in derivations).

   (a) *congruence of occurrences:* Roughly spoken, two single occurrences in a derivation are *congruent*, if they have necessarily the same shape due to the inference rules. As a consequence, we can only replace an occurrence in a derivation, if we simultaneously and uniformly replace a complete equivalence class with respect

[109] As we understand assumptions as occurrences of atomic derivations, the concept of an essential assumption is defined by relating occurrences in different derivations. This can be done with similar methods as the updates for the generation of complex derivations and the analogous methods used for dealing with proof conversions, as discussed in our project about the proper definition of proofs.

to the congruence of occurrences. Again, we obtain a multiple occurrence representing the complete equivalence class by merging the respective single occurrences.

(b) *localisation:* When introducing an implication, we are entitled to discharge assumptions having the shape of the antecedent of the introduced formula. As a consequence, occurrences in the discharged assumption are congruent to occurrences in the antecedent of the inferred implicative formula.

This means that besides the positions of such occurrences given by nominal derivations, we have to deal with a second specification of the respective occurrence, namely with its *localisation* in the antecedent of a specific formula. Such localisations of occurrences can be treated in a generalised theory of occurrences.

6. *proof strategy:* With similar methods as used in the proof of the subformula property by Prawitz, it is possible to show for normal derivations: if an occurrence of a complex term is not replaceable by a variable (due to the inference rules), then this occurrence is congruent to an occurrence in an undischarged assumption or in the conclusion of that derivation. As a consequence, we can transform every derivation into an equivalent (or stronger) pure derivation in normal form.

This formal result on the existence of pure derivations seems to be a good approach towards a clarification of the problem of pure proofs. Nevertheless, there remain some philosophical obstacles, in particular:

1. *defined notions:* We can avoid the use of defined notions in a formal language and replace them by abbreviations using only primitive notions.

   In set theory, for example, a formal language is sufficient, in which only the binary ϵ-symbol is available as non-logical symbol. All the notions usually defined, as the empty set (constant symbol), the power set function (function symbol) or the set inclusion (relation symbol), can be avoided by using the typical definitional clauses as abbreviations.

   As a consequence, there can be notions present in a proof (in a derivation), which are not represented by non-logical symbols, but by formula schemata.

In order to solve the underlying philosophical problem of pure proofs, we have to consider such and similar phenomena and provide some convincing solutions.

18.2 More Theories of Occurrences and Substitutions

In principle, it is possible to carry over the theory of occurrences and substitutions to any meaningful combination of syntactic entities. There are, at least, two reasons to investigate such a carry over:

1. *new phenomena:* There are some interesting phenomena, which are not yet investigated.

2. *new problems:* There are some interesting problems such that the solution of these problems demands the treatment of new types of occurrences.

The main motivation for the projects discussed above is given by the second reason. Subsequently, we discuss briefly some types of syntactic entities such that their definitions involve phenomena not present in a first order language. As a consequence, an investigation of the corresponding theories of occurrences seems to be interesting.[110]

Infinitary Languages: One essential property of first order languages is that their syntactic entities are generated in finitely many steps. Dropping this restriction, we obtain infinitary languages. We mention some applications of such infinitary objects:

1. *power series:* The infinite sums and, in particular, power series play a central role in analysis. In order to generate such power series, we have to drop the restriction that sums are generated in finitely many steps.

2. *infinitary formulae:* Introducing infinitary long formulae, the expressive power of first order logic can be increased. For example, the class of torsion groups can be axiomatised with the help of the following inifinite axiom:[111]

$$\forall x.(x = e \ \vee \ x \circ x = e \ \vee \ x \circ x \circ x = e \ \vee \ \ldots)$$

   In order to generate such axioms, we have to drop the restriction of finite generation with respect to the disjunction.

[110] We do not see any problem, in principle, in the investigation of these new kinds of occurrences; we assume that the investigation of these theories are suitable for student projects.

[111] Cf. the logic textbook [6, p. 136] by Ebbinghaus e.a. for more details.

Unique Readability: Every complex syntactic entity is generated out of uniquely determined subentities. This can be different, for example, with respect to regular languages as discussed in theoretical computer science. A simple example of such a language is given by the following regular expression:

$$a(ba)^*|(ab)^*a$$

Depending on the chosen generation of the word aba, for example, it is generated out of the words ab and a or out of the words a and ba.

Compositional Languages: For the definition of derivations, Zimmermann [37] uses special elimination rules, which he calls *elimination rules by composition*. According to Zimmermann, such inference rules are explained best with respect to the elimination rule for the disjunction. The usual three subderivations are presupposed:

$$\begin{array}{c} \mathbf{D}_0 \\ A \vee B \end{array} \quad ; \quad \begin{array}{c} A \\ \mathbf{D}_1 \\ C \end{array} \quad ; \quad \begin{array}{c} B \\ \mathbf{D}_2 \\ C \end{array}$$

The elimination by composition rule for disjunction allows to generate, for example, the following derivation:

$$\dfrac{\begin{array}{cc} & A \\ \mathbf{D}_0 & \mathbf{D}_1 \\ A \vee B & C \end{array}}{\begin{array}{c} B \\ \mathbf{D}_2 \\ C \end{array}}$$

We have to presuppose that there is at least one open assumption of B in $\mathbf{D}_2$ and to chose one, where the derivation above the double line is composed with the derivation below the double line.

Besides such subtleties, we may observe: the last inference step is represented by the double line; the conclusion of this inference step is not the formula below that double line, but the conclusion of the subderivation $\mathbf{D}_2$. Furthermore, the subderivation $\mathbf{D}_2$ is not a subtree (according to the usual definition), but located in the middle of the full derivation.

This means that the notion of a derivation (in a calculus with such compositional rules) cannot be given in the traditional way, but have to use non-trivial proof compositions. A detailed analysis of such derivations and of more elaborate concepts as the discharge function, subderivations, proof conversions etc. seems to be technically demanding and interesting.

Natural Languages: It is quite obvious that we may, in principle, apply the method of marking positions by nominal symbols also with respect to the syntactic entities of natural languages. A detailed analysis, whether and where this method may be used to achieve interesting results, is left to the kind reader more experienced in linguistics.

Another strategy to find new theories of occurrences is not to change the concept of the syntactic entities of the underlying language, but to vary the concept of occurrences.

Type-Insensitive Occurrences: The occurrences discussed in our investigations could be described as *type-sensitive* occurrences. The latter means that the shape of an occurrence is of the same syntactic type as the context (or of a subtype of this type). We can generalise the notion of an occurrence by dropping the restriction to such types (and loosing this way the central advantage of the presented approach to occurrences).

In order to do so, we have first to generalise the notion of the underlying syntactic entities. In the case of the standard terms of a formal language, for example, we could introduce first arbitrary strings over the extended alphabet (containing the nominal symbols). Standard terms would be defined as the strings satisfying the usual inductive definition. Nominal terms would be those strings, which are an elimination form of standard terms.

We illustrate this generalised notion of a nominal term; investigate the following string of symbols:

$$\mathtt{t} \;\doteq\; (0*)+0$$

Applying the respective general substitution function on $\mathtt{t}$ and, for example, on the string $+0$ results in the standard term $(0+0)+0$.[112]

Such a type-insensitive account to occurrences becomes useful, when discussing derivations (of the calculus of Natural Deduction): we can identify derivations being a *part* of another derivation without being a subderivation. Investigate the following generalised nominal derivation and standard derivation:

$$\mathtt{D} \doteq \dfrac{\dfrac{A \qquad A}{*}}{A} \quad ; \quad \mathbf{D} \doteq \dfrac{\dfrac{A \qquad A}{A \wedge A}}{A}$$

The nominal derivation $\mathtt{D}$ is an elimination form of the derivation $\mathbf{D}$ in which the atomic derivation $\mathbf{F} \doteq A \wedge A$ is actually eliminated in. While being a part of $\mathbf{D}$, the derivation $\mathbf{F}$ is obviously not a subderivation of $\mathbf{D}$.

[112] Actually, this is how Schütte defines nominal forms, namely as arbitrary strings over the extended alphabet. Cf. Schütte [29, p. 11].

Multi-Type Occurrences: Another canonical generalisation of the concept of occurrences is to drop the restriction that the sequence of shapes (of multi-shape occurrences) contains only one type of syntactic entities. Such a generalisation of the notion of occurrences seems only interesting, if there are interesting problems, which can be solved on this base.

Nested Occurrences: Due to our choice of representing positions via the nominal forms, we have a good concept of the lies-within relation, but we are not able to represent such nested occurrences simultaneously.

A first idea to change the situation are resolvable occurrences represented by an ordered quadruple as follows:

$$\langle 1 + (0 + 0), 0, *_0 + 0, 1 + *_0 \rangle$$

There are two strategies to obtain an occurrence (according to our definition): the intermediate nominal term $*_0 + 0$ can be used together with the second entry 0 to determine the shape $0 + 0$ as well as to determine the position $1 + (*_0 + 0)$ together with the fourth entry $1 + *_0$. In other words, resolving the quadruple results in the following both occurrences:

$$\langle 1 + (0 + 0), 0 + 0, 1 + *_0 \rangle \quad ; \quad \langle 1 + (0 + 0), 0, 1 + (*_0 + 0) \rangle$$

Observe that the second occurrence lies within the first; the quadruple represents the nested occurrences simultaneously.

A detailed analysis of this idea (or of better alternatives) can be interesting; in particular, such an analysis could result in a more general concept of occurrences.

18.3 Philosophical Discussions

Besides the formal character of our investigations, we consider the topics under discussion as philosophically relevant. We sketch some, which we consider to be interesting for further investigation.

Occurrences: In our investigations, we discussed the notion of occurrences from a formal point of view and provided a good formal conception of this notion. As already mentioned in the introduction, there is some philosophical literature discussing occurrences (in the context of types and tokens). We consider a our formal conception as a good starting point to a clarification of the philosophical questions concerning occurrences.

Formal Languages: We sketched in the preliminaries our own account to formal languages. Central difference to the usual approach is that we do not identify a formal language with some sets of syntactic entities, as, for example, the set of all formulae. In our conception, a formal language is identified with some underlying principles (maybe captured best by a signature) determining the syntactic entities available in the respective language. Our approach to formal languages is supported by our intuitions; nevertheless, we did not clarify our conception in all details and left some principle questions open.

Central philosophical problem is the identification of formal languages. In particular: should we identify languages differing only in the available logical symbols, or languages differing in the restrictions on the generation of their syntactic entities.[113] A more detailed investigation involving philosophical literature (if there is any) seems necessary.

Formal Languages (Infinite Terms): Our considerations about the identity of formal languages motivates the following thought experiment: consider a language of arithmetics, where we can apply the generation rules for terms infinitely often. Consequently, we can define as follows:

$$\omega = S^{\omega}(0)$$

It is interesting that this definition can be given without a change of the generation rules for formulae. If we allow, additionally, that the standard terms have no denotation, than we can add the following axiom to the usual axioms of arithmetics:

$$\forall x.x \neq S^{\omega}(0)$$

Using this axiom, we should be able to rule out the non-standard models of arithmetics. The latter means that we transcend the expressive power of standard first order languages via this extension of the generation rules for terms.

We emphasise that this approach to logic excludes the presupposition that every term has a denotation. (This is exactly, what we try to exclude by the suggested axiom.) This restriction is insofar interesting, as we can express in natural languages such non-existing concepts. Also, it seems that mathematicians have to deal with the analogous phenomenon, when they distinguish converging and non-converging sums. Finally, such an approach

[113] As an example for the latter question: should we identify a standard formal language of arithmetics with a language, where we allow that the terms are generated by infinitely many generation steps?

to logic seems to be an interesting challenge for the proof-theoretic semantics approach to meaning.

Informal Concepts of Mathematics: We have introduced suitable formal representatives of the informal concepts of occurrences and substitutions. This introduction is not only of its own value, but, more importantly, it permits the discussion of more elaborate informal concepts, as the calculations and the substitution functions. In particular, the formal representation of informal properties of these elaborate concept can contribute to the philosophical foundation of these notions.

We were able, for example, to uncover formal mechanisms allowing to identify *elementary* calculations; similarly, we suggest in the project "Pure Proofs" formal methods identifying *pure* derivations. We mention some more philosophically interesting concepts probably benefitting from the precise formal methods provided by a theory of occurrences:

1. *identity of proofs:* One important approach to the philosophical debate about the identity of proofs is Prawitz suggestion to identify proofs having the same normal form.[114] An investigation of pure proofs has impact on this philosophical position: pureness is a good reasons to distinguish normal proofs from their pure version.

2. *simplicity of proofs:* Another interesting philosophical problem is the simplicity of proofs, which is Hilbert's 24th (unpublished) problem.[115] Simplicity of a proofs seems to be a notion similar to pureness: intuitively clear, but demanding a formal foundation.

Implicit Substitution Functions: Functions, traditionally understood as substitutions functions, were classified as substitution functions via the concept of implicit substitution function. These are function having an explication method transforming them into explicit substitution function. The basic method of explication seems, in principle, suitable, but undesired functions are subsumed under the concept of substitution function. In order to improve the result and to rule out the undesired and pathological functions, some philosophically justified restrictions on the explication methods have to be provided.

[114]Motivated by ideas of Kreisel, Prawitz [24] introduced the field of general proof theory, in which the identity of proofs and related philosophical questions are investigated.

[115]Hilbert's unpublished notes about the problem of providing a criterion for the simplicity of proofs were discovered recently by Thiele [32]; maybe motivated by this discovery, there is some recent discussion about this problem. A survey of the problem and some recent contributions to the debate are found in Hipolito and Kahle [17].

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